Studies in CORE MATHEMATICS

[IN ENGLISH]

K. C. NAG

BOARD OF SECONDARY EDUCATION
WEST BENGAL

SYLLABUS FOR

MATHEMATICS (COMPULSORY)

Vide Notification No. Syl/1/62 dated 30th March

(This course is intended to be mainly a revision of the work done in earlier classes and reoriented to the use of Mathematics in daily life. The teacher is only expected to define the various terms used in the course-content and show their practical utility. It is not desired that he should burden the student with too many methematical details, methods and problems.)

(CLASS– IX)

Unit 1—ARITHMETIC

All questions should be straight forward. Application of Algebra should be permitted in Arithmetic.

Revision of previous work—Vulgar and Decimal fractions including Recurring Decimals ; Extraction of Square Root ; Square and Cubic measures ; Simple examples of Unitary Method including Time and Work, Time and Distance, Percentages and easy cases of Simple Interest. Simple ideas of Approximation (excluding Contracted Method and Infinite Series).

Compound Interest (calculation of interest only) ; Profit and Loss.

Unit 3—ALGEBRA

Revision of previous work—Directed Numbers ; Fundamental Laws ; Problems and Simple Equations ; the following formulæ with their applications : $(a+b)^2$, $(a-b)^2$, a^2-b^2, $(a+b)^3$, $(a-b)^3$, a^3+b^3, a^3-b^3 ;

Easy Factors ; H. C. F. ; L. C. M. ; Easy Fractions.

Simple simultaneous Equations involving two unknowns ;

Problems leading to Equations, Simple and Simultaneous ; Graphs of Simple Equations.

Unit 4—GEOMETRY

THEORETICAL

Revision of previous work as in The Board's Syllabus up to Class VIII.

To prove—

1. The opposite sides and angles of a parallelogram are equal, each diagonal divides the parallelogram into congruent triangles, and diagonals of a parallelogram bisect one another.

2. A quadrilateral is a parallelogram if—

(*i*) both pairs of opposite sides are equal, *or*

(*ii*) both pairs of opposite angles are equal, *or*

(*iii*) one pair of opposite sides are equal and parallel, *or*

(*iv*) its diagonals bisect one another.

3. If there are three *or* more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side and is equal to half of it.

The straight line joining the middle points of two sides of a triangle parallel to the third side and equal to half of it.

4. The formal proof should be preceded by practical work with squared paper in all the cases of this paragraph.

(*i*) Parallelograms on the same base and between the same parallels are equal in area.

(*ii*) Triangles on the same base (or on equal bases) and between the same parallels (*or* of the same altitude) are equal in area.

(*iii*) Equal triangles on the same base and on the same side of it are between the same parallels.

(*iv*) If a triangle and a parallelogram stand on the same base and are between the same parallels, the area of the triangle is half that of the parallelogram.

(*v*) In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

(*vi*) If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.

5. To prove—

The locus of points which are equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

The locus of points which are equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the two angles between the two given lines.

6. (*i*) The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.

(*ii*) The bisectors of the angles of a triangle are concurrent.

(*iii*) The medians of a triangle are concurrent.

PRACTICAL

1. Revision of previous work.

(i) Bisection of angles and straight lines.

(ii) Construction of a perpendicular to a straight line.

(iii) Construction of an angle equal to a given angle.

(iv) Construction of parallels to given straight lines.

(v) Construction of triangles with given parts.

(vi) Division of a straight line into a given number of equal parts.

2. Construction of quadrilaterals.

3. Construction of a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

4. Construction of a triangle equal in area to a given rectilineal figure.

CLASS X

Unit 1—ARITHMETIC

All questions should be straightforward. Applications of Algebra should be permitted in Arithmetic.

Ratio and Proportion ; simple examples on Unitary Method including direct Problems on Income-tax, Foreign Exchange and Draft ; Metric system dealing with topics of conversion. (Adequate practice should be given in the use of the metric system of weights and measures including area and volume).

Unit 2—STATISTICS

Frequency Tables ; Averages—Mean, Median and Mode ; Mean and Standard Deviations ; Graphical representations—Histogram, Frequency polygon.

(All data used for imparting the above-mentioned rudiments of Statistics should be collected by the pupils themselves. Examples : Weights, heights, ages of pupils, their school attendance and progress in studies, etc.)

Unit 3—ALGEBRA

Simple quadratic equations as can be solved by easy factorisation.

Graphical solutions of simultaneous Equations of the first Degree ; Ratio and Proportion.

Unit 4—GEOMETRY

THEORETICAL

1. **To prove—**

There is one circle, and only one, which passes through three *given points not in a* straight line.

2. **Axioms—**

In equal circles (or, in the same circle) equal chords cut off equal arcs and subtend equal angles at the centre and conversely.

To prove—

3. A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord and conversely.

4. In equal circles (or in the same circle) equal chords are equidistant from the centres and conversely.

5. The angles which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

6. Angles on the same segment of a circle are equal, and if the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

7. The angle in a semi-circle is a right angle ; the angle in a segment greater than a semi-circle is less than a right angle ; and the angle in a segment less than a semi-circle is greater than a right angle.

8. The opposite angles of any quadrilateral inscribed in a circle are supplementary and the converse.

The following theorems are also to be included :—

(i) The tangent at any point of a circle and its radius through the point are perpendicular to one another.

(ii) The two tangents to a circle from an external point are equal and they subtend equal angles at the centre.

(iii) If two circles touch, the point of contact lies in the straight line through the centres.

PRACTICAL

Simple cases of construction of circles : construction of Designs with Geometrical Figures.

Unit 5(a)—MENSURATION

Area of a Triangle ; Circumference and Area of a Circle ; Surface and Volume of Rectangular parallelopiped, Cylinder and Sphere.

Unit 5 (b)—GEOMETRY OF SPHERE

Elementary ideas of Geometry of a Sphere leading to the definition of Latitude, Longitude.

The following demonstrations and experiments are suggested for Class X, in connection with the different units, as indicated below :—

1. DEMONSTRATION & EXPERIMENTS

(Note : "D stand for *demonstration* and "E" for *experiments*).

Unit 1—ARITHMETIC

D. Explanation of Specimen Cheques ; Drafts ; Bills ; Foreign Currencies ; etc.

Unit 2—STATISTICS

E. Determination of weights, heights and ages of pupils and their Graphical Representations.

Unit 4—GEOMETRY

D. Explanation of Models of Geometrical Figures.

Unit 5 (a)—MENSURATION

E. Measurement of Areas of Rectangular Figures and Triangles ; Circumference and Area of a Circle.

Unit 5 (b)—GEOMETRY OF SPHERE

D. Geometry of Sphere.

CONTENTS

ARITHMETIC

Miscellaneous Examples on the first four rules

Examples (1)

1. The sum of two numbers is 2346 and one of them is twice the other. Find the numbers.

If one number $=1$,
the other $=2$,
$\therefore$ by addition, 3

$$\begin{array}{r|l|l} 3 & 2346 & 782 \\ & \underline{21} & \\ & 24 & \\ & \underline{24} & \\ & 6 & \\ & \underline{6} & \end{array}$$

$\therefore$ one number $=782$ and the other number $=782\times2=1564$.

2. The sum of two numbers is 1734 and their difference is 204. Find them.

Suppose the numbers to be x and y.

Then $x+y=1734$

and $x-y=\ 204$

$\therefore$ (adding) $2x=1938$

$\therefore$ $x=1938\div2=969$ and $y=1734-969=765$

$\therefore$ the required numbers $=969$ and 765.

3. The sum of two numbers is 5 times the smaller and the larger number is 420 ; find the other number.

The greater number $+$ the smaller number $=5$ times the smaller number,

$\therefore$ the greater number $=4$ times the smaller number.

$\therefore$ the required smaller number $=$ the greater number $\div 4$,
$=420\div4=105$.

4. The population of India being 315 millions and their total yearly income being 11340 million rupees, find the yearly income of an Indian on the average. [C. U. 1932]

The yearly income of 315 million men $=11340$ million rupees.

$\therefore$ the average yearly income of an Indian

$$=\frac{11340 \text{ millions}}{315 \text{ millions}} \text{ Rupees}=\frac{11346}{315} \text{ Rupees}=36 \text{ Rupees.}$$

5. Two numbers when divided by a certain divisor leave the remainders 4375 and 2986 respectively ; but when the sum of the two numbers is divided by the same divisor, the remainder is 2361. Find the divisor. [P. U. 1918]

If we divide the two numbers separately, the remainders are 4375 and 2986 respectively. $\therefore$ If their sum is divided by that divisor, there will remain (4375+2986) or 7361 as remainder. But here the remainder is 2361. $\therefore$ It is evident that 7361 is greater than the divisor by 2361 (i.e., if we divide 7361 by the divisor, the quotient will be 1 and the remainder left will be 2361).

$\therefore$ the required divisor $=7361-2361=5000$.

6. A and B together had Rs. 82, B and C Rs. 70, and A and C Rs. 76. Find how much each had.

The sum of rupees of A and B = 82
" " " " " B and C = 70
" " " " " A and C = 76

$\therefore$ Twice the sum of rupees of A, B and C = 228

$\therefore$ The sum of rupees of A, B and C $=228\div2=114$

$\therefore$ A had (114 − 70) or 44 rupees, B(114 − 76) or 38 rupees and C (114 − 82) or 32 rupees.

7. The quotient arising from the division of 9264 by a certain number is 17 and the remainder is 373. Find the divisor.

[C. U. 1929]

Divisor = (Dividend − Remainder) ÷ Quotient

Here dividend = 9264, quotient = 17 and remainder = 373.

$\therefore$ the required divisor $=(9264-373)\div17=8891\div17=523$.

8. If a number when divided by 56 leaves a remainder 29, what remainder will it leave when divided by 8 ? [C. U. 1927]

56 is a multiple of 8, $\therefore$ the part of the number divisible by 56 is also divisible by 8.

$\therefore$ When the whole number is divided by 8, the remainder will be the same as that when 29 is divided by 8.

$$\begin{array}{r|l|l} 8 & 29 & 3 \\ & 24 & \\ \hline & 5 & \end{array}$$

$\therefore$ the required remainder = 5.

9. A boy had to divide 7865321 by 254. He copied a figure wrong in the divisor and obtained 33612 as quotient with remainder 113. What mistake did he make? [C. U. '36]

Divisor=(Dividend − Remainder)÷Quotient.

∴ Here divisor=(7865321 − 113)÷33612

=7865208÷33612=234.

∴ The boy copied 234 in place of 254.

[N. B. Here the operation of division of 7865208 by 33612 should be shown.]

10. Divide Rs. 320 among A, B and C, so that A may get Rs. 10 more than B and B may get Rs. 7 less than C.

If B gets 1 share

C will get 1 share+Rs. 7.

and A will get 1 share+Rs. 10.

∴ The whole amount=3 shares+Rs. 17

∴ the amount of 3 shares=Rs. 320 − Rs. 17=Rs. 303.

∴ the value of 1 share=Rs. 303÷3=Rs. 101.

∴ A gets (Rs. 101+Rs. 10) or Rs. 111.

C gets (Rs. 101+Rs. 7) or Rs. 108

and B gets Rs. 101.

11. Divide 24680 rupees among A, B and C so that for every 2 rupees given to A, B gets 3 rupees and C gets 5 rupees. [C. U. '35]

Every time A gets Rs. 2,
B gets Rs. 3,
C gets Rs. 5

∴ Every time they three together get Rs. 10.
∴ Each will get his share (Rs. 24680÷Rs. 10) or 2468 times.
∴ A gets (Rs. 2×2468) or Rs. 4936,
B gets (Rs. 3×2468) or Rs. 7404.
and C gets (Rs. 5×2468) or Rs. 12340.

12. Find the least number of rupees that should be added to 149250 rupees to make the sum equally divisible among 4744 persons. [C. U. '34]

$$\begin{array}{r|l|l} 4744 & 149250 & 31 \\ & \underline{14232} & \\ & 6930 & \\ & \underline{4744} & \\ & 2186 & \end{array}$$

∴ the least number of rupees=Rs. (4744 − 2186)=Rs. 2558.

13. In a certain division sum the dividend is 37693, the quotient 52, and the remainder greater than 52 but less than 104. Find the divisor. [C. U. '35]

$$
\begin{array}{r|l|l}
52 & 37693 & 724 \\
 & \underline{364} & \\
 & 129 & \\
 & \underline{104} & \\
 & 253 & \\
 & \underline{208} & \\
 & 45 &
\end{array}
$$

Here the remainder is 45. But in the problem it is stated that the remainder is greater than 52 but less than 104. ∴ the last digit of the quotient will be 3 instead of 4 and then the remainder will be $53+45=97$.

∴ the reqd. divisor $=723$.

14. Spending annually Rs. 1200 for 7 years I had incurred some debt, which was subsequently cleared by reducing my yearly expenditure to Rs. 880 for 9 years. Find my annual income.

The total expenditure for the first 7 years
$=$Rs. $1200\times7=$Rs. 8400,

" " " " " next 9 years
$=$Rs. $880\times9=$Rs. 7920,

∴ " " " " " for 16 years
$=$Rs. $8400+$Rs. 7920
$=$Rs. 16320.

∵ After 16 years there is neither any debt nor any savings,

∴ the total income is equal to the total expenditure of 16 years.

∴ the total income of 16 years$=$Rs. 16320

∴ the required annual income$=$Rs. $16320\div16=$Rs. 1020.

15. If 109 is multiplied by a certain number it is increased by 2071. Find the number.

Here $109\times$ the reqd. number$=109+2071=2180$

∴ the reqd. number$=2180\div109=20$.

16. When a number is divided in succession by the factors 2, 3 and 7 of the divisor, the remainders are 1, 1 and 6 respectively, what is the complete remainder if it be divided by the divisor ?

[The rule : The complete remainder$=$1st remainder$+$(1st divisor$\times$2nd remainder)$+$(1st divisor$\times$2nd divisor$\times$3rd remainder)$+$(1st divisor$\times$2nd divisor$\times$3rd divisor$\times$4th remainder)$+\cdots$and so on if there be more factors.]

Here the complete remainder
$=1+2\times1+2\times3\times6=1+2+36=39$.

17. Is 823 a prime number ? [A. U. 1907]

[N. B. (1) *A prime number* is one which is divisible only by itself and by unity, *i.e.*, 1, but not by any other number. Thus, 1, 2, 3, 5, 7, 11, 13, etc. are prime numbers.

(2) *The rule for finding whether any number is prime or not.*

Divide the number successively by the prime numbers 2, 3, 5, etc. If it is found that the number is not divisible by any such prime number, then it is a prime number. But up to what greatest prime number as divisor should the operation of division be continued ? The rule is that while dividing on in this way you find the quotient less than the divisor, you should discontinue the operation of division.]

The last digit of 823 is 3, $\therefore$ at first sight it cannot be said whether it is prime or not. It should be tested by dividing it by the prime numbers 3, 7, 11, 13 etc. Proceeding thus it is found that 823 is not divisible by any of the prime numbers from 1 to 29. In dividing 823 by 29 we get the quotient 28, which is less than the divisor 29. $\therefore$ We should not divide it further by any other prime numbers greater than 29. $\therefore$ 823 is a prime number.

[Here the reason is that if the number would be divisible by any number greater than 29, its quotient would be less than 29 and 823 would be divisible also by that less quotient ($\because$ what is divisible by the divisor is also divisible by the quotient). But it has already been found that 823 is not divisible by any number less than 29.]

18. A and B had equal shares in a flock of sheep, of these B took 55 sheep and paid Rs. 20 to A who took only 35 sheep. Find the price of each sheep.

The total number of sheep $=55+35=90$.

$\therefore$ each will get 45 sheep, if they are equally divided.

$\therefore$ B has given A, Rs. 20 for 10 sheep he has taken from A's share.

$\therefore$ the price of one sheep=Rs. $20 \div 10 =$ Rs. 2.

19. A clock takes $3\frac{1}{3}$ seconds to strike the hour of 5 o'clock. How long does it take to strike 9 o'clock ? [D. B. '42]

There is 1 time interval between two stroke-sounds.

∴ There are 4 time intervals in between the 5 stroke-sounds and to strike 9 o'clock the time intervals are 8.

Now 4 intervals last $3\frac{1}{2}$ seconds.

∴ 8 intervals last $3\frac{1}{2}\times 2$ or 7 seconds.

20. Find the sum of all the numbers that can be formed by the digits 2, 3, 4, 5 taken all together, no digit occurring more than once in any number. [C. U. '50]

24 numbers can be formed with the digits 2, 3, 4, 5 taken all together. If we place 2 in the thousands' place and arrange the rest, there will be 6 numbers, viz. 2345, 2354, 2435, 2453, 2543, 2534.

Thus placing 3, 4, 5 respectively in the thousands' place we get in every case such 6 numbers. Now it is seen that in each of the thousands', hundreds', tens' and units' places there are six 2's, six 3's. six 4's and six 5's. Their sum is 84. ∴ The total sum of the 24 numbers $= 84 \times$ thousand $+ 84 \times$ hundred $+ 84 \times$ ten $+ 84 \times$ unity $= 84000 + 8400 + 840 + 84 = 93324$.

[**N. B.** The rule for finding how many numbers can be formed with a few given digits taken all together, no digit occurring more than once in any number is as follows :

If there be 2 digits, there will be 1×2 or 2 numbers. If there be 3 digits, there will be $1 \times 2 \times 3$ or 6 numbers. Thus in case of 4 digits there will be $1 \times 2 \times 3 \times 4$ or 24 numbers and so on.]

21. A man lost as much by selling 30 quintals of rice at Rs. 16 per quintal as he gained by selling 40 Qu. at Rs. 23 per quintal. What did each quintal of rice cost him ?

The selling price of 30 Qu. of rice $=$ Rs. $16 \times 30 =$ Rs. 480.

and „ „ „ „ 40 „ „ $=$ Rs. $23 \times 40 =$ Rs. 920.

∴ the total selling price of 70 quintals $=$ Rs. $(480 + 920)$

$=$ Rs. 1400.

Now in selling 70 quintal of rice there has been neither any gain nor any loss, ∴ the total cost price is equal to the total selling price.

∴ the cost price of 70 quintals of rice $=$ Rs. 1400.

∴ the cost price of 1 Qu. of rice $=$ Rs. $1400 \div 70 =$ Rs. 20.

22. Along a hedge 590 ft. long 60 trees are planted at equal distances, one tree being planted at each end of the hedge. Find the distance between any two consecutive trees. [Pat. U. '26]

Here trees are planted at equal distances and two trees are planted at the two ends of the hedge ;

$\therefore$ 60 trees have $(60-1)$ or 59 equal distances in between them.

$\therefore$ the distance between any two consecutive trees

$=590$ feet $\div 59 = 10$ feet.

Exercise 1

1. The sum of two numbers is 936 and one of them is half the other. Find the numbers.

2. The sum of two numbers is 166302 and their difference is 6616. Find their product. [D. B. '25]

3. The product of 37,131 and another number is 697968 ; find that number.

4. The quotient after division of a certain number by 372 is 273 and the remainder is 237. Find the number. [C. U. '17]

5. At a game of cricket A, B and C together score 108 runs, B and C together score 90 runs, and A and C together score 51 runs. Find the number of runs scored by each. [C. U. '29]

6. In dividing a number by 3, 5 and 7 in succession the remainders are 2, 4 and 4 respectively. What will be the remainder if it is divided by 105 ?

7. The divisor of a certain number is 25 times the quotient and 15 times the remainder. Find the number when the remainder is 375. [P. U. '29]

8. What is a prime number ? Is 1109 a prime number ? [D. B. '32]

9. In a division sum the divisor is 25 times the quotient and 5 times the remainder. If the quotient is 16, find the dividend.

10. A man spends in 3 months what he earns in two months. If his yearly savings be Rs. 1500, find his monthly income.

11. Divide Rs. 711 among A, B and C so that for every 2 rupees given to A, B may get 3 rupees and C 4 rupees.

12. The quotient is 7 times the divisor, which is 7 times the remainder and their sum is 741 ; find the dividend.

13. What number multiplied by 238 gives the same result as 408 multiplied by 350 ? [C. U. '22]

14. The product of two numbers is 47608946. One of them is 2149. Find the other. [C. U. '26]

15. In a question on multiplication of 8750 by 635 a candidate copied one figure wrongly and obtained 5993750 as his answer. What mistake did he commit in copying ? [C. U. '49 Sup.]

16. A and B together have Rs. 134 ; B and C have Rs. 100 ; and B has Rs. 58 more than C. How much does each of them have ? [E. B. S. B. '48]

17. Find the sum of all the possible numbers of 4 digits that can be formed with 4, 5, 6, 0 taken all together, no digit occurring more than once in any number.

18. Find the product of two numbers if their sum and difference be 8752 and 7946 respectively. [G. U. '49]

19. Find the sum of all the numbers which can be formed by the digits 3, 5, 7 taken altogether, no digit occurring more than once in any number. [G. U. '51]

20. A and B had equal shares in a flock of sheep ; of these B took 45 sheep and paid Rs. 40 to A who took only 35 sheep. Find the price of each sheep. [A. U.]

21. I lose as much by selling 60 cwt. of tea at £3 per cwt. as I gain by selling 40 cwt. at £4. 5s. per cwt. Find the cost price per cwt.

22. A clock takes 3 seconds to strike the hour of 3 o'clock. How long does it take to strike 7 o'clock ?

Miscellaneous Examples on H. C. F. and L. C. M. worked out

Examples (2)

1. What least number must be subtracted from 7923 to make the remainder exactly divisible by 205 ?

```
205 ) 7923 ( 38
      615
      ----
      1773
      1640
      ----
       133
```

∴ the required least number=133.

[N. B. To find what least number is to be subtracted from a given number to make the remainder exactly divisible by a number, divide the given number by the divisor and the remainder thus obtained will be the answer.]

2. What greatest number must be taken from 723596 so that the remainder may be divisible by 315 ?

Here it is evident that the remainder must be at least 315, so that it may be divisible by 315.

∴ The number to be subtracted from 723596 should be such that the remainder is 315.

∴ the required number $=723596-315=723281$.

3. What least number must be added to 7532 to make the sum divisible by 73 ?

```
73 ) 7532 ( 103
     73
     ----
      232
      219
      ---
       13
```

∴ the reqd. number $=73-13=60$.

[N.B. To find what least number is to be added to a given number to make the sum divisible by a number, subtract the remainder from the divisor and this difference will be the answer.]

4. What greatest number of 4 digits is divisible by 315 ?

The greatest number of 4 digits=9999.

```
315 ) 9999 ( 31
      945
      ----
       549
       315
       ---
       234
```

the required number $=9999-234=9765$.

[N. B. The greatest number of 4 digits is 9999. Dividing it by 315 we get the remainder 234. It being an excess, subtract it from 9999 and then the result will be divisible by 315.]

5. What least number of 5 digits is divisible by 39 ?

The least number of 5 digits = 10000.

$$\begin{array}{r|l}
39 & 10000 \; (\; 256 \\
 & \underline{78} \\
 & 220 \\
 & \underline{195} \\
 & 250 \\
 & \underline{234} \\
 & 16
\end{array}$$

[N. B. Here it will not do to subtract 16 from the dividend, because in that case the dividend will not be of 5 digits. So we have to see what should be added to the dividend to make it divisible by 39. (Vide the Example 3.)]

$39 - 16 = 23$

∴ the reqd. number $= 10000 + 23 = 10023$.

6. Find the greatest number that will exactly divide 2583 and 1230.

The H. C. F. of 2583 and 1230 will be the required greatest number.

∴ The required number = H. C. F. of 2583 and 1230 = 123.

[N. B. (1) To find the greatest number that will exactly divide several given numbers we shall find their H. C. F. for the answer. (2) To find the least number divisible by several given numbers we shall find their L. C. M. for the answer. (3) The H. C. F. or the L. C. M., as the case may be, should be actually worked out in each example.]

7. Find the least number divisible by 16, 21 and 28.

$$\begin{array}{r|l}
2 & 16,\ 21,\ 28 \\ \hline
2 & 8,\ 21,\ 14 \\ \hline
7 & 4,\ 21,\ 7 \\ \hline
 & 4,\ 3,\ 1
\end{array}$$

∴ the required number = the L. C. M. of the given numbers $= 2 \times 2 \times 7 \times 4 \times 3 = 336$.

8. By what greatest number must 73 and 129 be divided to leave the remainder 3 in each case ?

[If we divide 73 and 129 by the required greatest number, the remainder is 3 in each case. ∴ $(73 - 3)$ or 70 and $(129 - 3)$ or 126 must be divisible by that greatest number.]

$73 - 3 = 70$, $129 - 3 = 126$.

The H. C. F. of 70 and 126 = 14.

∴ the required number = 14.

9. Find the greatest number that will divide 291 and 311 leaving the remainders 3 and 5 respectively.

$291-3=288$, $311-5=306$.

Now, find the H. C. F. of 288 and 306. That will be the answer. $\therefore$ the required number$=18$.

10. Find the greatest number that will divide 86, 64 and 31 leaving the same remainder in each case.

Evidently 86 = a multiple of the reqd. number + the remainder.
64 = another multiple of the reqd. number + the same remainder,
31 = some other multiple of the reqd. number + the same remainder,

$\therefore$ the difference of any two of the numbers is divisible by the required number.

$86-64=22$, $86-31=55$, $64-31=33$.

Now, the H. C. F. of 22, 55 and $33=11$.

$\therefore$ the required number$=11$.

11. Find the least number which will leave a remainder 4 when divided by both 15 and 18. [C. U. 1923]

$$\begin{array}{r|l} 3 & 15,\ 18 \\ \hline & 5,\ \ 6 \end{array}$$

L. C. M.$=3\times5\times6=90$,

$\therefore$ the required number$=90+4=94$.

12. Find the least number which being increased by 1 will be exactly divisible by 22, 17, 33 and 102. [C. U. 1931]

$$\begin{array}{r|rrrr} 2 & 22, & 17, & 33, & 102 \\ \hline 3 & 11, & 17, & 33, & 51 \\ \hline 11 & 11, & 17, & 11, & 17 \\ \hline 17 & 1, & 17, & 1, & 17 \\ \hline & 1, & 1, & 1, & 1 \end{array}$$

Here the L. C. M.$=2\times3\times11\times17$ $=1122$. This is the least number divisible by the given numbers.

$\therefore$ the required number $=1122-1=1121$.

[N. B. We want such a number as being added to 1 will be divisible. We know that the L. C. M. is divisible by the given numbers. So we deduct 1 from the L. C. M ; because if 1 is added to 1121 the sum will be equal to the L. C. M. and therefore divisible by the given numbers.]

13. Find the least number which being diminished by unity will be exactly divisible by 22, 17, 33 and 102.

[Unity means 1. First find the L. C. M. of the given numbers. Then add 1 to it. This sum will be the answer, because if 1 is deducted from the sum it will be equal to the L. C. M. and therefore divisible by the given numbers.] [Ans. 1123]

14. Find the least number of 5 digits which has 53 for a factor. [C. U. 1934]

$$
\begin{array}{r|l}
53 \,)\, 10000 & (\, 188 \\
53 & \\
\hline
470 & \\
424 & \\
\hline
460 & \\
424 & \\
\hline
36 &
\end{array}
$$

$53-36=17$

$\therefore$ the required number $=10000+17=10017$.

15. Find the greatest number of 4 digits which is exactly divisible by 11, 44, 66, 88 and 99. [C. U. 1935]

$$
\begin{array}{r|l}
2 & 11,\ 44,\ 66,\ 88,\ 99 \\
\hline
2 & 11,\ 22,\ 33,\ 44,\ 99 \\
\hline
3 & 11,\ 11,\ 33,\ 22,\ 99 \\
\hline
11 & 11,\ 11,\ 11,\ 22,\ 33 \\
\hline
 & \qquad\qquad\quad 2,\ \ 3
\end{array}
$$

$$
\begin{array}{r}
792 \,)\, 9999 \,(\, 22 \\
792 \\
\hline
2079 \\
1584 \\
\hline
495
\end{array}
$$

The L. C. M. $=792$

$\therefore$ the required number $=9999-495=9504$.

16. Find the least number of 4 digits which is exactly divisible by 12, 16 and 18.

$$
\begin{array}{r}
144 \,)\, 1000 \,(\, 6 \\
864 \\
\hline
136
\end{array}
$$

The L. C. M. of 12, 16, 18 $=144$

$144-136=8$.

$\therefore$ the required number $=1000+8=1008$.

17. The product of two numbers is 864 and their L. C. M. is 72. Find their G. C. M. [C. U. '25]

[*Rule* : Product of two numbers $=$ Their H. C. F. $\times$ L. C. M.]

Here, L. C. M. $\times$ H. C. F. $=864$; but their L. C. M. $=72$

$\therefore$ the required H. C. F. $=864\div 72=12$.

18. The G. C. M. and L. C. M. of two numbers are 6 and 37674 respectively and one of the numbers is 414, find the other. [C. U. 1938]

The product of the two numbers $=$ G. C. M. $\times$ L. C. M.

$=6\times 37674$,

and one number $=414$,

$\therefore$ The required other number $=\dfrac{6\times \overset{91}{\cancel{37674}}}{\cancel{414}}=546$.

19. Three clocks chime at intervals of 12, 16 and 18 minutes. Having once chimed together, after what interval will they again chime together ?

[Here it is clearly understood that the required time will be divisible by 12, 16 and 18 ; otherwise the three clocks will not chime together after that interval.]

The L. C. M. of 12, 16 and 18=144,

∴ they will chime together again after an interval of 144 minutes or 2 hrs. 24 mins.

20. Four clocks are made to chime at intervals of 1 hr., 1 hr. 20 mins., 1 hr. 30 mins., and 1 hr. 40 mins. Having chimed together at 10 A.M., when will they next do so ?

[C. U. 1930]

1 hr.=60 mins., 1 hr. 20 mins.=80 mins.,

1 hr. 30 mins.=90 mins., 1 hr. 40 mins.=100 mins.

Now, the L. C. M. of 60 mins., 80 mins., 90 mins., and 100 mins.=3600 mins. The clocks will, therefore, chime together again after 3600 mins. or 60 hrs., *i.e.*, at 10 P.M. on the third day [on counting 60 hrs. from 10 A. M.].

21. Find the biggest pot that can be used in exactly measuring 2 kilolitres 52 litres and 1 kilolitre 5 hectolitres of milk.

2K*l*. 52 *l*.=2052 *l*. and 1 K*l*. 5 H*l*.=1500 *l*.

The H. C. F. of 2052 *l*. and 1500 *l*.=12 *l*.

∴ A pot whose capacity is 12 litres is the required biggest pot.

22. The bills of Rs. 4. 25 P. and Rs. 5. 50 P. are to be paid with the same kind of coin. Coin of what greatest value can be used ?

[Here the answer will be the sum of money that will exactly divide the sums of the two bills. So their G. C. M. must be the answer.] [Ans. Coin worth 25 P.]

23. Find the greatest fraction that will exactly divide $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$ and $\frac{8}{9}$: find also the least whole number which is exactly divisible by the above fractions.

[N. B. To find the greatest fraction that will exactly divide $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$ and $\frac{8}{9}$ we shall find their H. C. F. for the answer. And

to find the least fraction divisible by them we shall find their L. C. M. for the answer.]

The H.C.F. of the given fractions $= \dfrac{\text{H.C.F. of the numerators}}{\text{L.C.M. of the denominators}}$

and the L.C.M. of the given fractions

$$= \frac{\text{L. C. M. of the numerators}}{\text{H. C. F. of the denominators}}.$$

Here, the H.C.F. of $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$ and $\frac{8}{9} = \frac{2}{315}$, this is the first answer.

Again, their L. C. M. $= \frac{24}{1} = 24$, this is the second answer.

[N. B. In the second part the least whole number is wanted for the answer. If the L. C. M. obtained be a fraction instead of a whole number, then the least whole number which would be a multiple of that fraction would be the answer.]

24. Find the least integral number which is exactly divisible by $1\frac{5}{21}$ and $1\frac{6}{7}$. [D. B. 1940]

Here, the L. C. M. of $1\frac{5}{21}$ and $1\frac{6}{7} = \frac{26}{7}$, it is not a whole number. If we take at least 7 times this fraction, we get a whole number.

$\therefore$ The required number $= \frac{26}{7} \times 7 = 26$.

25. The circumference of the fore-wheel of a carriage is 2 m. 7 dm. and that of the hind-wheel is 4 m. 5 dm. Find the least distance in which they will make exact number of revolutions.

2m. 7 dm. = 27 dm.
and 4m. 5 dm. = 45 dm.

The L. C. M. of 27 dm. and 45 dm. = 135 dm.

$\therefore$ the required distance = 135 dm. = 13 m. 5 dm.

26. Find the least number which when divided by 16, 21 and 28, leaves the remainders 13, 18 and 25 respectively.

[Here it is apparent that each remainder is less than the divisor by 3. $\therefore$ if we add 3 to each remainder, it will be equal to the divisor and then divisible (*i.e.*, it will leave no remainder). $\therefore$ the required number will be less by 3 than the number divisible by 16, 21 and 28.]

$16 - 13 = 3$, $21 - 18 = 3$, $28 - 25 = 3$,

$$\begin{array}{r|rrr} 2 & 16, & 21, & 28 \\ \hline 2 & 8, & 21, & 14 \\ \hline 7 & 4, & 21, & 7 \\ \hline & 4, & 3, & 1 \end{array}$$

L. C. M. $= 2 \times 2 \times 7 \times 4 \times 3$
$= 336$ (it is divisible)

$\therefore$ the required number $= 336 - 3$
$= 333$.

27. What is the least number which being divided by 48, 64, 72 and 80 leaves the remainders 38, 54, 62 and 70 respectively ?

[C. U. '37]

48	64	72	80
38	54	62	70
10	10	10	10

Now, the L. C. M. of 48, 64, 72 and 80 = 2880,

∴ the required number = 2880 − 10 = 2870.

28. Find the groups of numbers between 500 and 1000 that have 163 as their G. C. M.

$$163\ )\ 500\ (\ 3$$
$$489$$
$$11$$

Here first find the multiples of 163 between 500 and 1000.

163×3 is less than 500.

Now, $163 \times 4 = 652$, $163 \times 5 = 815$, $163 \times 6 = 978$ and of these numbers the common factor is 163 ; 163×7 is greater than 1000, ∴ it will not be taken. Now, we are to find from these three numbers 652, 815 and 978 the groups of numbers that will have 163 for their H. C. F.

Of them 652 and 815, 815 and 978, or 652, 815 and 978—these three groups of numbers can have 163 for their H. C. F.

[The group (652 and 978) cannot be taken, because these two numbers have another common factor 2 besides 163, ∴ their H. C. F. will be 163×2 or 326.]

29. Find the number between 4000 and 5000 which is exactly divisible by 12, 18, 21 and 32.

L. C. M. of 12, 18, 21 and 32 = 2016.

Now find which multiple of it is between 4000 and 5000.

It is $2016 \times 2 = 4032$, and this is the required number.

30. A basket contains a number of mangoes ascertained to be between 1600 and 1700. If five of these mangoes are taken away, the remainder may be distributed equally among 4, 5, 6, 7 or 8 boys. Find the number of mangoes in the basket. [C. U. 1940]

The L. C. M. of 4, 5, 6, 7 and 8 = 840. Now find that multiple of 840 which lies between 1600 and 1700.

It is $840 \times 2 = 1680$.

∴ The required number = 1680 + 5 = 1685.

31. 945 pens and 1155 pencils may be equally divided among a certain number of boys. How many boys are there ? Give all possible answers.

$$\begin{array}{l} 945\,)\,1155\,(\,1 \\ \quad\;\; \underline{945} \\ \quad\;\; 210\,)\,945\,(\,4 \\ \qquad\qquad \underline{840} \\ \qquad\qquad 105\,)\,210\,(\,2 \\ \qquad\qquad\qquad\;\; \underline{210} \end{array}$$

$$\begin{array}{r|l} 5 & 105 \\ \hline 3 & 21 \\ \hline & 7 \end{array}$$

∴ the required number of boys $=3,\ 5,\ 7,\ 3\times5,\ 3\times7,\ 5\times7$ and 105, *i.e.* 3, 5, 7, 15, 21, 35 and 105.

[Since the two kinds of things can be equally divided among the boys, the numbers of the two kinds must be exactly divisible by the number of boys. ∴ the greatest number of boys will be the H. C. F. (*i.e.* 105) of those two numbers of things. Again, the two numbers which are divisible by 105 are also divisible by any factor of 105, ∴ all the factors of 105 will be the answer.]

32. Find the least number which when divided by 2, 3, 4, 5 and 6 leaves the remainder 1 in each case but is exactly divisible by 7. [D. B. 1933]

Here the L. C. M. of 2, 3, 4, 5 and $6=60$, ∴ the required number will be greater than a multiple of 60 by 1. Now we have to find how many times 60 will be added to 1 so that the sum may be divisible by 7. Now find the remainder obtained by dividing 60 by 7. The remainder will be 4. Now find how many times 4 is to be added to 1 so that the sum is divisible by 7. It is clearly seen that 5 times $4+1=21$, which is divisible by 7.

∴ the required number $=60\times5+1=301$.

33. Find the least number which being divided by 7, 9, 14, 21 and 35 leaves in each case a remainder 2 but is exactly divisible by 11. [C. U. 1942]

[Vide Ex. 32, Ans. 1892]

34. Find the number between 13000 and 14000 which when divided by 152 and 285 leaves in each case a remainder 31. [C. U. '43]

$$\begin{array}{r|l} 19 & 152,\ 285 \\ \hline & 8,\quad 15 \end{array}$$

∴ L.C.M. $=19\times8\times15=2280$. Find the multiple of 2280 that is between 13000 and 14000. It is $2280\times6=13680$.

∴ the required number $=13680+31=13711$.

35. What greatest number and what least number can be subtracted from 23759143 that the remainders may be exactly divisible by 24, 35, 91, 130 and 150 ? [C. U. 1896 and 1941]

The L. C. M. of 24, 35, 91, 130 and 150=54600 [Show the working out of the L. C. M. here.]

```
54600 ) 23759143 ( 435
        218400
        191914
        163800
         281143
         273000
           8143
```

∴ the required least number=8143, and the greatest number required =23759143 − 54600=23704543.

36. Find the greatest number of 5 digits that can be added to 8321 to make the sum exactly divisible by 15, 20, 24, 27, 32 and 36. [C. U. 1906]

The L. C. M. of 15, 20, 24, 27, 32 and 36=4320 [Here work out the L. C. M.]

```
         99999
          8321
4320 ) 108320 ( 25
         8640
         21920
         21600
           320
```

∴ the required number =99999 − 320=99679.

37. Find the greatest number of 4 digits and the least number of 5 digits that have 248 for their G. C. M. [C. U.' 44]

```
248 ) 9999 ( 40
      992
       79
```

∴ the reqd. greatest number =9999 − 79=9920.

```
248 ) 10000 ( 40
       992
        80
```

248 − 80=168

∴ the reqd. least number =10000+168=10168.

38. The G. C. M. of two numbers is 18 and the other two factors of their L. C. M. are 7 and 11. Find the numbers.

The reqd. numbers are 18×7 and 18×11, i.e., 126 and 198.

39. Find the least number of 6 digits which being divided by 12, 15 and 18 leaves the remainders 9, 12 and 15 respectively.

12 − 9=3, 15 − 12=3 and 18 − 15=3.

The L. C. M. of 12, 15, 18 = 180,

$180 - 100 = 80$.

$\therefore$ $100000 + 80 = 100080$, this is divisible by 12, 15 and 18,

$\therefore$ the reqd. number $= 100080 - 3 = 100077$.

$$\begin{array}{r} 180\;)\;100000\;(\;555 \\ \underline{900} \\ 1000 \\ \underline{900} \\ 1000 \\ \underline{900} \\ 100 \end{array}$$

40. The sum of two numbers is 136 and their G. C. M. is 17. What may be the numbers ?

The H. C. F. of the two numbers is 17. $\therefore$ They are divisible by 17, and $\therefore$ their sum is also divisible by 17. $136 \div 17 = 8$. $\therefore$ It is clear that the sum of the quotients obtained by dividing the two numbers by 17 is 8. Now find the pairs of two numbers having 8 for their sum—we have $8 = 1 + 7$; $8 = 2 + 6$; $8 = 3 + 5$ and $8 = 4 + 4$. Now of these only those pairs whose numbers are prime to each other are to be taken. $\therefore$ One pair of the required numbers is 17×1 and 17×7 or 17 and 119, and another pair of the required numbers is 17×3 and 17×5 or, 51 and 85.

41. The sum of two numbers is 1212 and their G. C. M. is 101. How many pairs of such numbers can be formed ? Form these pairs. [C. U. 1945]

$1212 \div 101 = 12$; now $12 = 1 + 11$; $12 = 2 + 10$; $12 = 3 + 9$; $12 = 4 + 8$; $12 = 5 + 7$; $12 = 6 + 6$. Here only 1, 11 and 5, 7 are prime to each other. $\therefore$ there will be two pairs of such numbers.

$\therefore$ One pair of the reqd. numbers $= 101 \times 1$ and 101×11
$= 101$ and 1111 ;

and another pair of the numbers $= 101 \times 5$ and 101×7
$= 505$ and 707.

42. The G. C. M. and L. C. M. of two nnmbers are 7 and 84 respectively ; find the numbers.

$$\begin{array}{r} 7\;)\;84\;(\;12 \\ \underline{7} \\ 14 \\ \underline{14} \end{array}$$

Here if we divide the L. C. M. by 7, the quotient is 12. $\therefore$ the product of the two quotients obtained by dividing the required two numbers by 7 will be 12.

Now, $12 = 3 \times 4$
$12 = 2 \times 6$
$12 = 1 \times 12$

Here 3 and 4, 1 and 12 are prime to each other. $\therefore$ the required two numbers $= 7 \times 3$ and $7 \times 4 = 21$ and 28. Again they may be 7×1 and 7×12 or 7 and 84.

43. In finding the G. C. M. of two numbers, the last remainder is 35 and the quotients are 1, 2, 1 and 3. Find the numbers. [C. S.]

```
A ) B ( 1
    ...
    C ) A ( 2
        ...
        D ) C ( 1
            ...
            35 ) D ( 3
                 D
                 ---
                 ×
```

The problems of this nature should be worked out from the end. Here the last remainder $=35$. $\therefore$ 35 is the H. C. F., i.e., the last divisor. The last divisor is 35 and the last quotient is 3. $\therefore$ the last dividend $=35\times3$ $=105$(D).

This 105 is the previous divisor(D) and then the quotient is 1 and the remainder is 35.

$\therefore$ the previous dividend (C) $=105\times1+35=140$.

Similarly, $A=C\times2+D=140\times2+105=385$

and $B=A\times1+C=385\times1+140=525$.

$\therefore$ the required numbers $=385$, 525.

44. In a division sum the dividend is 305165 and the successive remainders are 17, 27, 36 and 29 respectively. Find the divisor.

```
) 305165 (
   ...(1)
   -------
    17
    ...     (2)
   -------
    27
    ...     (3)
   -------
    36
    ...     (4)
   -------
    29
```

As there are four remainders, there must be four digits in the quotient. $\therefore$ the work of division begins with 305 as the first dividend. $\therefore$ in place of (1) there is $305-17=288$. Now take down 1 with the remainder 17 so as to form 171. $\therefore$ in place of (2) there is $171-27=144$. Now take down 6 with 27 to form 276. $\therefore$ in place of (3) there is $276-36=240$. Now take down 5 with 36 to form 365. $\therefore$ in place of (4) there is $365-29=336$. These 288, 144, 240 and 336 are multiples of the divisor and therefore divisible by the divisor. $\therefore$ their H. C. F. will be the required divisor.

```
144 ) 288 (2        144 ) 240 ( 1          48 ) 336 ( 7
      288                 144                   336
      ---                 ---                   ---
                           96 ) 144 ( 1
      H. C. F.=48               96
                                ---
                                48 ) 96 ( 2
                                     96
                                     ---
```

$\therefore$ the reqd. divisor $=48$.

45. A man bought a certain number of mangoes for Rs. 8. 16 P. and sold some of them for Rs. 6. 42 P. without making any profit. Find the least number of mangoes he may still have.

Here Rs. 8. 16 P. is the cost price of some whole number of mangoes and Rs. 6. 42 P. is also the cost price of some whole number of mangoes. ∴ the cost price of each mango will divide exactly both the given prices. ∴ the H. C. F. of Rs. 8. 16 P. and Rs. 6. 42 P. can be the maximum price of each mango. Rs. 8. 16 P.=816 P. ; Rs. 6. 42 P.=624 P. The H. C. F. of 816 P. and 642 P.=6 P. Now the man still has mangoes worth (816 – 642) or 174 P. The highest price of each mango being 6 P., the man has still at least (174 P.÷6 P.) or 29 mangoes left with him.

46. A labourer was engaged on daily wages for a number of days for Rs. 29. 25 P., but being absent on some of those days he was paid only Rs. 22. 50 P. Prove that his daily wages cannot be more than Rs. 2. 25 P. [C. U. '37]

Rs. 29. 25 P.=2925 P. ; Rs. 22. 50 P.=2250 P.

The H. C. F. of 2925 P. and 2250 P. will be the highest wages of the man. Their H. C. F.=225 P.=Rs. 2. 25 P.

∴ his daily wages cannot be more than Rs. 2. 25 P.

47. A man bought two heaps of mangoes, one for Rs. 19. 80 P. and the other for Rs. 34. 65 P. If the price of each mango be the same, and not less than 24 P. and not more than 36 P. find the total number of mangoes he bought.

Rs. 19. 80 P.=1980 P. ; Rs. 34. 65P.=3465 P. Their H.C.F. =495 P. ∴ the highest price of one mango is 495 P. and any factor of 495 P. may be the price of each mango. Now it is given that the price of each mango will be between 24 P. and 36 P. ∵ one factor of 495 is 33, ∴ 33 paise will be the cost price of each mango, because it is between 24 P. and 36 P. But the total price of the mangoes=3465 P.+1980 P.=5445 P.

∴ the required number of mangoes=5445 P.÷33 P.=165.

48. Rs. 2. 50 P., Rs. 3. 50 P. and Rs. 4. 50 P. are distributed amongst a number of men, women and children and the total number of persons are as small as possible ; find the number of persons.

Here each person has got equal share. ∴ the three sums of money will be divisible by the amount received by each of them.

Again, the number of persons being as small as possible the share of each person will be the H. C. F. of these three sums of money. Now the H. C. F. of Rs. 2. 50 P., Rs. 3. 50 P., and Rs. 4. 50 P. $=50$ P.

$\therefore$ the number of men $=250$ P. $\div 50$ P. $=5$;

the number of women $=$ Rs. 350 P. $\div 50$ P. $=7$;

and the number of children $=$ Rs. 450 P. $\div 50$ P. $=9$.

$\therefore$ the total number of persons $=5+7+9=21$.

49. Find the number nearest to one lac that is exactly divisible by 2, 3, 4, 5, 6 and 7.

Here the L. C. M. of 2, 3, 4, 5, 6 and $7=420$.

If we divide 100000 by 420, the remainder is 40.

$\therefore$ the required number $=100000-40=99960$.

[**N. B.** Here we may have a number divisible by the given numbers, if we add $(420-40)$ or 380 to 1 lac ; but in that case its difference from 1 lac being 380, it cannot be the required number nearest to one lac. The answer should be given finding correctly the nearest number.]

50. In going up a flight of stairs if a boy goes up 2 at a time there is 1 over, if 3 at a time there are 2, if 4 at a time there are 3 and if 5 at a time there are 4 over. Find the least number of stairs in the staircase.

Here it is found that if we divide the number of stairs by 2, 3, 4 and 5, the remainders are 1, 2, 3 and 4 respectively.

$\therefore$ the least number of steps $=$ (the L. C. M. of 2, 3, 4, 5) -1
$=60-1=59$. [Vide Ex. 29.]

51. The product of two numbers is 142884 and their L. C. M. is 2268. Find the numbers.

The H. C. F. of the two numbers
$=$ their product $\div$ L.C.M. $=142884\div2268=63$.

```
63 ) 2268 ( 36
     189
     ----
      378
      378
     ----
```

Now $36=1\times36$

$36=2\times18$

$36=3\times12$

$36=4\times9$

$36=6\times6$

Of the factors of 36, 1 and 36 and also 4 and 9 are prime to each other. $\therefore$ the required two numbers are either 63×1 and 63×36, *i.e.*, 63 and 2268, or 63×4 and 63×9, *i.e.*, 252 and 567.

52. The product of two numbers is 12960 and their G.C.M. is 36 ; how many pairs of such numbers can be formed ? Form them.

H. C. F. $\times$ L. C. M. = Product of the two numbers.

$\therefore$ 36 $\times$ L. C. M. = 12960, $\therefore$ L. C. M. = $\frac{12960}{36}$ = 360.

Now the L. C. M. 360 divided by the H. C. F. 36 gives the quotient 10. $\therefore$ the product of the two quotients obtained by dividing the two required numbers by the H.C. F. 36, is 10.

Now we have to find two numbers, prime to each other, having 10 as their product. $10 = 1 \times 10$ and $10 = 2 \times 5$; so there may be two pairs of numbers. One pair = 36×1 and 36×10 = 36 and 360, and the other pair = 36×2 and 36×5 = 72 and 180.

*53. Divide 1904 into two parts so that their G. C. M. and L. C. M. may be 28 and 32340 respectively. [Oxford U.]

The quotient obtained by dividing 1904 by 28 is 68. $\therefore$ the sum of the quotients obtained by dividing the required two parts by 28 is 68. Again the L. C. M. 32340 when divided by the H. C. F. 28 will give the product of those two quotients. $32340 \div 28 = 1155$; $\therefore$ the product of those two quotients = 1155.

$\therefore$ we have $x + y = 68$ and $xy = 1155$.

Now $(x-y)^2 = (x+y)^2 - 4xy = 68^2 - 1155 \times 4 = 4$, $\therefore x - y = 2$.

$x + y = 68$

$x - y = 2$

$\therefore$ $2x = 70$ $\therefore$ $x = 35$.

$\therefore$ one quotient = 35

and the other ,, = 68 − 35 = 33

$\therefore$ the required two parts = 28×33 and 28×35

= 924 and 980.

54. Find two numbers of 4 digits whose L. C. M. and G.C.M. are 13923 and 119 respectively. [C. S.]

$13923 \div 119 = 117$,

$117 = 1 \times 117$

$117 = 3 \times 39$

$117 = 9 \times 13$

The two reqd. numbers are of 4 digits, $\therefore$ the pair of 9 and 13 is to be taken.

$\therefore$ the required numbers = 119×9 and 119×13

= 1071 and 1547.

55. Two numbers, each of 3 digits, are such that their G. C. M. is 16 and L. C. M. is 11760. Find them. [I. P. S. '38]

L. C. M. $\div$ H. C. F. $= 11760 \div 16 = 735$.

$$\begin{aligned} 735 &= 1 \times 735 \\ &= 5 \times 147 \\ &= 15 \times 49 \\ &= 105 \times 7 \\ &= 21 \times 35 \end{aligned}$$

Of the above pairs 105 and 7 and also 21 and 35 are not prime to each other. $\therefore$ there can be three pairs of numbers; but the numbers 1×16 and 735×16 or 5×16 and 147×16 are not of 3 digits.

$\therefore$ the reqd. numbers $= 15 \times 16$ and $16 \times 49 = 240$ and 784.

56. Three numbers are such that their L. C. M. is 924 and the G. C. M. of each pair is 11; find their product.

$\because$ the G. C. M. of any two numbers $= 11$.

$\therefore$ the G. C. M. of three numbers $= 11$.

Suppose, the three numbers to be $11x, 11y, 11z$ [where x, y, z are prime to each other]

The L. C. M. of $11x$, $11y$, $11z = 11xyz = 924$,

$\therefore$ $xyz = 924 \div 11 = 84$.

$\therefore$ the required product $= 11 \times 11 \times 11 \times xyz$

$= 11 \times 11 \times 11 \times 84 = 111804$.

57. The G. C. M., L. C. M. and the difference of two numbers are 9, 936 and 45 respectively. Find the numbers.

$$9 \,)\, 936 \,(\, 104 \qquad\qquad 9 \,)\, 45 \,(\, 5$$

$$\begin{array}{r} 9 \\ \hline 36 \\ 36 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 45 \\ \hline \end{array}$$

Now resolve 104 into two factors whose difference is 5. $104 = 8 \times 13$ and the difference of 8 and 13 is 5.

$\therefore$ the required numbers $= 9 \times 8$ and $9 \times 13 = 72$ and 117.

58. The circumference of a circular path is 12 kilometres. From a certain place on the path A, B & C begin to walk at 2, 3 and 4 km. per hour respectively. When will they again meet together at the starting place ?

A takes $(12 \div 2)$ or 6 hours, B $(12 \div 3)$ or 4 hours and C $(12 \div 4)$ or 3 hours to walk once round a circular path of 12 kilometres.

$\therefore$ A, B and C return to the starting place at intervals of 6, 4 and 3 hours respectively.

The L. C. M. of 6, 4 and 3 hours=12 hours.

$\therefore$ They will again meet together at the starting place after 12 hours.

59. Find the least number which when divided by 12, 15 and 20 leaves the remainders 3, 6 and 11 respectively, but is exactly divisible by 19.

[Vide examples Nos. 26 and 32] $12-3=9$, $15-6=9$, $20-11=9$. The L. C. M. of 12, 15 and $20=60$. The required number will be less than 60 or any multiple of 60 by 9 and again it will also be divisible by 19.

$$19\,)\,60\,(\,3$$
$$\underline{57}$$
$$3$$

Now find how many times 60, diminished by 9, will be divisible by 19.

Now find how many times the remainder 3 deducted by 9 will be divisible by 19. It is seen that $3\times3-9=0$ and 0 is divisible by 19 ($\because$ 0 is divisible by any number).

$\therefore$ The required number$=60\times3-9=171$.

*60. The sum of two numbers is 96 and their L. C. M. is 280. Find the numbers.

The H. C. F. of any two numbers will also be the H. C. F. of their sum and their L. C. M.

$\therefore$ here the H. C. F. of the numbers=the H. C. F. of the given sum and the given L. C. M.=the H. C. F. of 96 and 280=8.

$280 \div 8=35$. Now resolve 35 into two factors such that their sum when multiplied by the H. C. F. 8 will give the product 96. $35=1\times35$, $35=5\times7$.

Now see that the sum of 5 and 7 muliplied by 8 will give 96 as the product.

$\therefore$ the required two numbers $=5\times8$ and $7\times8=40$ and 56.

61. Find three largest numbers such that their G. C. M. is 8 and L. C. M. is 1320.

$1320\div8=165$; $165=5\times3\times11$. Now the product of any two of the numbers 5, 3 and 11 multiplied by 8 will be one required number.

$\therefore$ The three required numbers $=5\times3\times8$, $5\times11\times8$ and $3\times11\times8=120$, 440 and 264.

H. C. F. and L. C. M. of fractions

$$\text{H. C. F. of fractions}=\frac{\text{H. C. F. of numerators}}{\text{L. C. M. of denominators}}$$

$$\text{L. C. M. of fractions}=\frac{\text{L. C. M. of numerators}}{\text{H. C. F. of denominators}}$$

[**N. B.** In such examples (1) first reduce the mixed fraction into an improper fraction, (2) if there be a whole number, write it in the form of a fraction with 1 as its denominator and (3) reduce the given fractions into their lowest terms. Then follow the above process.]

Example. Find the H.C.F. and the L.C.M. of 6, $\frac{9}{132}$ and $1\frac{10}{11}$.

Here $6=\frac{6}{1}$ [written in the form of a fraction.]

$\frac{9}{132}=\frac{3}{44}$ [reduced to the lowest terms.]

$1\frac{10}{11}=\frac{21}{11}$ [reduced to an improper fraction.]

Now, the H. C. F. of the numerators 6, 3 and $21=3$, and the L. C. M. of the denominators 1, 44 and $11=44$.

$\therefore$ the required H. C. F. $=\frac{3}{44}$.

[To find the L.C.M.] Again, the L. C. M. of the numerators 6, 3 and $21=42$ and the H. C. F. of the denominators 1, 44 and $11=1$.

$\therefore$ the required L. C. M. $=\frac{42}{1}=42$.

H. C. F. and L. C. M. of decimals

[To find the H. C. F. and L. C. M. of decimal fractions the above process may be followed on reducing them into vulgar fractions.]

The general Method. First write the given numbers so that all of them may have the same number of decimal places (putting 0's at the end, if necessary). Then find the H.C.F. or the L.C.M. (as the case may be) of them as if they were integers (i. e. removing the decimal points). Now, in the result thus obtained, put the decimal point to the left of as many digits as the equal number of decimal places above.

Example. Find the H. C. F. and L. C. M. of 1·6, ·24 and ·036.

The given numbers are equivalent to 1·600, ·240 and ·036.

The H. C. F. of 1600, 240 and 36 = 4 ; their L. C. M. = 14400.

∴ the H. C. F. required = ·004

and the L. C. M. required = 14·400 = 14·4.

Exercise 2

1. Find the least number which is exactly divisible by the first nine integers. [C. U. 1920]

[Here the answer will be the L.C.M. of numbers from 1 to 9.]

2. Find the greatest number that will divide 2300 and 3500, and leave the remainders 32 and 56 respectively.

[Vide Example 9.] [C. U. 1927]

3. Find the least number which when diminished by 39 is exactly divisible by 32, 40, 48, 56 and 64. [C. U. '34]

[Vide Example 15. Here the L. C. M. of the numbers + 39 will be the answer.]

4. Find the greatest number of 5 digits exactly divisible by 12, 16, 18 and 21.

5. Find the least number of 6 digits that has 433 for a factor.

[Vide Example 14.] [C. U. '36]

6. The G. C. M. and L. C. M. of two numbers are 90 and 1080 respectively and one of them is 270. Find the other.

[C. U. 1939 Sup.]

7. Five bells toll at intervals of 9, 12, 18, 20 and 30 mins. respectively, beginning together. When will they next toll together and how many times will each toll during the interval ? [D. B. 1938]

8. The circumferences of the fore-wheel and the hind-wheel of a carriage are 9 ft. 11 in. and 12 ft. 9 in. respectively. Find the least distance over which the carriage must travel in order that both the wheels may make a complete number of revolutions. [C. U. 1917]

9. What groups of numbers between 300 and 500 have 63 for their G. C. M. ? [See Ex. 28.]

10. Among a certain number of children 91509 mangoes and 83721 oranges may be equally divided. How many are the children ? Give all possible answers. [D. B. 1930]

11. A number lies between 1600 and 2000 ; the number is known to be divisible by 102 and 36 ; find the number.

[D. B. 1939]

[See Ex. 29]

12. What greatest number and what least number can be subtracted from 53790823 in order that the remainders may be exactly divisible by 24, 35, 63, 91 and 520 ? [D. B. 1935]

13. What is the greatest number of 5 digits which can be added to 8509 so that the sum may be exactly divisible by 20, 27, 32 and 36 ? [D. B. '35]

[Vide Example 36]

14. The G. C. M. and L. C. M. of two numbers are 18 and 108 respectively. How many pairs of such numbers can be formed ? Form them.

15. (i) 64329 is divided by a certain number. If 175, 114 and 213 are respectively the first, the second and the third or final remainders in the operation of division, find the quotient.

[C. U. 1939]

[Vide Example 44.]

(ii) Find the least number which is divisible by 19 but leaves the remainder 6 when divided by 16, 18 and 20.

16. A carpenter was engaged for a number of days at Rs. 25.50 P., but being absent for some days he was paid only Rs. 13.50 P., show that his daily wages could not exceed Re. 1.50 P.

17. What number nearest to 1000 is exactly divisible by 60 ?

18. In finding the G. C. M. of two numbers the last divisor is found to be 49 and if the successive quotients are 17, 3 and 2 ; find the numbers. [Civil Service Ex.]

[Vide Ex. 43. Such sums are to be worked out from the end.]

19. The product of two numbers is 896 and their G. C. M. is 4. Find the numbers.

20. A, B, C and D can go round a circular path in 8, 9, 10 and 12 minutes respectively. They start together from the same place and in the same direction. In what time will they next meet at the same place ?

21. A merchant has three kinds of wine ; of the first 403 gallons, of the second 434 gallons and of the third 465 gallons. What is the least number of full casks of equal size in which these can be stored without mixing ? [A. U. 1906]

22. In a long division sum the quotient consists of two figures. If the dividend is 34256 and the two remainders are 305 and 560, find the divisor and the quotient. [P.S.C.]

23. A company of sepoys proceed in 5 equal rows and after sometime arrange themselves into 7 equal rows. Find the least number above 1000 which the company may contain. [C.U. '51]

24. Find the least number which when divided by 14, 18 and 21 leaves the remainders 3, 7 and 10 respectively, but is exactly divisible by 17.

25. The sum of two numbers is 100 and their L. C. M. is 624. Find the numbers.

26. The product of two numbers is 2744 and their highest common factor is 7. Find the numbers, if each of them be greater than 7. [D. B. 1948]

27. Find the least number of 5 digits which when divided by 4, 6, 10 and 15 leaves in each case the same remainder 3. [C. U. '49]

28. An integral number of seers of salt can be had for Rs. 5. 10 as. as well as for Rs. 7. 5 as. Find the price per seer, if it is between 4 as. and 5 as. [D. B. '49]

[The H. C. F. of Rs. 5. 10 as. and Rs. 7. 5 as.$=$9 as. Now see what factor of 9 as. lies between 4 as. and 5 as. It is $4\frac{1}{2}$ ans.]

29. Find the greatest number that will divide 80, 191 and 265, leaving the same remainder in each case.

30. Find the least number which when divided by 252 and 378 will leave the remainders 244 and 370 respectively. [W. B. S. F. '53]

31. Find the greatest number of 5 digits which is exactly divisible by 25, 36, 48 and 120. [G. U. '54]

32. Find the number nearest to ten million which is divisible by 8, 12, 20 and 24. [U. U. '48]

33. What number between 12000 and 15000 when divided by 15, 25, 35 and 48 leaves the remainder 3 in each case? [G. U. '55]

34. In finding the G. C. M. of two numbers by the usual division-process, the successive quotients are found to be 5, 3 and 2 and the last divisor is 21. Find the numbers. [G. U. '55]

35. Find the sum of all numbers lying between 2000 and 4000 whose G. C. M. is 379. [U. U. '55]

36. The G. C. M., L. C. M. and the difference of two numbers are 9, 1008 and 81 respectively. Find the numbers.

37. Find the G. C. M. and L. C. M. of 4, $\frac{16}{27}$ and $1\frac{31}{81}$.

38. Find the G. C. M. and L. C. M. of 2·1, 2·8 and 3·5.

SIMPLIFICATION OF FRACTIONS

N. B. :—(1) In simplifying a vulgar fraction involving the signs +, −, ×, ÷, and 'of', the operation of the signs should be done in the following order :—First, the operation of 'of', then of division and then of multiplication and last of all the operation of addition and subtraction. Fractions connected by 'of' should be considered as one single quantity and should be dealt with before performing the other operations. The operation of 'of' is like that of multiplication. If, however, only the signs of '÷' and '×' are involved in an expression, the operations of the signs may be done from the left to the right as they occur.

(2) [ ], { }, ( ), —, these are the four kinds of brackets used. To simplify the fractions involving these brackets begin the simplification work from the innermost pair of brackets. If there are the sigins 'of', ÷, ×, +, − within a pair of brackets, follow the rule no. (I) to simplify the portion within the brackets. If there is no sign between two pairs of brackets, they are to be multiplied together and treated as one single quantity.

(3) The numerator of a fraction is treated as a dividend and the denominator as a divisor. As for example :—

(A) $\frac{\frac{1}{3}}{\frac{4}{9}}=\frac{1}{3}\div\frac{4}{9}=\frac{1}{\cancel{3}}\times\frac{\overset{3}{\cancel{9}}}{4}=\frac{3}{4}$; (B) $\frac{3}{\frac{5}{6}}=\frac{\frac{3}{1}}{\frac{5}{6}}=\frac{3}{1}\times\frac{6}{5}=\frac{18}{5}$

(C) $\frac{\frac{5}{6}}{10}=\frac{\cancel{5}}{6}\times\frac{1}{\underset{2}{\cancel{10}}}=\frac{1}{12}$

[In (B) the numerator 3 is a whole number and should be taken as $\frac{3}{1}$] [In(C) the denominator 10 is to be taken as $\frac{10}{1}$.]

(4) If in a fraction both the numerator and the denominator are compound quantites of the same kind, it is equivalent to (*i. e.*, the quotient will be) an abstract number (fraction or integer). Again if the numerator only be a compound quantity, the quotient will be a compound quantity of the same kind.

Examples : (A) $\frac{1}{2}$ of Rs. $\frac{5}{2}$ = Rs. $\frac{5}{4}$ = Re. 1. 50 P.

(B) $\frac{4}{5}\times\frac{2\text{ Hg. }1\text{ Dg.}}{2\text{ Dg. }1\text{ g.}}=\frac{4}{5}\times\frac{210\text{ g.}}{21\text{ g.}}=8$ [But not 8 g.]

Caution. When the numerator and the denominator of a fraction are compound quantities expressed in terms of Rs. P. or Kg. g. etc, boys often give the answer in terms of those units. But this is a gross mistake and the result on simplification must be an abstract number (fractional or integral).

(5) **To convert a decimal fraction into an equivalent vulgar** fraction take the figures to the right of the decimal point as the numerator, and for the denominator write 1 followed by as many 0's as there are figures to the right of the decimal point.

As, $\cdot71 = \frac{71}{100}$; $1\cdot2 = \frac{12}{10}$; $\cdot04 = \frac{4}{100}$.

(6) **The rule for reducing a recurring decimal to a vulgar fraction :** Take the whole given number as an integral number and subtract from it the non-recurring part, then place the difference as the numerator ; for the denominator write as many nines as there are figures in the period followed by as many 0's as there are non-recurring figures. Thus,

$$(A)\quad \cdot\dot{7} = \frac{7}{9},\ (B)\ \cdot1\dot{2}\dot{7} = \frac{127-1}{990} = \frac{126}{990},\ (C)\ \cdot02\dot{3}\dot{7} = \frac{237-2}{9900} = \frac{235}{9900}$$

$$(D)\quad 2\cdot1\dot{3}0\dot{4} = \frac{21304-21}{9990} = \frac{21283}{9990}.$$

[In the last sum there is "304" in the period or the recurring part and only 1 in the non-recurring part ; $\therefore$ the denominator contains three nines followed by one zero.]

Examples (3)

Simplify :—

1. $3\frac{4}{7}$ of $2\frac{2}{5} \times \frac{7}{6} \div \frac{7}{8}$ of $\frac{9}{10} = \frac{\overset{5}{\cancel{25}}}{7}$ of $\frac{12}{\cancel{5}} \times \frac{7}{6} \div \frac{7}{8}$ of $\frac{9}{10}$

$$= \frac{60}{7} \times \frac{7}{6} \div \frac{63}{80} = \frac{\overset{10}{\cancel{60}}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{6}} \times \frac{80}{63} = \frac{800}{63} = 12\tfrac{44}{63}.$$

[Here the operation of 'of' is treated first and then the division work is done.]

2. $1\frac{\frac{2}{3}-\frac{1}{6}}{\frac{2}{3}+\frac{1}{4}} \div \left(\frac{1}{11}+\frac{5}{11}\right) = 1\frac{\frac{4-1}{6}}{\frac{8+3}{12}} \div \left(\frac{1+5}{11}\right) = 1\frac{\frac{3}{6}}{\frac{11}{12}} \div \frac{6}{11}$

$$=1\frac{3\times \overset{2}{\cancel{12}}}{\cancel{6}\times 11} \div \frac{6}{11} = 1\frac{6}{11}\div\frac{6}{11} = \frac{17}{11}\times\frac{11}{6}=\frac{17}{6}=2\frac{5}{6}.$$

[Here the first part is the integer 1 followed by the next part $\frac{6}{11}$, $\therefore$ we have $1\frac{6}{11}$.]

3. $5-5\times\frac{2+1\frac{1}{5}(2+1\frac{1}{5})}{1\frac{1}{5}+2(2+1\frac{1}{5})}$ [A. U. 1896] $=5-5\times\frac{2+\frac{6}{5}(3\frac{1}{5})}{\frac{6}{5}+2(3\frac{1}{5})}$

$$=5-5\times\frac{2+\frac{6}{5}\times\frac{16}{5}}{\frac{6}{5}+2\times\frac{16}{5}} = 5-5\times\frac{2+\frac{96}{25}}{\frac{6}{5}+\frac{32}{5}} = 5-5\times\frac{\frac{146}{25}}{\frac{38}{5}} = 5-\cancel{5}\times\frac{\overset{73}{\cancel{146}}\times\cancel{5}}{\underset{\cancel{5}}{\cancel{25}}\times\underset{19}{\cancel{38}}}$$

$$=5-\frac{73}{19}=\frac{95-73}{19}=\frac{22}{19}=1\frac{3}{19}.$$

4. $12\div\cfrac{1}{7-\cfrac{1}{1-\cfrac{1}{1+\frac{1}{2}}}}$ of $19\frac{1}{5}=12\div\cfrac{1}{7-\cfrac{1}{1-\cfrac{1}{\frac{3}{2}}}}$ of $\frac{96}{5}$

$$=12\div\cfrac{1}{7-\cfrac{1}{1-\frac{2}{3}}} \text{ of } \frac{96}{5}=12\div\cfrac{1}{7-\cfrac{1}{\frac{1}{3}}} \text{ of } \frac{96}{5}=12\div\frac{1}{7-3} \text{ of } \frac{96}{5}$$

$$=12\div\frac{1}{\cancel{4}} \text{ of } \frac{\overset{24}{\cancel{96}}}{5}=12\div\frac{24}{5}=12\times\frac{5}{24}=\frac{5}{2}=2\frac{1}{2}.$$

5. $\frac{5\frac{5}{8}}{6\frac{3}{7}}$ of $\frac{6\frac{7}{11}}{9\frac{1}{8}}\div\frac{8}{9}\left(2\frac{3}{11}+\frac{13}{22}\right)$ of $\frac{90\text{ P.}}{\text{Re. }1.50\text{P.}}$

$$=\frac{\frac{45}{8}}{\frac{45}{7}} \text{ of } \frac{\frac{73}{11}}{\frac{73}{8}}\div\frac{8}{9}\left(\frac{25}{11}+\frac{13}{22}\right) \text{ of } \frac{\overset{3}{\cancel{90}}\text{ P.}}{\underset{5}{\cancel{150}}\text{ P.}}=\frac{\cancel{45}\times7}{8\times\cancel{45}} \text{ of } \frac{\cancel{73}\times\cancel{8}}{11\times\cancel{73}}\div\frac{8}{9}\left(\frac{63}{22}\right) \text{ of } \frac{3}{5}$$

$$=\frac{7}{11}\div\frac{8\times\overset{21}{\cancel{63}}}{\underset{3}{\cancel{9}}\times22} \text{ of } \frac{\cancel{3}}{5}=\frac{7}{11}\div\frac{8\times21}{22\times5}=\frac{\cancel{7}}{\cancel{11}}\times\frac{\overset{2}{\cancel{22}}\times5}{\underset{4}{\cancel{8}}\times\underset{3}{\cancel{21}}}=\frac{5}{12}.$$

[**N. B.** $\frac{8}{9}$ being followed by the part within brackets the whole compound fraction is to be treated as a single term. It is $\frac{63}{22}$ on simplifying the part within brackets. Then $\frac{8\times63}{9\times22}$ is written in one line. If the sign $\div$ were followed by $\div\frac{8}{9}\times(\frac{25}{11}+\frac{13}{22})$, the operation of division by $\frac{8}{9}$ would be done first on inverting $\frac{8}{9}$ and the part within brackets would remain separated.]

6. $\dfrac{\text{3 cwt. 3 qr. 14 lb.}}{\text{2 cwt. 1 qr. 20 lb.}}$ of £7. 18s. 8d. [C. U. 1912]

$$=\frac{\frac{31}{8}\text{ cwt.}}{\frac{17}{7}\text{ cwt.}}\text{ of }£\frac{119}{15}=\frac{31\times7}{8\times\cancel{17}}\text{ of }£\frac{\overset{7}{\cancel{119}}}{15}=£\,\frac{1519}{120}=£12.\ 13s.\ 2d.$$

[The numerator and the denominator of the first fraction are expressed in terms of the same units ; $\therefore$ it is an abstract fraction. The answer is expressed in pounds, shillings and pence, because the last part of the expression is a compound quantity.]

7. $\dfrac{\text{Re. 1. 9 as.}}{\text{Rs. 6. 4 as.}}$ of 3 guineas. [C. U. 1918]

$$=\frac{\cancel{25}\text{ as.}}{\underset{4}{\cancel{100}}\text{ as.}}\text{ of 63s.}=\tfrac{1}{4}\text{ of 63s.}=\tfrac{63}{4}\text{s.}=15\tfrac{3}{4}\text{ s.}$$

$=$15s. 9d. [N. B. 1 guinea $=$ 21 shillings.]

8. $\cdot75\times\cdot75+\cdot25\times\cdot25+2\times\cdot75\times\cdot25$ [C. U. 1940]

$=(\cdot75)^2+(\cdot25)^2+2\times\cdot75\times\cdot25=(\cdot75+\cdot25)^2=(1)^2=1.$

[Putting a for $\cdot75$ and b for $\cdot25$ we get the expression a^2+b^2+2ab. It is equal to $(a+b)^2$.]

9. $\dfrac{2\cdot79\times2\cdot79-\cdot21\times\cdot21}{2\cdot79-\cdot21}=\dfrac{a^2-b^2}{a-b}$

[Putting a for $2\cdot79$ and b for $\cdot21$.]

$$=\frac{(a+b)(a-b)}{a-b}=a+b=2\cdot79+\cdot21=3.$$

10. $\dfrac{1\cdot49\times14\cdot9-\cdot41\times4\cdot1}{14\cdot9-4\cdot1}=\dfrac{10(1\cdot49\times14\cdot9-\cdot41\times4\cdot1)}{10(14\cdot9-4\cdot1)}$

[Here the numerator and the denominator are multiplied by 10 without altering the value of the fraction.]

$$=\frac{14\cdot9\times14\cdot9-4\cdot1\times4\cdot1}{10(14\cdot9-4\cdot1)}=\frac{(14\cdot9)^2-(4\cdot1)^2}{10(14\cdot9-4\cdot1)}$$

$$=\frac{(14\cdot9+4\cdot1)(14\cdot9-4\cdot1)}{10(14\cdot9-4\cdot1)}=\frac{14\cdot9+4\cdot1}{10}=\frac{19}{10}=1\cdot9.$$

11. Find the simplest value of :
$7.1245 \times 7.1245 \times 7.1245 - 3 \times 7.1245 \times 6.1245 - 6.1245 \times 6.1245 \times 6.1245.$ [D. B. '48]

Suppose, $a = 7.1245$ and $b = 6.1245$,

$\therefore \quad a - b = 7.1245 - 6.1245 = 1.$

Now, the given expression $= a^3 - 3ab - b^3 = a^3 - 3ab \times 1 - b^3$

$= a^3 - 3ab(a-b) - b^3 \quad [\because \; a - b = 1]$

$= (a-b)^3 = (1)^3 = 1.$

12. Prove that $2.87\dot{7}\dot{9} = \frac{2879-28}{990}$. [E. B. S. B. '48]

$1000 \times 2.8\dot{7}\dot{9} = 1000 \times 2.8797979\ldots = 2879.7979\ldots$

Again, $\underline{10 \times 2.8\dot{7}\dot{9} = \quad 10 \times 2.8797979\ldots = \quad 28.7979\ldots}$

$\therefore$ (subtracting) $990 \times 2.8\dot{7}\dot{9} \quad = \quad 2879 - 28$

$\therefore \quad 2.8\dot{7}\dot{9} = \frac{2879-28}{990}.$

[Vide the rule for reducing a recurring decimal to a vulgar fraction. Mark here that subtracting 10 times $2.8\dot{7}\dot{9}$ from a thousand times $2.8\dot{7}\dot{9}$ we have got $(1000-10)$ or 990 times $2.8\dot{7}\dot{9}$.]

13. $\dfrac{.4\dot{4}2857\dot{1} + .5\dot{5}7142\dot{8}}{.2\dot{2}8571\dot{4} + .7\dot{7}1428\dot{5}}.$ [A. U. '48]

The given expression $= \dfrac{4.\dot{4}2857\dot{1} + 5.\dot{5}7142\dot{8}}{2.\dot{2}8571\dot{4} + 7.\dot{7}1428\dot{5}}$

[multiplying numerator and denominator by 10]

$= \dfrac{4\frac{3}{7} + 5\frac{4}{7}}{2\frac{2}{7} + 7\frac{5}{7}} = \frac{10}{10} = 1.$

[Here it is also easy to simplify the expression without reducing it to a vulgar fraction.

The sum of the two terms of the numerator $= 9.\dot{9}9999\dot{9} = 10$; the sum of the denominator $= 10$. Remember that $7.\dot{9} = 8$, $2.\dot{9} = 3$. When the recurring part contains the figure 9 only, the recurring part should be omitted and the preceding figure increased by unity.]

14. $\dfrac{1\frac{1}{2} + 2\frac{2}{3} + 3\frac{3}{4}}{\frac{1}{1\frac{1}{2}} + \frac{1}{2\frac{2}{3}} + \frac{1}{3\frac{3}{4}}} \div \dfrac{.2\dot{1}}{.034\dot{8}}.$ [C. U. '48 Suppl.]

The given expression $= \dfrac{\frac{3}{2} + \frac{8}{3} + \frac{15}{4}}{\frac{2}{3} + \frac{3}{8} + \frac{4}{15}} \div \dfrac{\frac{21-2}{90}}{\frac{348-34}{9000}} = \dfrac{\frac{18+32+45}{12}}{\frac{80+45+32}{120}} \div \dfrac{\frac{19}{90}}{\frac{314}{9000}}$

$= \dfrac{\frac{95}{12}}{\frac{157}{120}} \div \dfrac{19 \times 9000}{90 \times 314} = \dfrac{95 \times 120}{12 \times 157} \times \dfrac{90 \times 314}{19 \times 9000} = 1.$

15. $\dfrac{0\cdot\dot{5}\dot{2}}{0\cdot\dot{1}5\dot{4}} \div \dfrac{26\cdot\dot{2}\dot{6}}{4\cdot\dot{9}0\dot{4}} + \dfrac{2}{1+\dfrac{3}{1+\cdot\dot{3}}}$ [C. U. 1933]

$$= \frac{\frac{52}{99}}{\frac{154}{999}} \div \frac{\frac{2626-26}{99}}{\frac{4904-4}{999}} + \frac{2}{1+\frac{3}{1-\frac{1}{3}}} = \frac{\frac{52}{99}}{\frac{154}{999}} \div \frac{\frac{2600}{99}}{\frac{4900}{999}} + \frac{2}{1+\frac{3}{\frac{2}{3}}}$$

$$= \frac{52\times999}{99\times154} \div \frac{2600\times999}{99\times4900} + \frac{2}{1+\frac{3\times3}{1\times2}} = \frac{\overset{2}{\cancel{52}}\times\cancel{999}}{\cancel{99}\times\underset{\underset{11}{22}}{\cancel{154}}} \times \frac{\overset{7}{\cancel{99}\times\cancel{4900}}}{\cancel{2600}\times\cancel{999}} + \frac{2}{\frac{11}{2}}$$

$$= \frac{7}{11} + \frac{2\times2}{1\times11} = \frac{7}{11} + \frac{4}{11} = 1.$$

16. $\dfrac{15\cdot\dot{6}+7-\cdot\dot{3}}{3\times7\cdot\dot{4}\times\cdot25} + \left\{37+\dfrac{3\cdot7\dot{0}3\dot{7}}{100}\right\}\times0\cdot27.$ [C. U. 1934]

$$= \frac{\frac{156-15}{9}+7-\frac{3}{9}}{3\times\frac{74-7}{9}\times\frac{25}{100}} + \left\{37+\frac{\frac{37037-37}{9990}}{100}\right\}\times\frac{27}{100}$$

$$= \frac{\frac{\overset{47}{\cancel{141}}}{\cancel{9}}+7-\frac{1}{3}}{\cancel{3}\times\frac{67}{\underset{3}{\cancel{9}}}\times\frac{1}{4}} + \left\{37+\frac{\frac{37000}{9990}}{\frac{100}{1}}\right\}\times\frac{27}{100} = \frac{\frac{67}{3}}{\frac{67}{12}} + \left\{37+\frac{\cancel{37000}\times1}{\underset{27}{\cancel{9990}\times\cancel{100}}}\right\}\times\frac{27}{100}$$

$$= \frac{\cancel{67}\times\overset{4}{\cancel{12}}}{\cancel{3}\times\cancel{67}} + \left\{37+\frac{1}{27}\right\}\times\frac{27}{100} = 4+\frac{999+1}{27}\times\frac{27}{100}$$

$$= 4+\frac{\overset{10}{\cancel{1000}}}{\cancel{27}}\times\frac{\cancel{27}}{\cancel{100}} = 4+10 = 14.$$

17. $\dfrac{1\cdot\dot{3}\times1\cdot\dot{3}+1\cdot\dot{3}-1}{1\cdot\dot{3}\times1\cdot\dot{3}+1\cdot\dot{3}+1} = \dfrac{a^3-1}{a^2+a+1}$ [suppose, $a=1\cdot\dot{3}$.]

$$= \frac{(a-1)(a^2+a+1)}{(a^2+a+1)} = a-1 = 1\cdot\dot{3}-1 = \cdot\dot{3}.$$

18. $\dfrac{\frac{2}{3}+\frac{3}{4}}{\frac{5}{6}\div\frac{7}{8}}$ of $1\cdot\dot{6}$ g. $-\ 0\cdot125$ of $0\cdot1\dot{6}$ of 23 g. [C. U. 1919]

$=\dfrac{\frac{8+9}{12}}{\frac{\cancel{5}}{\underset{3}{\cancel{6}}}\times\frac{\overset{4}{\cancel{8}}}{7}}$ of $1\frac{2}{3}$ g. $-\ \dfrac{\cancel{125}}{\underset{8}{\cancel{1000}}}$ of $\dfrac{16-1}{90}$ of 23 g.

$=\dfrac{\frac{17}{12}}{\frac{20}{21}}$ of $\dfrac{5}{3}$ g. $-\ \dfrac{1}{8}$ of $\dfrac{\cancel{15}}{\underset{6}{\cancel{90}}}$ of 23 g. $=\dfrac{17\times\overset{7}{\cancel{21}}}{12\times\underset{4}{\cancel{20}}}$ of $\dfrac{\cancel{5}}{\cancel{3}}$ g. $-\ \dfrac{23}{48}$ g.

$=\dfrac{119}{48}$ g. $-\ \dfrac{23}{48}$ g. $=\dfrac{119-23}{48}$ g. $=\dfrac{\overset{2}{\cancel{96}}}{\cancel{48}}$ g. $=2$ g.

19. $\frac{2}{3}\div\frac{4}{9}$ of $7\frac{1}{2}+999\frac{494}{495}\times 99$ [C. U. 1942]

$=\dfrac{2}{3}\div\dfrac{4\times15}{9\times2}+\left(1000-\dfrac{1}{495}\right)\times 99$ $\left[\because\ 999\dfrac{494}{495}=999+\dfrac{495-1}{495}=999+\dfrac{\cancel{495}}{\cancel{495}}-\dfrac{1}{495}=1000-\dfrac{1}{495}\right]$

$=\dfrac{\cancel{2}}{\cancel{3}}\times\dfrac{\overset{\cancel{3}}{\cancel{9}}\times\cancel{2}}{\underset{2}{\cancel{4}}\times\underset{5}{\cancel{15}}}+99000-\dfrac{\cancel{1}}{\underset{5}{\cancel{495}}}\times\cancel{99}=\dfrac{1}{5}+99000-\dfrac{1}{5}=99000.$

20. $\dfrac{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}}{\frac{1}{2\frac{1}{2}}-\frac{1}{3\frac{1}{2}}+\frac{1}{4\frac{1}{2}}}\times\dfrac{\cdot0005\dot{3}}{\cdot432\text{ of Re. 1. 9 as.}}+\dfrac{\text{Rs. 2. 14as. }0\cdot96\text{p.}}{\cdot00\dot{3}\dot{5}}$ [C. U. 1945]

$=\dfrac{\frac{6-4+3}{12}}{\frac{1}{\frac{5}{2}}-\frac{1}{\frac{7}{2}}+\frac{1}{\frac{9}{2}}}\times\dfrac{\frac{53}{99000}}{\frac{432}{1000}\times\text{Rs.}\frac{25}{16}}\times\dfrac{\frac{\text{Rs. }72}{25}}{\frac{35}{9900}}$ $\left[\ \cdot96\text{ p.}=\dfrac{96}{100}\text{ p.}=\dfrac{\overset{\overset{2}{\cancel{8}}}{\cancel{96}}}{\underset{25}{\cancel{100}}\times\cancel{12}}\text{a.}\ \right]$

$=\dfrac{\frac{5}{12}}{\frac{2}{5}-\frac{2}{7}+\frac{2}{9}}\times\dfrac{53\times\overset{\overset{\cancel{4}}{\cancel{100}}}{\cancel{1000}}\times16}{\underset{\cancel{10}}{\cancel{99000}}\times\underset{\underset{3}{\cancel{108}}}{\cancel{432}}\times25}\times\dfrac{\overset{2}{\cancel{72}}\times\cancel{9900}}{\cancel{25}\times35}$

[Here the answer is an abstract fraction, the numerator and the denominator being expressed in rupees.]

$=\dfrac{\frac{5}{12}}{\frac{106}{315}}\times\dfrac{53\times16\times2}{3\times25\times35}=\dfrac{\cancel{5}\times\overset{\overset{\cancel{35}}{\cancel{105}}}{\cancel{315}}}{\underset{\cancel{3}}{\cancel{12}}\times\underset{2}{\cancel{106}}}\times\dfrac{\cancel{53}\times\overset{4}{\cancel{16}}\times\cancel{2}}{\cancel{3}\times\underset{5}{\cancel{25}}\times\cancel{35}}=\dfrac{4}{5}=\cdot8.$

21. $\dfrac{\cdot321 \times \cdot321 - \cdot179 \times \cdot179}{\cdot321 - \cdot179}$ of Rs. 5. $= \dfrac{a^2 - b^2}{a - b}$ of Rs. 5

[Suppose, $a = \cdot321$ and $b = \cdot179$] $= (a+b)$ of Rs. 5.

$= (\cdot321 + \cdot179)$ of Rs. $5 = \cdot5$ of Rs. $5 = \frac{1}{2}$ of Rs. 5=Rs. 2. 50P.

22. Express Rs. 28. 8 as. as a fraction of $\frac{2}{3}$ of Rs. 187. 5a.

Rs. 28. 8 as. = Rs. $28\frac{1}{2}$ = Rs. $\frac{57}{2}$;

$\frac{2}{3}$ of Rs. 187. 5 as. = Rs. $187\frac{5}{16} \times \frac{2}{3}$ = Rs. $(\frac{2997}{16} \times \frac{2}{3})$ = Rs. $\frac{999}{8}$

$\therefore$ the reqd. fraction $= \dfrac{\frac{57}{2}}{\frac{999}{8}} = \frac{57 \times 8}{2 \times 999} = \frac{76}{333}$.

[In such sums to express one quantity as the fraction of another, first express both the quantities in terms of the same unit and then divide the measures of the first by the measure of the second in the form of the numerator and denominator of a fraction.]

23. What part of $\frac{3}{8}$ of 5 cwt. is $\frac{1}{100}$ of a ton ? [P. U.]

$\frac{3}{8}$ of 5 cwt. $= \frac{5 \times 3}{8}$ cwt. $= \frac{15}{8}$ cwt.

and $\frac{1}{100}$ of 1 ton = 20 cwt. $\times \frac{1}{100} = \frac{1}{5}$ cwt.

$\therefore$ the reqd. fraction $= \dfrac{\frac{1}{5}}{\frac{15}{8}} = \frac{1 \times 8}{5 \times 15} = \frac{8}{75}$.

24. Express 1 second as the decimal of 1 hour.
[C. U. 1911, 1919]

1 hr. = 3600 sec., $\therefore$ $\dfrac{1 \text{ sec.}}{1 \text{ hr.}} = \dfrac{1 \text{ sec.}}{3600 \text{ sec.}} = \dfrac{1}{3600} = \cdot0002\dot{7}$ Ans.

```
3600)1·0000(·00027̇
      7200
     -----
     28000
     25200
     -----
      2800
```

25. Express $\dfrac{0\cdot3\dot{5}}{0\cdot0\dot{8}}$ of Rs. 5. 5 as. 4 p. as the decimal of Rs. 58. 10 as. 8 p. [C. U. 1935]

$\dfrac{\cdot3\dot{5}}{\cdot0\dot{8}}$ of Rs. 5. 5 as. 4 p. $= \dfrac{\frac{35-3}{90}}{\frac{8}{90}}$ of Rs. $5\frac{1}{3}$ = Rs. $\left(\dfrac{\overset{4}{\cancel{32}} \times \cancel{90}}{\cancel{90} \times \cancel{8}} \times \dfrac{16}{3}\right)$ = Rs. $\frac{64}{3}$

Again, Rs. 58. 10 as. 8 p. = Rs. $58\frac{2}{3}$.

$\therefore$ the reqd. decimal $= \dfrac{\text{Rs. } \frac{64}{3}}{\text{Rs. } 58\frac{2}{3}}$

$= \dfrac{\text{Rs. } \frac{64}{3}}{\text{Rs. } \frac{176}{3}} = \dfrac{\overset{4}{\cancel{64}} \times \cancel{3}}{\cancel{3} \times \underset{11}{\cancel{176}}} = \dfrac{4}{11} = \cdot\dot{3}\dot{6}$.

26. What decimal of a rupee must be added to ·045 of 4as. 8p. so that the sum may be an anna ? [C. U. 1936]

·045 of 4 as. 8 p. $= \frac{45}{1000} \times \frac{14}{3}$ as. $= \frac{21}{100}$ as.

1 as. $- \frac{21}{100}$ as. $= \frac{79}{100}$ as. Now, see what decimal of Re. 1 is $\frac{79}{100}$ as.

The reqd. decimal $= \frac{\frac{79}{100} \text{ as.}}{\text{Rs. } 1} = \frac{\frac{79}{100} \text{ as.}}{16 \text{ as.}} = \frac{79}{1600} = {}^{\cdot}049375.$

27. Express $1\frac{3}{4}$ kg. as the decimal of 1 quintal 12 kg.

$$\frac{1\frac{3}{4} \text{ kg.}}{1 \text{ quintal } 12 \text{ kg.}} = \frac{\frac{7}{4} \text{ kg.}}{112 \text{ kg.}} = \frac{7}{4 \times 112} = \frac{1}{64} = {}^{\cdot}015625.$$

28. Express £3. 15s. 4d. as the decimal of Rs. 100 (£1 = Rs. 15). [D. B. 1933]

£3. 15s. 4d. $= £\frac{113}{30} =$ Rs. $(\frac{113}{30} \times 15) =$ Rs. $\frac{113}{2}$ [$\because$ £1 = Rs. 15]

Now, $\frac{\text{Rs. } \frac{113}{2}}{\text{Rs. } 100} = \frac{113}{200} = {}^{\cdot}565.$

29. A post has half of its length in mud, one-third of its length in water, and ten metres above water. Find the whole length of the post.

There is $(\frac{1}{2} + \frac{1}{3})$ or $\frac{5}{6}$ of the post in mud and water.

$\therefore$ there is $(1 - \frac{5}{6})$ or $\frac{1}{6}$ of the post above water.

$\therefore$ $\frac{1}{6}$ of the whole length of the post = 10 metres.

$\therefore$ the required whole length of the post $= 10 \text{ m.} \div \frac{1}{6}$

$= 10 \text{ m.} \times 6 = 60$ metres.

Exercise 3

Simplify :—

1. $(6\frac{1}{4} - 3\frac{7}{8})$ of $(\frac{2}{3} - \frac{3}{5}) \div \{5 - (2\frac{1}{3} - 1\frac{1}{2})\}$
2. $(\frac{2}{7} - \frac{2}{49})$ of $(\frac{1}{3} - \frac{1}{9}) \div (\frac{2}{3} - \frac{2}{9})$ of $(\frac{1}{7} - \frac{1}{49})$. [P. U. 1910]
3. $1 \div [1 + 1 \div \{1 + 1 \div (1 + 1 \div 2)\}]$ [B. U. 1910]

[The portion within first brackets $= 1 + 1 \div 2 = 1 + 1 \times \frac{1}{2}$

$= 1 + \frac{1}{2} = \frac{3}{2}$.]

4. $\dfrac{2\frac{3}{8}+5\frac{7}{8}}{1\frac{1}{2}-\frac{4}{9}}\div\left(\dfrac{3\frac{1}{2}}{4}\text{ of }\dfrac{5}{8}\right)\times\dfrac{2\frac{3}{16}}{32}$ [C.U. 1923]

5. $\left\{2\frac{3}{4}+\frac{5}{9}\text{ of }\dfrac{7}{3\frac{4}{5}}-\dfrac{1\frac{2}{9}}{2\frac{1}{2}}\right\}\div 1\frac{77}{228}$ [P.U. 1919]

6. $\dfrac{1\frac{1}{4}-\frac{5}{12}}{1\frac{1}{4}+\frac{5}{12}}+\dfrac{9\times 5}{14\times 3}\text{ of }\dfrac{7}{8}-\dfrac{11\frac{1}{4}}{15}$ [D.B. 1935]

7. $\dfrac{2\frac{13}{16}\text{ of }\frac{1}{8}}{\frac{3}{8}\text{ of }\frac{5}{16}\div 5\frac{1}{6}}\div\left(1\frac{15}{16}\text{ of }\frac{1}{8}\right)$ [C.U. 1922]

8. $\left(4\frac{2}{3}-1\frac{1}{8}\right)\times\left(3\frac{1}{2}-\frac{8}{9}\right)\div\left(13\frac{1}{3}+7\frac{1}{2}\right)\text{ of }\dfrac{3\frac{1}{3}}{1\frac{1}{9}}$ [C.U. 1887]

[In this sum first perform the operation of 'of' with the last bracket, and put the result after the sign $\div$.]

9. $6-6\times\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{3}{4}-\frac{1}{8}}}}$

10. $3-\cfrac{11}{5+\cfrac{1}{5-\cfrac{1}{1+\frac{1}{4}}}}$

10. $8-8\times\cfrac{2\frac{1}{3}-1\frac{2}{7}}{2-\cfrac{1}{6-\frac{1}{6}}}$ [C. U. 1879]

12. $\dfrac{3\frac{1}{5}+2\frac{7}{11}}{4\frac{7}{10}-1\frac{17}{22}}\div\cfrac{5}{11+\cfrac{7}{8+\frac{5}{9}}}-4\dfrac{5}{7\frac{2}{3}}$ [C.U. 1933]

[N. B. $4\dfrac{5}{7\frac{2}{3}}=4\dfrac{5}{\frac{23}{3}}=4\dfrac{5\times 3}{23}=4\frac{15}{23}=\frac{107}{23}$]

13. $\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{4+\cfrac{5}{6\frac{1}{3}}}}}+\frac{3}{2}\div\frac{5}{6}\text{ of }\frac{3}{2}\times 1\frac{1}{4}-\frac{1}{11}\left(10+\frac{13}{30}\right)$ [C. U. 1934]

14. $\cfrac{5\frac{1}{2}}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}-\dfrac{\frac{2}{5}\div\frac{3}{5}\text{ of }\frac{2}{5}}{\frac{2}{5}\div\frac{2}{5}\times\frac{2}{5}}\times\dfrac{2}{5}+\left(\dfrac{\frac{14}{15}}{1\frac{11}{15}}+7\frac{6}{13}-6\frac{56}{91}\right)$ [C.U. 1936]

15. $\dfrac{\text{Rs. 4. 50 P.}}{\text{Rs. 7. 20 P.}}-\dfrac{\text{1 hr. 16 min. 45 sec.}}{\text{2 hr. 7 min. 55 sec.}}$

16. $\dfrac{3\frac{3}{4}+7\frac{5}{12}}{8\frac{5}{6}-4\frac{2}{3}}-4\frac{1}{5}\div\dfrac{2\frac{3}{4}}{1\frac{5}{6}}$ of $\dfrac{14 \text{ Dm.}}{9 \text{ Dm. } 9\text{m.}}$

17. $\left(5\frac{5}{6}-\frac{5}{12}\right)$ of $\left(\dfrac{\frac{5}{6}}{3\frac{1}{3}}\div\frac{7}{8}\text{ of }\frac{4}{5}\right)\div\frac{5}{7}$ of $\dfrac{6 \text{ g. } 3 \text{ dg.}}{9 \text{ dg.}}$

18. $\dfrac{\frac{3}{5}+\frac{7}{8}}{\frac{8}{9}+\frac{9}{10}}$ of $\dfrac{13 \text{ s. } 5 \text{ d.}}{9 \text{ s. } 10 \text{ d.}}\div\frac{3}{5}(\frac{3}{7}+\frac{2}{3})$ of $\dfrac{6 \text{ Dg. } 3 \text{ g.}}{8 \text{ Dg. } 3 \text{ g.}}$

[C. U. 1899]

19. $\dfrac{2\frac{3}{4}}{5\frac{1}{6}}$ of $\frac{3}{4}(\frac{7}{9}+\frac{1}{12})\div\dfrac{5\frac{7}{8}}{7\frac{1}{4}}$ of $\dfrac{2 \text{ m. } 9 \text{ dm.}}{4 \text{ m. } 7 \text{ dm.}}$ [C.U. 1933]

20. $\dfrac{22\cdot5}{1\cdot5}\times\dfrac{10\cdot5}{\cdot35}\times\dfrac{79\cdot2}{13\cdot2}$ of $\dfrac{8\cdot52}{2\cdot13}\div3$ of $\dfrac{7\cdot5}{\cdot15}$

21. $\dfrac{1}{1+\dfrac{1}{3+\frac{1}{4}}}\div\dfrac{5\cdot2}{\cdot051}$ [D.B. '28] 22. $\dfrac{\cdot\dot{8}\dot{1}\times\cdot005}{\cdot45}$ [C.U. '10]

23. $\dfrac{\cdot\dot{2}\dot{4}}{\cdot125}$ of $\dfrac{3\cdot125}{2\cdot16}\div\dfrac{187\cdot5}{3\cdot42}$ of $\dfrac{2\cdot\dot{2}}{1\cdot5}$ [C.U. 1886]

24. $\dfrac{\cdot\dot{2}\times\cdot\dot{2}\times\cdot\dot{2}+\cdot0\dot{2}\times\cdot0\dot{2}\times\cdot0\dot{2}}{\cdot\dot{2}\times\cdot\dot{2}-\cdot\dot{2}\times\cdot0\dot{2}+\cdot0\dot{2}\times\cdot0\dot{2}}$

25. $\dfrac{2\cdot4\dot{6}-2\cdot\dot{3}\dot{0}}{\cdot\dot{3}+\cdot1\dot{2}\dot{7}}+\dfrac{4\frac{1}{3}}{19}$

[C.U. 1912]

26. $\dfrac{3+\frac{1}{3}}{3+\dfrac{1}{3+\dfrac{1}{3+\frac{2}{3}}}}+\dfrac{9}{17}\cdot\dfrac{7 \text{ quintal } 60 \text{ kg.}}{6 \text{ quintal } 30 \text{ kg.}}+\dfrac{5\frac{5}{6}\div\frac{3}{5}\times\frac{4}{5}}{5\frac{2}{5}\div\frac{4}{9}\text{ of }\frac{2}{5}}$

[Here is a point (·) after $\frac{9}{17}$. This point (·) sign indicates the sign of multiplication.]

27. $\dfrac{\frac{3}{5}}{1+\dfrac{2}{3+\frac{2}{5}}}+\dfrac{5\frac{6}{7}}{2\frac{5}{6}}\div\dfrac{6}{7}\div5+\dfrac{0\cdot003}{0\cdot07}$ of $\dfrac{\text{Rs. } 25\frac{3}{8}}{\text{Rs. } 8.\ 50 \text{ P.}}$

28. $\dfrac{1}{1+\dfrac{1}{6\div\frac{2}{3}+6\div2\div3}}+\dfrac{4}{7}\times\dfrac{0\cdot3\times\text{Rs. } 3.\ 4 \text{ as. } 6 \text{ p.}}{0\cdot0\dot{3}\times\cdot25\times\text{Rs. } 72.\ 3 \text{ as.}}\div$

$(\frac{1}{2}\div\frac{2}{5}\text{ of }\frac{5}{7}-\frac{1}{2}\div\frac{2}{5}\times\frac{5}{7})$ [C. U. 1939]

29. $\frac{8}{3}\times\frac{0\cdot8\dot{5}}{1\cdot2}-7\cdot\dot{1}4285\dot{7}\times1\cdot875$ [C. U. 1941]

[N. B. $\cdot\dot{1}4285\dot{7}=\frac{1}{7}$]

30. $\dfrac{5}{5+\dfrac{5}{5+\frac{1}{5}}}\times\dfrac{6\frac{1}{8}}{5\frac{1}{8}}+\dfrac{5\cdot208\dot{3}\text{ of Rs. 3. 84 P.}}{1000\times\frac{4}{17}\times\cdot1\dot{8}\text{ of 45P.}}$

31. $\dfrac{\cdot0\dot{0}7\dot{4}\times\cdot135}{\cdot0\dot{0}\dot{8}\times\cdot09}+\dfrac{3\frac{1}{2}\div2\frac{1}{3}\times1\frac{1}{2}}{3\frac{1}{2}\div2\frac{1}{3}\text{ of }1\frac{1}{2}}\cdot\dfrac{5}{18}$ [C.U. 1944]

32. $\cdot\dot{6}\times\dfrac{\cdot426\times\cdot426-\cdot174\times\cdot174}{\cdot426-\cdot174}$ of Re. 1. 4 as.

33. $\dfrac{2\cdot8\text{ of }2\cdot2\dot{7}}{1\cdot3\dot{6}}+\left\{\dfrac{4\cdot\dot{4}-2\cdot8\dot{8}}{1\cdot3+2\cdot62\dot{9}}\right\}$ of $8\cdot2$ [M,U. 1878]

34. $\dfrac{2\frac{1}{5}-1\frac{2}{5}}{3\frac{1}{5}+1\frac{2}{5}}\div\dfrac{\frac{2}{7}-\frac{1}{8}}{\frac{2}{7}+\frac{1}{8}}+\dfrac{\cdot05\times\cdot7}{\cdot071}$ of $\dfrac{\text{Rs. 2. 7 as.}}{\text{Re. 1. 11 as.}}$ [P.U. 1925]

35. $\dfrac{\cdot49}{2\cdot1}$ of $\dfrac{(3\frac{1}{3}-2\frac{1}{2})\div\frac{5}{6}\text{ of }\frac{5}{8}}{2\frac{2}{3}\div(\frac{1}{2}+\frac{1}{4})}$ of £46.

36. Rs. 2. 4 as. $\div\dfrac{\frac{8}{19}\text{ of }\dfrac{\frac{25}{32}}{2\frac{1}{4}-\frac{7}{12}}\text{ of £5}-\frac{25}{57}\text{ of 1 guinea}}{\text{£93. 8s. }2\frac{8}{19}\text{d.}}$

[P.U. 1903]

37. $\dfrac{\cdot67\times\cdot67\times\cdot67-\cdot001}{\cdot67\times\cdot67+\cdot067+\cdot01}+\dfrac{\cdot57}{1+\dfrac{1}{3-\frac{1}{14}}}$ [C.U. 1928]

[Hints : Put a for $\cdot67$, b for $\cdot1$; Now, $\cdot001=\cdot1\times\cdot1\times\cdot1=b^3$

$\cdot067=\cdot67\times\cdot1=ab$; $\cdot01=\cdot1\times\cdot1=b^2$.

$\therefore$ the first part$=\dfrac{a^3-b^3}{a^2+ab+b^2}=a-b=\cdot67-\cdot1=\cdot57$]

38. What decimal of a maund is a chhatak ? [C.U. '13]

39. Reduce 1 yard to the decimal of a mile. [C.U. '33]

40. What decimal of a rupee is a pie ? [C.U. '11]

41. What decimal of a sovereign is a penny ? [C. U. '16]

42. What decimal of Rs. 8 must be added to Rs. 3. 75 P. so that the sum may be Rs. 5 ?

43. Find the value of $\frac{(3\cdot47)^2-(2\cdot53)^2}{\cdot94}$ of £1. 5s.
If Re. 1 = 1s. 6d., express the value in rupee. [D. B. 1940]

44. Simplify $\dfrac{£44}{11+\dfrac{1}{7+\dfrac{3}{8\frac{1}{4}}}}\div\frac{1}{5}$ of £1. 13s. 4d. [A. U. 1904]

45. A man travelled $\frac{3}{7}$ of his journey by boat, $\frac{2}{5}$ by rail and walked 12 miles. How far did he go? [D. B. 1925]

46. Simplify—

$$\frac{1}{1+\dfrac{1}{5+\frac{2}{3}}}\div\frac{1\cdot1\dot{3}}{2}\times\frac{\cdot14\times\cdot12\times\cdot02+\cdot04\times\cdot16\times\cdot01}{\cdot01\times\cdot2\times\cdot1}$$

[C. U. 1946]

47. Simplify—

$$\frac{\cdot0347\times\cdot0347\times\cdot0347+(\cdot9653)^3}{(\cdot0347)^2-\cdot347\times\cdot09653+(\cdot9653)^2}$$

[P. U. 1932]

[$\cdot0347=a$, $\cdot9653=b$. $\therefore$ $\cdot347\times\cdot09653=\cdot0347\times\cdot9653=ab$]

48. Express $0\cdot16$ of $2\cdot5$ litres $+\,0\cdot1\dot{6}$ of $2\cdot6$ $l.$ as the fraction of 2 D$l.$ Convert the fraction into a recurring decimal. [C. U. 1920]

49. $\dfrac{1\frac{2}{3}+\frac{1}{2}}{6\frac{1}{2}\div\frac{3}{4}}\div\dfrac{\cdot625}{13\cdot125}$ of $\dfrac{2\frac{3}{5}}{1\frac{3}{5}}\times\dfrac{3\cdot\dot{5}\times\cdot05}{2\cdot\dot{3}\times2\cdot5}$ [C. U. 1947]

50. $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\dfrac{1}{4\frac{1}{2}}+\dfrac{1}{3\frac{1}{2}}-\dfrac{1}{2\frac{1}{2}}}\div\dfrac{\cdot02\dot{8}\times\cdot09\times3\cdot5}{2\cdot2\dot{6}\times\cdot008\times\cdot1}$ [C. U. 1948]

51. $\dfrac{2\frac{1}{2}\text{ of }\frac{5}{8}\div3\frac{1}{8}\times6\frac{2}{5}}{3\frac{4}{5}+2\frac{3}{5}-1\frac{7}{15}}\div\left\{\dfrac{10\cdot9\dot{3}\text{ of (Rs. 3. 12 as.)}}{17\cdot\dot{7}\text{ of (Rs. 3. 13 as. 6 ps.)}}\right\}$

[C. U. 1949]

52. $\dfrac{5}{5+\dfrac{5}{5+\frac{1}{5}}}+\dfrac{\text{Re 1. 40 P.}}{\text{Re 1. 24 P.}}\div\frac{1}{7}(2\cdot\dot{4}+4\cdot\dot{6})^2$ [G. U. 1949]

53. $\dfrac{3\cdot\dot{3}}{6\cdot0625}$ of $\dfrac{9\cdot7}{2\cdot4\dot{2}}\div\dfrac{2\cdot5}{1\cdot09}(7\cdot25+2\cdot75)\times\dfrac{8\text{ g.}}{25\text{g. }60\text{cg.}}$

54. $\dfrac{\frac{1}{2}\div\frac{2}{3}\times\frac{3}{4}}{\frac{1}{2}\div\frac{2}{3}\text{ of }\frac{3}{4}}\times\dfrac{13\text{a. }4\text{p.}}{5\text{a. }10\text{p.}}\times(1\cdot4-2\cdot\dot{3}+1\cdot\dot{6})$ [G. U. '53]

55. $\dfrac{\frac{1}{3}+\frac{1}{4}+\frac{1}{6}}{3\frac{1}{3}+4\frac{1}{4}+6\frac{1}{6}} \div \dfrac{\cdot 13\dot{8} \div \cdot 3\dot{6} \text{ of } 1\cdot\dot{6}}{\cdot 3\dot{1}\dot{8} \div \cdot 2\dot{5}\dot{4}}$ [G. U. '48]

56. $\dfrac{5\frac{5}{8}}{6\frac{3}{7}}$ of $\dfrac{6\cdot\dot{6}\dot{8}}{9\cdot 125} \div \frac{8}{9}\left(2\frac{8}{11}+\frac{13}{22}\right)$ of $\dfrac{8\text{s. }3\text{d.}}{£1.\ 13\text{s.}}$ [G. U. '50]

57. Simplify—

$$1-\cfrac{2}{3+\cfrac{4}{5-\cfrac{6}{7+\frac{8}{9}}}} \div 2\cdot 08\dot{3} \text{ of } \frac{2 \text{ cwt. } 2 \text{ qr. } 21 \text{ lb.}}{10 \text{ cwt. } 2 \text{ qr. } 11 \text{ lb.}}$$ and reduce the result to the decimal of $1\cdot 1$.

UNITARY METHOD

Examples (4)

1. If 15 men can do a piece of work in 10 days, in how many days can 6 men do twice that work ?

15 men can do the work in 10 days,

$\therefore$ 1 man ... ... 10×15...,

$\therefore$ 6 men ... ... $\dfrac{10\times 15}{6}$...,

$\therefore$ 6 ... ... twice the work in $\dfrac{10\times 15}{6}\times 2$ or 50 days.

2. A loaf worth 4 P. weighs 5 grams, when wheat is Rs. 12 a quintal. How much will the loaf worth 5 P. weigh when wheat is Rs. 25 a quintal ?

When wheat is Rs. 12 a quintal, the 4 P. loaf weighs 5 grams,

$\therefore$ ... Re. 1 ... ... 5×12...,

$\therefore$ ... Re. 1 ... 1 P. ... ... $\dfrac{5\times 12}{4}$...,

$\therefore$ ... Rs. 25 ... 1 P. ... ... $\dfrac{5\times 12}{4\times 25}$...,

$\therefore$ ... Rs. 25 ... 5 P. ... ... $\dfrac{5\times 12\times 5}{4\times 25}$

or 3 grams.

[N.B. The weight of the loaf will be decreased 2, 3,...times, as the price of wheat will be increased 2, 3,...times and vice versa.]

3. The net income of a man is Rs. 630 after paying an income-tax at 3 pies in the rupee. Find his gross income.

The man has to pay an income-tax at 3 p. in the rupee or in 192 p., $\therefore$ his net income is (192 − 3) or 189 p. or Re. $\frac{189}{192}$ or Rs. $\frac{63}{64}$ if his gross income be Re. 1.

If the net income be Rs. $\frac{63}{64}$, the gross income = Re. 1,

$\therefore$ ... ... Re. 1 ... ... = Rs. $\frac{1 \times 64}{63}$,

... ... Rs. 630 ... ... = Rs. $\frac{1 \times 64}{63} \times 630$,

= Rs. 640.

4. A contractor engaged to finish 6 kilometres of railway in 200 days, but after employing 140 men for 60 days, he found that only one and a half kilometres were completed. How many additional men must be then engaged that the work may be finished within the given time? [C. U. 1910]

200 days − 60 days = 140 days.

6 km. − $1\frac{1}{2}$ km. = $4\frac{1}{2}$ km.

In 60 days $1\frac{1}{2}$ km. of the railway is constructed by 140 men,

$\therefore$ ... 1 ... ... $140 \times \frac{2}{3}$...,

$\therefore$ 1 ... 1 ... ... $140 \times \frac{2}{3} \times 60$...,

$\therefore$ 1 ... $4\frac{1}{2}$ ... ... $\frac{140 \times 2 \times 60 \times 9}{3 \times 2}$...,

... 140... $4\frac{1}{2}$ ... ... $\frac{140 \times 2 \times 60 \times 9}{3 \times 2 \times 140}$ men

or 180 men.

$\therefore$ (180 − 140) or 40 additional men should be engaged.

5. If 50 men can do a piece of work in 12 days working 8 hrs. a day, how many hours a day would 60 men have to work in order to do another piece of work twice as great in 16 days? [D. B. 1930]

To work 8 hrs. a day for 12 days is to work for a total period of 12 × 8 or 96 hrs.

$\therefore$ 50 men can do the work in 96 hrs.,

$\therefore$ 1 man ... ... 96×50...,

$\therefore$ 60 men ... ... $\dfrac{96 \times 50}{60}$ or 80 hrs.

$\therefore$ 60 ... ... twice the work in 80×2 or 160 hours.

$\therefore$ to complete the work in 16 days, they must work $\frac{160}{16}$ or 10 hours a day.

6. A garrison of 420 men have food enough to last them 35 days. After '5 days they are re-inforced by 210 men bringing no food with them. How much longer will the food last? [C. U. '18]

After 5 days the food left will last 420 men $(35-5)$ or 30 days, but then the number of men becomes $(420+210)$ or 630.

The food left will last 420 men 30 days more,

$\therefore$ ... ... 1 man 30×420....,

$\therefore$ ... ... 630 men $\dfrac{30 \times 420}{630}$ or 20 days more.

7. If 5 men and 9 boys could do a piece of work in 17 days, in how many days could 9 men and 12 boys do it, the work of 2 men being equal to that of 3 boys? [C. U. '46]

Here 3 boys can do as much work as 2 men,

$\therefore$ 1 boy ... ... $\frac{2}{3}$ man,

$\therefore$ 9 boys ... ... $\frac{2}{3} \times 9$ or 6 men.

Again, 1 boy ... ... $\frac{2}{3}$ man

$\therefore$ 12 boys ... ... $\frac{2}{3} \times 12$ or 8 men.

$\therefore$ 5 men and 9 boys ... $(5+6)$ or 11 men

and 9 men and 12 boys ... $(9+8)$ or 17 men.

Now 11 men can do the work in 17 days,

$\therefore$ 1 man ... ... 17×11...,

$\therefore$ 17 men ... ... $\dfrac{17 \times 11}{17}$ or 11 days.

$\therefore$ the required time $=11$ days.

8. If 8 men or 17 boys can do a piece of work in 26 days, how many days will it take 4 men and 24 boys to do a piece of work $50 \times 0.0\dot{9}$ times as great ? [C. U. '37]

Here 8 men can do as much work as 17 boys.

$\therefore$ 4 men ... ... $\frac{17}{2}$...,

$\therefore$ 4 men+24 boys ... ... $(\frac{17}{2}+24)$ or $\frac{65}{2}$ boys.

17 boys can do the work in 26 days,

$\therefore$ 1 boy ... ... 26×17...,

$\therefore$ $\frac{65}{2}$ boys ... ... $\frac{26 \times 17 \times 2}{65}$ or $\frac{68}{5}$ days.

$\therefore$ To complete the work $50 \times 0.0\dot{9}$ or $\frac{50 \times 9}{90}$ or 5 times as great they will take $\frac{68}{5} \times 5$ or 68 days.

9. If 5 guns, firing 3 rounds in 5 minutes, kill 1200 men in 4 hrs., how many men will be killed in 3 hrs. by 7 guns firing 4 rounds in six minutes ?

[N. B. Quicker the firing, the greater will be the number killed.]

5 guns firing 3 rounds in 5 mins. kill in 4 hrs. 1200 men

$\therefore$ 1 gun ... ... $\frac{1200}{5}$...

$\therefore$ 1 gun ... ... 1 min. ... $\frac{1200}{\cancel{5}} \times \cancel{5}$...

$\therefore$ 1 gun ... 1 round in 1 min. ... $\frac{1200}{3}$ or 400...

$\therefore$ 1 gun ... 1 round in 1 min. ... 1 hr. $\frac{400}{4}$ or 100...

$\therefore$ 7 guns firing 1 round in 1 min. kill in 1 hr. 100×7 men,

$\therefore$ 7 guns ... ... 6 min. ... ... $\frac{100 \times 7}{6}$...,

$\therefore$ 7 guns ... 4 rounds in 6 min. ... ... $\frac{100 \times 7 \times 4}{6}$...,

$\therefore$ 7 guns firing 4 rounds in 6 min. kill in 3 hrs. $\frac{100 \times 7 \times 4 \times 3}{6}$

or 1400 men.

10. If 6 horses cost as much as 24 cows, 10 cows as much as 8 buffaloes, 4 buffaloes as much as 15 asses, 8 asses as much as 32 sheep, and if the price of 9 sheep be Rs. 25, find the cost of a horse.

The cost of 6 horses = the cost of 24 cows,

$\therefore$ the cost of 1 horse = the cost of $\frac{24}{6}$ cows.

The cost of 10 cows = the cost of 8 buffaloes,

$\therefore$ the cost of 1 cow = the cost of $\frac{8}{10}$ buffaloes.

The cost of 4 buffaloes = the cost of 15 asses,

$\therefore$ the cost of 1 buffalo = the cost of $\frac{15}{4}$ asses.

The cost of 8 asses = the cost of 32 sheep,

$\therefore$ the cost of 1 ass = the cost of $\frac{32}{8}$ sheep.

The cost of 9 sheep = Rs. 25,

$\therefore$ the cost of 1 sheep = Rs. $\frac{25}{9}$.

$\therefore$ the cost of 1 horse $=(\frac{24}{6}\times\frac{8}{10}\times\frac{15}{4}\times\frac{32}{8}\times\frac{25}{9})$ rupees.

$=$ Rs. $\frac{400}{3}$ = Rs. $133\frac{1}{3}$.

11. If 12 chairs and 4 tables cost Rs. 520, find the cost of 10 chairs and 7 tables, the cost of 8 chairs being equal to that of 6 tables.

The cost of 8 chairs = the cost of 6 tables,

$\therefore$	...	1 chair =	...	$\frac{6}{8}$	or	$\frac{3}{4}$ table,
$\therefore$	...	12 chairs =	...	$\frac{3}{4}\times12$	or	9 tables,
and	...	10 chairs =	...	$\frac{3}{4}\times10$	or	$\frac{15}{2}$ tables,

$\therefore$ 12 chairs + 4 tables = (9 + 4) or 13 tables,

and 10 chairs + 7 tables $=(\frac{15}{2}+7)$ or $\frac{29}{2}$ tables,

Now, the cost of 13 tables = Rs. 520.

$\therefore$ ... $\frac{29}{2}$ tables = Rs. $(\frac{520}{13}\times\frac{29}{2})$ or Rs. 580.

12. A contractor undertook to do a piece of work in 17 days and engaged 25 men for the purpose. After 6 days he found it necessary to put 10 more men on the work and then he had it done one day too soon. How many days behind hand would he have been without the additional men ?

The work has been finished in (17 − 1) or 16 days.

The work left after 6 days has been done by (25 + 10) or 35 men in (16 − 6) or 10 days.

35 men can do the work left in 10 days,

$\therefore$ 1 man ... ... 10×35 ...,

$\therefore$ 25 men ... ... $\frac{10 \times 35}{25}$ or 14 days,

$\therefore$ the whole work was completed in $(14+6)$ or 20 days in all.

$\therefore$ He would have been $(20-17)$ or 3 days behind hand without the additional men.

Exercise 4

1. If 24 men can do a piece of work in 15 days, working $8\frac{1}{2}$ hrs. a day, how many men will be required to do another piece of work twice as great in 17 days, working 6 hrs. a day? [C. U. '16]

2. A man undertakes to do a piece of work in 10 days, and employs upon it 12 men. At the end of 6 days, the work is only half done; find the additional number of men he must employ in order to do the work in time. [C. U. '51 ; D. B. '25]

3. A contractor undertakes to execute a certain work in a given time; he employs 55 men, who work 9 hrs. daily; when $\frac{3}{4}$ of the time has expired, he finds that only $\frac{3}{7}$ of the work is done; how many men must he now employ 11 hrs. a day to fulfil his contract? [C. U. '11 ; B. C. S. '34]

4. A garrison of 2200 men has provisions for 50 days. At the end of 17 days reinforcement arrives and the provisions last for 20 days more. What is the reinforcement? [D. B. '40]

5. If the wages of 45 women amount to 207 dollars in 48 days, how many men must work 16 days to earn $76\frac{2}{3}$ dollars, the daily wages of a man being double those of a woman?

6. 8 men or 12 women can do a piece of work in 10 days; how long will 4 men and 16 women take to finish it? [D. B. '24]

7. If a garrison of 750 men have provisions for 20 weeks, how long will the provisions last if at the end of 4 weeks they are reinforced by 450 men? [C. U. '27]

8. A besieged garrison of 4000 men had provisions left for 56 days. After a week 500 men broke through the enemy's line. For how long will the provisions last the remainder of the garrison? [D. B. '39]

9. A contractor undertakes to execute a certain piece of work in a given time and engages 60 men who work 8 hours daily. When three-fourths of the time has expired it is found that only two-fifths of the work has been done. How many men, all working 10 hours a day, must he engage now so as to finish the work in the stipulated time? [W. B. S. F. '53]

10. If a battery of 6 guns, firing 3 rounds in 10 minutes, will breach a certain work in 60 hours; how many guns must be employed for the same purpose firing 2 rounds in 5 minutes in order that the breach may be made in 15 hours? [D. B. '41]

11. A contractor undertook to build a house in 21 days and engaged 15 men for the purpose. After 15 days he found it necessary to put 9 more men on the work and then he had it finished one day too soon. How many days behind hand would he have been without the additional men? [B. C. S. '38]

12. A contractor undertakes to do a piece of work in 38 days. By employing 60 men on it, he does $\frac{3}{7}$ of it in 22 days. How many additional men must he employ to finish it in time? [D. B. '37]

13. A man can walk 600 miles in 35 days, resting 9 hrs. a day; how long will he take to walk 375 miles, if he rests 10 hrs. a day and walks $1\frac{1}{2}$ times as fast as before? [C. U. 1888]

14. If the loaf worth 4 P. weighs 50 grams when wheat is 60 P. per kilogram, what ought the loaf worth 5 P. to weigh when wheat is at Re. 1. 25 P. per kilogram?

15. If 8 men or 15 women can earn Rs. 120 in 30 days, how much can 21 men and 24 women earn in 45 days? [C. U. 1907]

16. If 12 men can do a piece of work in 30 days of 9 hours each, how many men will take to do ten times the work if they work 24 days of 5 hours each? [C. U. 1948]

17. If 9 lbs. of rice cost as much as 4 lbs. of sugar and 14 lbs. of sugar are worth as much as $1\frac{1}{2}$ lbs. of tea and 2 lbs. of tea are worth 5 lbs. of coffee, find the cost of 11 lbs. of coffee if $2\frac{1}{2}$ lbs. of rice cost $6\frac{1}{4}$ d. [B. U.]

18. 100 labourers can dig a trench in 150 days. If 20 of them leave after working 50 days, in how many days more will the remaining men complete the work ? [G. U. '55]

19. If 15 chairs and 2 tables cost Rs. 330, find the cost of 7 chairs and 4 tables, the cost of 10 chairs being equal to that of 6 tables.

AVERAGE

The average or mean value of any number of quantities of the same kind is their sum divided by their number. Suppose the ages of Ram, Hari and Jadu to be 35 yrs., 15 yrs. and 13 years respectively. What is their average age ? Here 35 yrs., 15 yrs. and 13 yrs. are 3 quantities of the same kind. The sum of those quantities is 63 years. $\therefore$ their average age is (63 yrs.$\div$3) or 21 years. As there are three men, the sum of the ages is divided by 3. Note that the average age multiplied by the number of men gives the sum of their ages.

Examples [5]

1. A man bought 10 litres of milk at 16 P. per litre and mixed the same with 6 litres of water ; what was the average price of the mixture per litre ?

The cost price of 10 litres of milk$=$16 P.$\times$10$=$160 P. 6 litres of water being mixed with it, the quantity of mixture is 16 litres. $\therefore$ The price of 16 litres of the mixture$=$160 P. ($\because$ water has no value),

$\therefore$ the average price of the mixture per litre$=$160 P.$\div$16$=$10 P.

2. The average income of a man for the first four days is Rs. 45 and that of the next two days is Rs. 36. If his average income of the first 7 days be Rs. 40, find his income of the 7th day.

The total income of the first four days$=$Rs. 45$\times$4$=$Rs. 180.

" " " the next two days$=$Rs. 36$\times$2$=$Rs. 72.

$\therefore$ " " " the first 6 days$=$Rs. 180$+$Rs. 72$=$Rs.252

But " " " 7 days$=$Rs. 40$\times$7$=$Rs. 280.

$\therefore$ The income of the 7th day$=$Rs. 280 $-$ Rs. 252$=$Rs. 28.

3. The average age of 5 boys is 9 yrs. and when their father is included, their average age is 16 yrs. What is their father's age ? [M. E. 1933]

The sum of the ages of 5 boys = 9 yrs. × 5 = 45 yrs.

The sum of the ages of 5 boys and their father (*i.e*, 6 men)

= 16 yrs. × 6 = 96 yrs.

∴ the age of the father = 96 yrs. − 45 yrs. = 51 yrs.

4. What is the average of the even numbers from 1 to 20 ?

2+4+6+8+10+12+14+16+18+20 = 110.

There are 10 numbers here. ∴ their average = 110 ÷ 10 = 11.

5. The average weight of A, B and C is 45 kg., that of A and B is 40 kg., and of B and C 43 kg. ; what is the weight of B ?

The total weight of A, B and C = 45 kg. × 3 = 135 kg.

... ... A and B = 40 kg. × 2 = 80 kg.

∴ the weight of C = 135 kg. − 80 kg. = 55 kg.

Again, the weight of B and C = 43 kg. × 2 = 86 kg.

∴ the weight of B = 86 kg. − 55 kg. = 31 kg.

6. The average attendance of the boys of a school for the 6 days of the week is 270. If the average attendance of the first 4 and the last 3 days be 260 and 285 respectively, find the number of boys present on Thursday.

The first 4 days are the days from Monday to Thursday and the last 3 days are the days from Thursday to Saturday.

Thus Thursday is taken twice here.

The total attendance of the boys for the first 4 days

= 260 × 4 = 1040

... ... for the last 3 days = 285 × 3 = 855

∴ The total attendance of the first 4 and the last 3 days

= 1040 + 855 = 1895.

∴ 1895 is the total attendance of 7 days, Thursday being taken twice. From 1895 subtract the total attendance of 6 days i.e., 270 × 6 or 1620 and you will get the attendance on Thursday.

∴ the number of boys present on Thursday.

= 1895 − 1620 = 275.

7. The average age of four children and their mother is 2 years less than that of the children and their father. If the father is 58 years old, find the mother's age.

Here the number of persons in both cases is 5 and the first average is 2 years less than that of the second,

$\therefore$ the total age in the first case is 2 yrs. $\times$ 5 or 10 years less than that in the second case.

$\therefore$ the age of the mother = 58 yrs. − 10 yrs. = 48 years.

8. There are 24 students in a class and one of them who is 17 years old leaves the class and his place is filled up by a new-comer; if the average is thereby lowered by one month, find the age of the new-comer. [C. U. 1943]

Here the number of men in both cases is 24.

$\therefore$ the average age being lowered by one month, the total age is lowered by 24 months or 2 yrs.

$\therefore$ the age of the new-comer = 17 yrs. − 2 yrs. = 15 years.

9. Out of 11 cows one worth Rs. 25 was removed and a horse was taken in. If their average price is thereby increased by Rs. 5, find the price of the horse.

The average price of 11 animals being increased by Rs. 5, the total price is increased by Rs. 5 $\times$ 11 or Rs. 55.

$\therefore$ the price of the horse = Rs. 25 + Rs. 55 = Rs. 80.

10. Of three numbers the first is twice the second and the second is 3 times the third. If their average is 100, what are the numbers ?

The second number = 3 $\times$ the third number and the first number = 2 $\times$ the second number = 6 $\times$ the third number.

$\therefore$ the first + the second + the third number
= 10 $\times$ the third number.

Again, the sum of the three numbers = 100 $\times$ 3 = 300.

$\therefore$ 10 times the third number = 300,

$\therefore$ the third number = 30, then the second number = 30 $\times$ 3 = 90, and the first number = 90 $\times$ 2 = 180.

11. A cricketer scored 85 runs in the 17th innings and thereby increased the average run of the previous 16 innings by 3. What was the average run in the 17 innings? [U. P. '30]

The average run being increased by 3, the cricketer had to make 17×3 or 51 runs more in the 17th innings than the average run of the first 16 innings.

$\therefore$ the average run of the first 16 innings $= (85 - 51) = 34$.

$\therefore$ the average run in the 17 innings $= 34 + 3 = 37$.

12. A man walked to a station at 6 miles an hour and came back at the rate of 2 miles per hour. What was the average rate for the journey?

The man takes $\frac{1}{6}$ hr. to go a distance of 1 mile. He takes $\frac{1}{2}$ hr. to return the same distance.

$\therefore$ he takes $(\frac{1}{6} + \frac{1}{2})$ or $\frac{2}{3}$ hr. for every two miles.

$\therefore$ he takes $\frac{2}{3 \times 2}$ or $\frac{1}{3}$ hr. for 1 mile on the average.

$\therefore$ the required average rate for the journey is 3 miles per hr.

13. The average monthly salary of each of A and B is Rs. 64, that of B and C Rs. 50 each and of C and A Rs. 70 each. Find the actual salary of each. [C. U. 1945]

The total monthly salary of A and B = Rs. 64×2 = Rs. 128

... ... B and C = Rs. 50×2 = Rs. 100

... ... C and A = Rs. 70×2 = Rs. 140

$\therefore$ Twice the total monthly salary of A, B and C = Rs. 368

$\therefore$ the total monthly salary of A, B and C = Rs. $368 \div 2$

= Rs. 184

$\therefore$ monthly salary of A = Rs. 184 − Rs. 100 = Rs. 84

... ... B = Rs. 184 − Rs. 140 = Rs. 44

... ... C = Rs. 184 − Rs. 128 = Rs. 56.

14. A train travels from Howrah to Burdwan at the rate of 55 kilometres per hour and returns at the rate of 66 kilometres per hour. Find the average rate of the whole journey.

The train takes $\frac{1}{55}$ hr. to go 1 km. and it takes $\frac{1}{66}$ hr. to return 1 km.

$\therefore$ it takes $(\frac{1}{55} + \frac{1}{66})$ or $\frac{1}{30}$ hr. for a total journey of 2 km.

$\therefore$ it takes $(\frac{1}{30} \times \frac{1}{2})$ or $\frac{1}{60}$ hr. on the average for a journey of 1 km.

$\therefore$ the average rate of the whole journey is 60 km. per hour.

Alternative method : The L. C. M. of 55 and 66=330. Suppose the distance between the two places to be 330 km.

It takes $(330 \div 55)$ or 6 hrs. to travel a distance of 330 km. and $(330 \div 66)$ or 5 hrs. to return.

$\therefore$ it takes $(6+5)$ or 11 hrs. altogether for a journey of $(330 \text{ km.} \times 2)$ or 660 km.

$\therefore$ the average rate of the whole journey is $(660 \text{ km.} \div 11)$ or 60 km. per hour.

[N. B. Here it will be a mistake to find the average by dividing the sum of 55 and 66 by 2.]

15. The average rainfall at Gauhati from 1st to 7th July was 5·25 inches and that from 1st to 9th July was 5·45 inches. If the rainfall on the 9th was 6·2 inches, find that on the 8th.

[G. U. '55]

The total rainfall of 9 days from 1st to 9th July

$=5{\cdot}45 \text{ inches} \times 9 = 49{\cdot}05$ inches.

The total rainfall of 7 days from 1st to 7th July

$=5{\cdot}25 \text{ inches} \times 7 = 36{\cdot}75$ inches.

$\therefore$ the total rainfall of 8th and 9th July

$=(49{\cdot}05 - 36{\cdot}75)$ inches.

$=12{\cdot}3$ inches.

But the rainfall on the 9th July $=6{\cdot}2$ inches (given)

$\therefore$ the required rainfall on the 8th July

$=(12{\cdot}3 - 6{\cdot}2)$ inches $=6{\cdot}1$ inches.

16. A and B have each taken 8 wickets for 28. A then takes 1 for 35 and B, 4 for 56. Who has now the better average ?

[B. C. S. '47]

A has taken $(8+1)$ or 9 wickets for the total $(28+35)$

or 63 runs.

$\therefore$ the average of A's wicket-taking

$=63 \div 9 = 7$ (i.e., 1 wicket for 7 runs.)

Again, B has taken $(8+4)$ or 12 wickets for the total $(28+56)$ or 84 runs.

$\therefore$ the average of B's wicket-taking $=84 \div 12 = 7$ (i.e., 1 wicket for 7 runs.)

$\therefore$ A and B have the same average.

17. The average of 8 results is 30 ; that of the first two is $21\frac{1}{2}$, and of the next three is $24\frac{2}{3}$; the sixth is less than the seventh by 3 and less by 9 than the eighth. Find the last result.

The total of 8 results $=30\times8=240$

The sum of the first two $=21\frac{1}{2}\times2=43$

The sum of the next three $=24\frac{2}{3}\times3=74$,

$\therefore$ the sum of the first five $=43+74=117$,

$\therefore$ the sum of the last three i.e., of the 6th, 7th and 8th

$=240-117=123$.

Let the 6th result be x.

$\therefore$ $x+(x+3)+(x+9)=123$,

or, $3x=123-12=111$, $\therefore$ $x=37$.

$\therefore$ The 8th result $=37+9=46$.

18. Each of 53 persons contributed to a fund as many annas as he had rupees with him. The total contribution was Rs. 327. 8 as., If 15 of them had Rs. 12 each, and 18 of them Rs. 10 each, how much had each of the remaining 20 persons ?

[D. B. '48]

15 persons had Rs. 12 each, $\therefore$ they have contributed 12 as. each

$\therefore$ the total subscription of 15 persons

$=12$ as. $\times15=$ Rs. 11. 4 as.

In the same way, the total subscription of 18 persons, each contributing 10 as. $=10$ as. $\times18=$ Rs. 11. 4 as.

$\therefore$ the total subscription of $(15+18)$ or 33 persons

$=$ Rs. 11. 4 as. $+$ Rs. 11. 4 as. $=$ Rs. 22. 8 as.

$\therefore$ the total subscription of the remaining $(53-33)$ or 20 persons $=$(Rs. 327. 8 as. $-$ Rs. 22. 8 as.)$=$ Rs. 305 $=4880$ as.

$\therefore$ each person has contributed 4880 as. $\div20$ or 244 ans.

$\therefore$ each of the remaining 20 persons had Rs. 244.

Exercise 5

1. In a class there are 40 boys and their average age is 16 years. One boy aged 17 years leaves the school and another joins and the average age becomes 15·875. Find the age of the new boy. [C. U. 1949]

2. B was born when A was 9 yrs. old and C was born when B was 3 yrs. old. What will be their average age when C will be 4 yrs. old ?

3. The average age of a class of 30 boys is 10·5 years and if their teacher be included the average is 11 years. What is the teacher's age ?

4. The average of ten numbers is 1·015102, that of the first six is 1·01267 and of the last five is 1·01688. What is the sixth number ? [U. P. 1927]

5. A train travels from Calcutta to Madhupur at the rate of 45 miles per hour and returns from Madhupur to Calcutta at the rate of 36 miles per hour. Find the average rate for the whole journey. [C. U. '42]

6. The average salary per head of all the workers of an institution is Rs. 60. The average salary per head of 12 ocfliers is Rs. 400. The average salary per head of the rest is Rs. 56. Find the total number of workers in the institution. [I. P. S. '40]

7. The average weight of 7 men is decreased by 3 kg. when one of them who weighs 140 kg. is replaced by a new man ; find the weight of the new man.

8. A train travels from Dacca to Mymensing at the rate of 30 km. per hour and returns at the rate of 40 km. an hour. Find the average rate for the whole journey.

9. In a school the average age of 41 boys in the first class is 6·25 yrs., that of the 36 boys in the second class is 7·31 yrs. and of the 25 boys in the third class is 8·93 yrs. Find the average age of the boys in the school. [U. U. '51]

10. The average of 8 results is 20 ; that of the first two is $15\frac{1}{2}$, and of the next three is $21\frac{1}{3}$; the sixth is less than the seventh by 4 and less by 7 than the eighth. Find the last. [I. P. S. '33]

SQUARE ROOT

Examples (6)

Ex. 1. Find the square root of :—

```
   | 19'36' | 44
   | 16     |
84 |  336   |
   |  336   |
```

[**N.B.** First mark the digit 6 in the units' place and then mark off the digits to the left into periods of 2.]

∴ The square root=44.

Thus the first period (*i.e.*, the portion from the left to the first marked digit) is 19. Now, as $4^2=16$ and $5^2>19$, so 4 is the nearest square root of this first period. Put 4 as the first figure of the square root. Then subtract the square of 4 *i.e.*, 16 from the first period 19 and bring down the second period (36) after the remainder 3. Thus we have 336 as the new dividend. Double the figure 4 and put 8 in the place of divisor. Next we divide 33 of the number 336 (omitting the last figure) by 8. The quotient is 4 which is suggested as the second figure of the square root. Place it to the right of the trial divisor 8 (which is twice the first part of the root). Then multiply the divisor as it now stands (84) by the second figure of the root *i.e.*, (4). Now subtract this product 336 from 336 and there being no remainder, we conclude that 44 is the square root of 1936.]

Ex. 2.

```
   | 3610000 | 1900
   | 1       |
29 | 261     |
   | 261     |
```

∴ the square root=1900.

[Here the square root of the number obtained by bringing down figures up to the second period 61 is found to be 19 and there is no remainder. There are 4 more zeroes forming two periods in the number. ∴ the required square root has two 0's after 19, *i.e.*, it will be 1900.]

Ex. 3.

```
    | 91204 | 302
    | 9     |
602 |  1204 |
    |  1204 |
```

∴ the square root=302.

[Here the quare root of the first period is 3. On subtracting its square from 9 we get no remainder. The second period 12 is brought down and twice 3 or 6 is taken as a trial divisor. Now omitting the last figure of 12 we get 1, which

cannot be divided by 6. So place 0 in the square root making it 30. Now, doubling 30, the trial divisor is 60. Then bringing down the third period 04 we have 1204. Omitting the last two figures of 1204 we get 12, which being divided by 6 gives 2. Now put 2 as the third or last figure of the square root as well as to the right of the trial divisor, and then multiply 602 by 2. There being no remainder now, the sq. root is 302.]

Ex. 4.

$$\begin{array}{r|l|l} & 15'\cdot21' & 3\cdot9 \\ & 9 & \\ \hline 69 & 621 & \\ & 621 & \\ \hline \end{array}$$

$\therefore$ the square toot $=3\cdot9$.

[Here mark off the pairs of digits on either side, right and left, of the decimal point beginning with 5 in the units' place. 1 in the hundredth place is marked next here. In the root the decimal point must be placed immediately after dealing with the integral part (here 15) of the given number.]

Ex. 5.

$$\begin{array}{r|r|l} & \cdot00'02'56' & \cdot016 \\ & 1 & \\ \hline 26 & 156 & \\ & 156 & \\ \hline \end{array}$$

$\therefore$ the square root $=\cdot016$.

[There is no digit in the units' place here. So the operation is done as before to the right of the decimal point. The square root begins with the decimal point as there is no integral part in the given number. For 00 (two zeroes) in the first period take one 0 in the square root. Then the operation begins with the second period '02'.]

Ex. 6. $\frac{16}{169}$. The square root $=\sqrt{\frac{16}{169}}=\frac{\sqrt{16}}{\sqrt{169}}=\frac{4}{13}$. Ans.

[The radical sign $\sqrt{}$ indicates the square root of a number. The square root of a vulgar fraction is the square root of its numerator divided by the square root of its denominator.]

Ex. 7. Find the square root of $9+\cfrac{1}{1+\cfrac{1}{7+\frac{1}{6}}}$. [C.U. 1828]

[The fraction simplified is $\frac{484}{49}$. Its square root $=\frac{\sqrt{484}}{\sqrt{49}}=\frac{22}{7}=3\frac{1}{7}$]

Ex. 8. Of what number is the square root $\cdot03$?

The required number $=\cdot03\times\cdot03=\cdot0009$.

Ex. 9. Find the square root of $\frac{3}{4}$ to 2 places of decimals.

$$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{\sqrt{4}}=\frac{1{\cdot}73}{2}={\cdot}86.$$

$$\begin{array}{r|l|l}
 & 3 & 1{\cdot}73 \\
 & 1 & \\ \hline
27 & 200 & \\
 & 189 & \\ \hline
343 & \;\;1100 & \\
 & \;\;1029 & \\ \hline
 & \;\;\;\;\;71 &
\end{array}$$

[Here the square root of the denominator is 2. But the numerator 3 is not a perfect square, ∴ its square root is obtained only up to 2 places of decimals.]

Ex. 10. Find the square root of $\frac{5}{7}$ to 2 places of decimals.

$$\sqrt{\frac{5}{7}}=\sqrt{\frac{5\times 7}{7\times 7}}=\sqrt{\frac{35}{49}}=\frac{\sqrt{35}}{\sqrt{49}}=\frac{5{\cdot}91}{7}={\cdot}84.$$

$$\begin{array}{r|l|l}
 & 35' & 5{\cdot}91 \\
 & 25 & \\ \hline
109 & 1000 & \\
 & \;\;981 & \\ \hline
1181 & \;\;\;1900 & \\
 & \;\;\;1181 & \\ \hline
 & \;\;\;\;\;719 &
\end{array}$$

[If the denominator be not a perfect square it is advantageous to make it so by multiplying it by some least possible number. Here 7 multiplied by 7 will make a perfect square. ∴ both the numerator and the denominator are multiplied by 7. This will not change the value of the given fraction. Hence we get the fraction $\frac{35}{49}$, the square root of whose denominator is 7 and that of the numerator to 2 places of decimals is 5·91. ∴ the required square root $=\frac{5{\cdot}91}{7}={\cdot}84.$]

Ex. 11. Find the square root of ·2$\dot{3}$ to 2 places of decimals.

${\cdot}2\dot{3}={\cdot}2333\ldots$

$$\begin{array}{r|l|l}
 & {\cdot}23'33'\ldots & {\cdot}48 \\
 & 16 & \\ \hline
88 & \;733 & \\
 & \;704 & \\ \hline
 & \;\;\;29 &
\end{array}$$

∴ the square root $={\cdot}48$.

Ex. 12. Find the missing digit in 2220* so that the number may be a perfect square.

$$
\begin{array}{r|l|l}
 & 2'22'0*' & 149 \\
 & 1 & \\
\hline
24 & 122 & \\
 & 96 & \\
\hline
289 & 260* & \\
 & 2601 & \\
\hline
\end{array}
$$

$\therefore$ ther equired missing digit $=1$.

[Here 0* is placed against the second remainder 26 and twice 14, i.e., 28 is placed as a trial divisor. Now it is evident that if 9 be the last figure in the square root and the trial divisor and the divisor so formed is multiplied by 9, the product is 2601, whose first three digits are the same as the first three digits of 260*. $\therefore$ 1 should be placed in the place of *.]

Ex. 13. The product of two numbers is 1296 and one of them is 16 times the other. Find the numbers.

Let one number be x. Then the other number $=16x$.

$\therefore$ $x\times16x=1296$, or $16x^2=1296$, or, $x^2=81$. $\therefore$ $x=\sqrt{81}=9$.

$\therefore$ one number $=9$, and the other number $=9\times16=144$.

Ex. 14. What least number must be subtracted from 12773 to make the remainder a perfect square?

[In finding the square root of 12773 we have 4 as the last remainder. $\therefore$ 4 should be subtracted from the number to make the remainder a perfect square.]

Ex. 15. What least number must be added to 6700 to make the sum a perfect square?

$$
\begin{array}{r|l|l}
 & 67'00' & 81 \\
 & 64 & \\
\hline
161 & 300 & \\
 & 161 & \\
\hline
 & 139 & \\
\end{array}
$$

$81+1=82$

The given number is greater than 81^2. $\therefore$ when the required least number is added to the given number the sum must be 82^2.

$$82^2=6724,$$

$\therefore$ the required number $=6724-6700=24$.

Ex. 16. By what least number must 192 be multiplied so that the product may be a perfect square ?

$$\begin{array}{r|r} 2 & 192 \\ \hline 2 & 96 \\ \hline 2 & 48 \\ \hline 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline & 3 \end{array}$$

$192 = 2^2 \times 2^2 \times 2^2 \times 3$. Of these factors only 3 is not a perfect square. $\therefore$ 192 must be multiplied by 3 to make the product a square number.

Ex. 17. By what least number must 1260 be divided to make the quotient a square number ?

$1260 = 2^2 \times 3^2 \times 5 \times 7$; here of the factors 5 and 7 are not square numbers. $\therefore$ if the number be divided by 5×7 or 35, the quotient is $2^2 \times 3^2$, which is a square number.

Ex. 18. A general wishing to arrange his men, who were 335250 in number, into a solid square found that there were 9 men over. How many were there in the front ? [C. U. 1911]

When the men were arranged into a solid square, there were 9 men over. $\therefore$ the solid square was formed with the number of men obtained by subtracting 9 from the given number.

$\therefore$ the number of men in the front $= \sqrt{335241} = 579$.

[Here the square root should be actually worked out.]

Ex. 19. A teacher in arranging 722 students into a solid square found that 7 boys were wanting. Find the number of boys in each row.

$722 + 7 = 729$. $\therefore$ the number of boys in each row $= \sqrt{729} = 27$.

Ex. 20. A subscription of Rs. 104. 4 P. was raised for a fund, each man subscribing as many paise as there were subscribers. Find the number of subscribers

$$\begin{array}{r} \text{Rs. } 104.\ 4 \text{ P.} \\ \times 100 \\ \hline 10400 \\ 4 \\ \hline 10404 \text{ P.} \end{array}
\qquad
\begin{array}{r|l|l} & 1'04'04' & 102 \\ & 1 & \\ \cline{2-2} 202 & 404 & \\ & 404 & \\ \cline{2-2} \end{array}$$

$\therefore$ the number of subscribers $= 102$.

Ex. 21. A number of boys raised Re. 1. 44 P. by subscription among themselves. Each boy contributed the same number of paise as the number of boys. Find each boy's contribution.

Re. 1.44P. $= 144$ P., and $\sqrt{144} = 12$ $\therefore$ Each boy contributed 12P.

Ex. 22. A certain number of boys spent Rs. 3660. 25 P. each spending as many 25 Paisa-pieces as there were boys. Find the number of boys. [C. U, '40]

$$
\begin{array}{r|l|l}
 & 1'46'41' & 121 \\
 & 1 & \\
\hline
22 & \ \ 46 & \\
 & \ \ 44 & \\
\hline
241 & \ \ 241 & \\
 & \ \ 241 & \\
\hline
\end{array}
$$

Rs. 3660. 25 P.
$=(3660\times4+1)$ pieces of 25 P.
$=14641$ pieces of 25 P.

$\therefore$ the required number of boys $=\sqrt{14641}=121$.

Ex. 23. A certain number of boys spent Rs. 81, each spending twice as many two-anna pieces as there were boys altogether. How many boys were there ? [C. U. 1926]

Rs. $81=81\times8$ or 648 two-anna pieces.

$648\div2=324$. $\therefore$ the required number of boys $=\sqrt{324}=18$.

[Each boy spends twice as many two-anna pieces as there are boys, $\therefore$ Rs. 81 is to be reduced to 648 two-anna pieces by multiplying 81 by 8 and then 648 is to be divided by 2. Thus in case of spending thrice as many two-anna pieces, 648 is to be divided by 3.]

Ex. 24. A regiment of soldiers can be arranged into 10, 15 or 25 equal rows and also into a solid square. Find the least number of soldiers in the regiment. [P. U. '35]

The L. C. M. of 10, 15 and $25=150$, which is not a perfect square. $\therefore$ 150 men cannot be arranged into a solid square. Now we have to find the least multiple of 150 that is a perfect square. $150=5^2\times2\times3$

$\therefore$ the required number of soldiers $=5^2\times2^2\times3^2=900$.

Ex. 25. Of three numbers the product of the first two is 28, that of the second and third is 77 and that of the third and first is 44. Find them.

Let the three numbers be x, y and z respectively.

$$
\left.\begin{array}{ll}
\therefore\ xy=28 & \ldots(1) \\
\quad\ \ yz=77 & \ldots(2) \\
\text{and } zx=44 & \ldots(3)
\end{array}\right\}
$$

Now multiplying (1), (2) and (3) together we get

$x^2y^2z^2=28\times77\times44=4\times7\times7\times11\times4\times11=4^2\times7^2\times11^2$.

$\therefore\ xyz=4\times7\times11 \quad \ldots \quad \ldots(4)$

Dividing (4) by (1) we get $z=11$, *i.e.*, the third number$=11$,

$\therefore$ the first number$=44\div11=4$

and the second number$=77\div11=7$.

Ex. 26. The product of two numbers is 1575 and their quotient is $\frac{9}{7}$. Find the numbers.

Let x and y be the two numbers.

Then $xy=1575$ ... ...(1) and $\dfrac{x}{y}=\dfrac{9}{7}$ ... ...(2)

Now, multiplying (1) by (2) we have, $xy\times\dfrac{x}{y}=\overset{225}{\cancel{1575}}\times\dfrac{9}{\cancel{7}}$,

or, $x^2=225\times9=15^2\times3^2$, $\therefore$ $x=15\times3=45$;

$y=1575\div45=35$. $\therefore$ the two numbers are 45 and 35.

Ex. 27. What time will it take to walk round a square field 8·1 acres in area at 4 miles per hour? [P. U. '32]

The area of the field$=8·1$ acres

$=8·1\times4840$ sq. yds$=81\times484$ sq. yds.

$\therefore$ Each side of the square field

$=\sqrt{81\times484}$ yards$=\sqrt{9^2\times22^2}$ yds.$=9\times22$ yds.

$\therefore$ the four sides of the field$=(9\times22\times4)$ yds.

It takes 1 hr. to walk 4 miles or 1760×4 yards.

$\therefore$ the time required to walk round the field is

$=\dfrac{9\times22\times\cancel{4}}{\cancel{4}\times\underset{80}{\cancel{1760}}}$ hrs. $=\dfrac{27}{4}$ minutes$=6$ minutes 45 seconds.

Exercise 6

Find the square root of :—

1. 1522756 [C. U. '22] 2. 2819041 [C. U. '23]
3. 1000014129 [C. U. '18] 4. 6256586734489 [C. U. '10]
5. 5322249 6. 144000000 7. 170·485249 [C. U. '15]

8. $29\cdot192409$ [C. U. '13] 9. $2919\cdot46783041$ [C. U.'11]

10. $\cdot00105625$ [M. E. '29]

11. $3\frac{13}{81}$. 12. $\frac{6}{8\frac{1}{6}}$. 13. $\dot{1}1\cdot\dot{1}$ 14. $\frac{32\cdot\dot{4}}{72\cdot\dot{9}}$ [M.E.'32]

15. (a) What number multiplied by itself will give $109\frac{628}{729}$? [P. U. '25]

(b) Find the square root of 2 to 7 places of decimals. [D. B. '33]

(c) Extract the sq. root of $\frac{7}{12}$ to 4 places of decimals.

16. Find the square root of $\cdot4$ to 3 places of decimals. [D. B. '40]

17. Find the square root of $1+(\cdot046)^2$ to 4 places of decimals. [C. U. '28]

18. Calculate to five places of decimals the square root of $1+(0\cdot067)^2$. [C. U. '39]

19. What least number must be added to 8275 to make the sum a square number ?

20. Find the square root of $1-(0\cdot00135)^2$ to 4 places of decimals. [C. U. '26]

21. A party of boys spent Rs. 10920. 25 P., each spending as many 25 pasia-pieces as there were boys ; what was the number of boys in the party ?

22. The subscription to a certain fund amounted to Rs. 380. 25 P. and each person subscribed as many paise as there were subscribers. Find the number of subscribers. [C. U. 1900]

23. Determine the square root of 1225 by resolving it into its prime factors. [E. B. S. B. '48]

[Hints : $1225=5\times5\times7\times7=5^2\times7^2$.]

24. Find the least number that must be added to $153\cdot140025$ to make it a perfect square. [Pat. U. '20]

SQUARE MEASURE

N. B. (1) In any rectangle, measure of area=measure of length × measure of breadth ;

more briefly, **area=length × breadth.**

Whence, length = area ÷ breadth, breadth = area ÷ length.

The area of the floor, the ceiling and each wall of a room and of a field and a sheet of paper, etc. which are rectangular surfaces = length × breadth.

(2) The length and the breadth of a square are equal. ∴ If the area of a square be known, the square root of the area will give the length of each side of the square.

(3) The sum of the lengths of the sides a rectilineal figure is called its perimeter.

∴ **the perimeter of a rectangle=2(length+breadth),**

and the perimeter of a square=a side × 4.

(4) 4 square feet and 4 ft. square are not same. Four square feet denotes an area 4 times as large as a square foot ; four feet square denotes the area of a square whose each side is 4 feet (*i.e.*, whose area is 4 ft. × 4 ft. or 16 square feet).

(5) The length, breadth and height are said to be the dimensions of a room.

(6) **The area of four walls of a room**=perimeter × height
= 2(length+breadth) × height.

(7) 144 square inches = 1 square foot,
9 square feet = 1 square yard,
$30\frac{1}{4}$ square yards = 1 square pole,
40 square poles = 1 rood,
4 roods or 4840 square yards = 1 acre.

(8) If the total expense of covering a place be divided by the expenses per square yard or square metre we get the area of the place in square yards or square metres respectively.

Examples [7]

1. The area of a rectangular field is $\frac{1}{2}$ acre and its breadth is 40 yards. Find the length of the field.

The area of the field $=\frac{1}{2}$ acre $=\frac{1}{\cancel{2}} \times \overset{2420}{\cancel{4840}}$ sq. yards.

$\therefore$ The length $=2420$ sq. yds. $\div 40$ yds.

$$=\frac{\overset{121}{\cancel{2420}}}{\underset{2}{\cancel{40}}}\text{ yds.}=60\text{ yds. }1\text{ ft. }6\text{ in.}$$

2. How much fencing will be required to fence a square field of 10 acres ?

The area of the square field $=10$ acres $=48400$ sq, yds.

$\therefore$ Each side of the field $=\sqrt{48400}$ yds. $=220$ yds.

$\therefore$ the length of fencing $=$ the perimeter of the field

$=220$ yds. $\times 4=880$ yds.

3. Find the area of the ceiling of a room which required 900 tiles, each 2 metres square.

Area of each tile $=2$ m. square $=2$ m. $\times 2$ m. $=4$ sq. m.

$\therefore$ area of the ceiling $=$ area of 900 tiles

$=4$ sq. m. $\times 900=3600$ sq. metres.

4. The circumference of a steam roller is 2 metres and it is 6 decimetres wide. How much ground will it pass over in 120 complete revolutions ?

The area of the ground passed over in one revolution of the steam roller $=$ the circumference $\times$ breadth

$=2$ m. $\times \frac{6}{10}$ m. $=\frac{6}{5}$ sq. m.

$\therefore$ in 120 revolutions the roller passes over $\frac{6}{5}\times 120$ sq. m. or 144 sq. m. of ground.

5. A field, 120 metres by 30 metres, is exchanged for a square field of equal area. Find the length of the latter.

Here the area of the square field $=$ the area of the given field

$=120$ m. $\times 30$ m. $=3600$ sq.m.

$\therefore$ the length of the square $=\sqrt{3600}$ m. $=60$ metres.

6. How many tiles, each 14″ by 12″, will be required for the floor of a room $70' \times 9'$?

[C. U. 1864]

Area of each tile $=14$ in. $\times 12$ in. $=168$ sq. in. $=\frac{168}{144}$ sq. ft.

Area of the floor $=70$ ft. $\times 9$ ft. $=630$ sq. ft.

$\therefore$ The number of tiles required $=630\div\frac{168}{144}=\overset{90}{\cancel{630}}\times\frac{\overset{6}{\cancel{144}}}{\underset{7}{\cancel{168}}}=540.$

7. The length of a rectangular field is four times its breadth. Its area is 90 acres. Find its length. [D. B. '32]

Area $=90$ acres $=4840\times 90$ sq. yds.

or, length $\times$ breadth $=4840\times 90$ sq. yds.

or, $4\times$ breadth $\times$ breadth $=4840\times 90$ sq. yds.

($\because$ length $=4\times$ breadth)

or, $4(\text{breadth})^2=4840\times 90$ sq. yds.

or, $(\text{breadth})^2=\dfrac{4840\times 90}{4}$ sq. yds. $=1210\times 90$ sq. yds.

$\therefore$ breadth $=\sqrt{1210\times 90}$ yds. $=330$ yds.

$\therefore$ the required length $=330$ yds. $\times 4=1320$ yds.

8. The sides of a rectangle are as 4 : 3 and its area is 2028 sq. metres. Find its length and breadth.

Here $\dfrac{\text{length}}{\text{breadth}}=\dfrac{4}{3}$, $\therefore$ length $=\frac{4}{3}\times$ breadth.

Now, following the method of the example 7 we have length $=52$ yds., breadth $=39$ yds.

9. A room is twice as long as it is wide. If it costs Rs. 32 to pave its floor at 25 P. per sq. metre, find its length and breadth.

[N. B. The area is obtained by dividing the total expenses by the expenses per sq. m. If the expenses per sq. m. or sq. ft. be given, the area will be in sq. m. or sq. ft. as the case may be.]

Here, the area of the floor $=(\text{Rs. } 32\div\text{Re. }\frac{1}{4})$ sq. m. $=128$ sq.m.

or, length $\times$ breadth $=128$ sq. m.

$\therefore$ $2\times$ breadth $\times$ breadth $=128$ sq. m.

or $2\times(\text{breadth})^2=128$ sq. m., or, $(\text{breadth})^2=64$ sq.m.

$\therefore$ breadth $=\sqrt{64}$ m. $=8$ metres

and length $=8$ m. $\times 2=16$ metres

10. A room, 27 ft. long, requires Rs. 58. 8 as. to cover its floor with carpet at Re. 1. 2 as. per sq. yd. Find the breadth of the room.

Area of the floor $=$ Area of the carpet

$=(\text{Rs. } 58\frac{1}{2}\div\text{Rs. } 1\frac{1}{8})$ sq. yds.

$=52$ sq. yds. $=52\times 9$ sq. ft.

$\because$ Its length $=27$ ft., $\therefore$ the breadth $=\frac{52\times 9}{27}$ ft. $=17$ ft. 4 in.

11. A rectangular court, three times as long as it is broad, is paved with 2028 stones, each $1\frac{1}{2}$ decimetres square. Find the length of the court.

Area of each stone $= \frac{3}{2}$ dm. $\times \frac{3}{2}$ dm. $= \frac{9}{4}$ sq. dm.

$\therefore$ the area of the court

$=$ the area of 2028 stones $= \frac{9}{4}$ sq. dm. $\times 2028$

$\therefore$ length $\times$ breadth $= 9 \times 507$ sq. dm.

or, $3\,(\text{breadth})^2 = 9 \times 507$ sq. dm. ($\because$ length $= 3 \times$ breadth).

or, $(\text{breadth})^2 = \dfrac{\overset{3}{\cancel{9}} \times 507}{\cancel{3}}$ sq. dm. $= 1521$ sq. dm.

$\therefore$ breadth $= \sqrt{1521}$ dm. $= 39$ dm.

$\therefore$ the length $= 39$ dm. $\times 3 = 11$ m. 7 dm.

12. The cost of turfing a square field at 25 P. per sq. metre is Rs. 225. Find the perimeter of the field.

Area of the square field $=$ (Rs. $225 \div$ Re. $\frac{1}{4}$) sq. m. $= 900$ sq. m.

$\therefore$ each side of the field $= \sqrt{900}$ m. $= 30$ m.

$\therefore$ its perimeter $= 30$ m. $\times 4 = 120$ metres.

13. The cost of carpeting a room is Rs. 120. Had its length been 3 m. less than it was, the cost would have been Rs. 105. Find the length.

If the length be 3 metres less, it costs Rs. $(120 - 105)$ or Rs. 15 less.

$\therefore$ Rs. 15 is the cost when the length is 3 m.

$\therefore$ Re. 1 ... ... $\frac{3}{15}$ m.

$\therefore$ Rs. 120 ... ... $\frac{3}{15} \times 120$ or 24 m.

$\therefore$ the length reqd. $= 24$ metres.

14. The perimeter of a square is equal to that of a rectangle whose length is 48 ft. and is 3 times its breadth. How many stones, each $18'' \times 8''$, will be required to pave it ? [D. B. '35]

The length of the rectangle $= 48$ ft.,

$\therefore$ its breadth $= 48$ ft. $\div 3 = 16$ ft.

$\therefore$ its perimeter $= 2\,(48 + 16)$ ft. $= 128$ ft.

$\therefore$ the perimeter of the square $= 128$ ft.

$\therefore$ each side of the square $= (128 \div 4)$ ft. $= 32$ ft.

$\therefore$ its area $= 32 \times 32$ sq. ft. $= 32 \times 32 \times 144$ sq. in.

Again, the area of each stone $= 18'' \times 8'' = 144$ sq. in.

$\therefore$ The number of stones $= 32 \times 32 \times 144 \div 144 = 32 \times 32 = 1024$.

15. A lawn is half as long again as it is broad and the cost of levelling it at $31\frac{1}{4}$ P. per square metre, is Rs. 1470. What would be the cost of fencing it at Rs. 4 per metre ?

$$31\tfrac{1}{4}\text{ P.}=\text{Rs. }\tfrac{125}{4\times100}=\text{Re. }\tfrac{5}{16}.$$

Area of the lawn $=(\text{Rs. }1470\div\text{Re. }\frac{5}{16})$ sq. m. $=294\times16$ sq. m.

$\therefore$ length $\times$ breadth $=294\times16$ sq. m.

or, $\frac{3}{2}\times$ breadth $\times$ breadth $=294\times16$ sq. m.

or, $\frac{3}{2}(\text{breadth})^2=294\times16$ sq. m.

or, $(\text{breadth})^2=\dfrac{294\times16\times2}{3}$ sq. m. $=98\times16\times2$ sq. m.

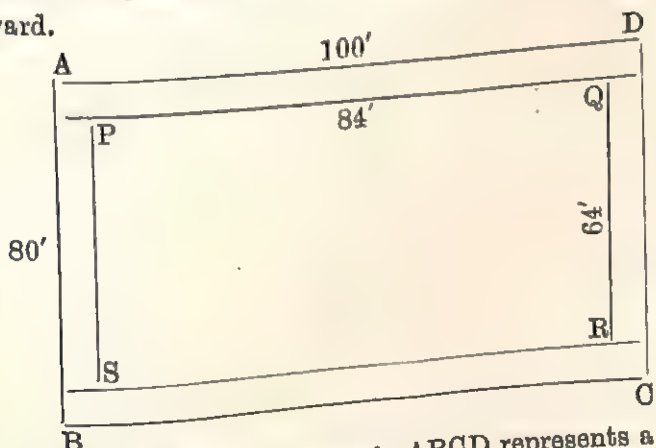
$\therefore$ breadth $=\sqrt{196\times16}$ m. $=14\times4$ m. $=56$ m.,

$\therefore$ length $=56$ m. $\times\frac{3}{2}=84$ m.

$\therefore$ the total length of fencing
$=$ the perimeter of the lawn
$=2(84+56)$ m. $=280$ metres.

$\therefore$ the reqd. cost $=$ Rs. $4\times280=$ Rs. 1120.

16. A rectangular court-yard 100 ft. long by 80 ft. wide has within it a gravel path 8 ft. wide running round it. Find the area of the path and the cost of gravelling it at 5 as. 3 p. per sq. yard. [C. U. '12]



In the diagram the rectangle ABCD represents a court-yard.

Its area $=100$ ft. $\times80$ ft. $=8000$ sq. ft.

PQRS represents the inner court-yard excludnig the path.

$PQ=100'-8'\times2=84'$ and $PS=80'-8'\times2=64'$ (PQ is less than AD by $8'\times2$ or $16'$, *i.e.*, by $8'$ on each side and PS is also less than AB by $16'$).

$\therefore$ area of the inner court-yard PQRS $= 84' \times 64' = 5376$ sq.ft.

$\therefore$ Area of the path $=$ ABCD $-$ PQRS

$= 8000$ sq. ft. $- 5376$ sq. ft. $= 2624$ sq. ft.

Again, the cost reqd. $= 5$ as. 3 p. $\times \frac{2624}{9} =$ Re. $\frac{21}{64} \times \frac{2624}{9}$

$=$ Rs. $\frac{287}{3} =$ Rs. 95. 10 as. 8 p.

[2624 sq. ft. is reduced to sq. yds. as cost per sq. yd. is given].

Second Method. Produce PQ and SR both ways to cut AB and CD so that the path is divided into 4 paths (vide the diagram).

The two opposite paths along the length are equal, each 100′ long and 8′ wide.

Again, the two paths along the breadth are also equal, each (80′ − 16′) or 64′ long and 8′ wide.

$\therefore$ area of the whole path $= 100' \times 8' \times 2 + 64' \times 8' \times 2$

$= 1600$ sq. ft. $+ 1024$ sq. ft. $= 2624$ sq. ft.

17. A square grass-plot, whose side is 600 m. is bordered on the outside by a path 10 m. wide. Find the cost of gravelling the path at Rs. 2. 50 P. per 100 sq. metres.

Here, the path being outside the grass-plot the two paths along the length are each (600 m. + 20 m.) or 620 m. long and 10 m. wide. Again, the two paths along the breadth are each 600 m. long and 10 m. wide.

$\therefore$ the area of the path $= 620$ m. $\times 10$ m. $\times 2 + 600$ m. $\times 10$ m. $\times 2$

$= 24400$ sq. m.

$\because$ cost of 100 sq. m. $=$ Rs. $\frac{5}{2}$,

$\therefore$ ... 1 sq. m. $=$ Rs. $\frac{5}{2 \times 100}$,

$\therefore$ ... 24400 sq. m. $=$ Rs. $\frac{5 \times 24400}{2 \times 100} =$ Rs. 610.

18. How many paving stones, each 2 ft. long and 1 ft. 6 in. wide, will be required for paving a road 30 ft. wide surrounding the outside of a square park whose area is 40 acres ? [D.B. '35]

Area of the square park $= 40$ acres $= 40 \times 4840$ sq. yds.

$\therefore$ its each side $= \sqrt{40 \times 4840}$ yds. $= 440$ yds.

The road is 30 ft. or 10 yds. wide.

Now, the area of the path

$=460$ yds.$\times 10$ yds.$\times 2+440$ yds.$\times 10$ yds.$\times 2=18000$ sq. yds.

Area of each stone$=2$ ft.$\times\frac{3}{2}$ ft.$=3$ sq. ft.$=\frac{1}{3}$ sq. yd.

$\therefore$ The reqd. number of stones$=(18000$ sq. yds.$\div\frac{1}{3}$ sq. yds.)

$=54000.$

19. A rectangular court, 90 ft. long by 75 ft. wide, has within it a gravel path, 6 ft. wide, running round it. Find the total cost of turfing the court at 3 as. 6 p. and of gravelling the path at 7 as. 6 p. per sq. yard. [C. U. '37]

Area of the court with its path$=90$ ft.$\times 75$ ft.$=6750$ sq. ft.

Area of the court excluding the path to be turfed

$=(90$ ft.-12 ft.$)\times(75$ ft.-12 ft.$)=78$ ft.$\times 63$ ft.

$=4914$ sq. ft.$=\frac{4914}{9}$ sq. yds.

$\therefore$ the cost of turfing the court

$$=3\tfrac{1}{2}\text{ as.}\times\frac{4914}{9}=\text{Rs. }\frac{7}{\underset{16}{\cancel{32}}}\times\frac{\overset{\overset{273}{\cancel{2457}}}{\cancel{4914}}}{\cancel{9}}=\text{Rs. }119.\ 7\text{ as.}$$

Again, area of the path

$=6750$ sq. ft.-4914 sq. ft.$=1836$ sq. ft.$=204$ sq. yds.

$\therefore$ the cost of gravelling the path$=$Re. $\frac{15}{32}\times 204=$Rs. 95. 10 as.

$\therefore$ the total cost$=$Rs. 119. 7 as.$+$Rs. 95. 10 as.$=$Rs. 215. 1a.

20. The inside measurements of a room are 42 ft. 6 in. and 22 ft. 6 in. The walls are 2 ft. 3 in. thick and there is a verandah all round 10 ft. 6 in. wide. Find the cost of paving the verandah with tiles $4\frac{1}{2}$ in. by 3 in. and costing 9 pies each. [C. U. '45]

Length of the inner side of the room$=42$ ft. 6 in.

and its breadth$=22$ ft. 9 in. ;

The wall being 2 ft. 3 in. thick its length on the outside

$=42$ ft. 6 in$+4$ ft. 6 in.$=47$ ft.

and breadth$=22$ ft. 9 in.$+4$ ft. 6 in.$=27$ ft. 3 in.

$\therefore$ the area of the verandah (vide second method of Ex. 16)

$=68$ ft.$\times\frac{21}{2}$ ft.$\times 2+\frac{109}{4}$ ft.$\times\frac{21}{2}$ ft.$\times 2$

$=1428$ sq. ft. $+\frac{2289}{4}$ sq. ft.$=\frac{8001}{4}$ sq. ft.

Area of each tile $=\frac{9}{2}$ in. $\times 3$ in. $=\frac{27}{2}$ sq. in.

$$=\frac{27}{2\times 144}\text{ sq. ft.}=\frac{3}{32}\text{ sq. ft.}$$

$\therefore$ the number of tiles $=\frac{8001}{4}\div\frac{3}{32}=\frac{8001}{4}\times\frac{32}{3}=2667\times 8$

$\therefore$ the total cost $=2667\times 8\times 9$ pies $=$ Rs. 1000. 2 as.

21. Find the cost of carpeting a room, 18 m. by 15 m. with carpet 15 dm. wide and costing Rs. 20 per metre.

Area of the carpet $=$ area of the floor $=18$m. $\times 15$ m. $=270$ sq.m.

$\therefore$ length of the carpet $=$ Area $\div$ breadth

$$=270\text{ sq. m.}\div\frac{3}{2}\text{m.}=\frac{270\times 2}{3}\text{m.}=180\text{ metres.}$$

$\therefore$ the total cost $=$ Rs. $20\times 180=$ Rs. 3600.

22. What biggest size of square stones may be used to pave a court 452 metres long and 404 metres wide ?

Here the stones used are of square size, so there will be an integral number of stones used ; $\therefore$ the length and breadth of the court are exactly divisible by the length of each side of the stone. Now, we have to find what greatest length will exactly divide 452 metres and 404 metres. The H. C. F. of 452 metres and 404 metres $=4$ metres. $\therefore$ 4 m. $\times 4$ m. or 4 metres square is the biggest size of square stone that may be used.

23. What will be the cost of paper, 20 inches wide, at $3\frac{1}{2}$ d. a yard for the walls of a room 21 ft. long, 15 ft. wide and 10 ft. high ? [C. U. 1918]

Area of the four walls $=2(21$ ft. $+15$ ft.$)\times 10$ ft. $=720$ sq. ft. ;

$\therefore$ the area of paper $=720$ sq. ft.

Breadth of paper $=20$ in. $=\frac{20}{12}$ ft.,

$\therefore$ length of paper $=720$ sq. ft. $\div\frac{20}{12}$ ft. $=\frac{720\times 12}{20}$ft. $=144$ yds.

$\therefore$ the reqd. cost $=\frac{7}{2}$ d. $\times 144=$ £. 2. 2 s.

24. Find the cost of papering the walls of a room 12 ft. 6 in. long, 7 ft. 6 in. wide and 12 ft. high with half-anna postage stamps measuring $\frac{15}{16}$ in. by $\frac{3}{4}$ in. [C. U. 1915]

Area of the four walls $=2(12$ ft. 6 in. $+7$ ft. 6 in.$)\times 12$ ft.

$=2\times 20$ ft. $\times 12$ ft. $=480$ sq. ft. $=480\times 144$ sq. in.

Area of each stamp $= \frac{15}{16}$ in. $\times \frac{3}{4}$ in. $= \frac{45}{64}$ sq. in.

$\therefore$ number of stamps $= 480 \times 144 \div \frac{45}{64} = \dfrac{\overset{96}{\cancel{480}} \times \overset{16}{\cancel{144}} \times 64}{\underset{\cancel{9}}{\cancel{45}}}$

Price of each stamp $= \frac{1}{2}$ *a.* $=$ Re. $\frac{1}{32}$.

$\therefore$ the total cost $= 96 \times 16 \times 64 \times \frac{1}{32}$ rupees $=$ Rs. 3072.

25. What will be the height of a room, 20 metres long by 16 metres wide, so that the area of its floor and ceiling may be together equal to the area of its walls ?

Area of the floor = Area of the ceiling.

$\therefore$ area of the 4 walls = area of the floor + area of the ceiling
= area of the floor $\times 2 = 20$ m. $\times 16$ m. $\times 2 = 640$ sq. metres.

Again, area of 4 walls = 2(length + breadth) $\times$ height

$\therefore$ 2(length + breadth) $\times$ height = 640 sq. m.

or 2(20 m. + 16 m.) $\times$ height = 640 sq. m.

or 72 m. $\times$ height = 640 sq. m.

$\therefore$ height $= \frac{640}{72}$ m. $= \frac{80}{9}$ m. $= 8\frac{8}{9}$ metres.

26. The length of a square room is 16 metres and the cost of papering the walls at 50 P. a sq. metre is Rs. 400. Find the height of the room.

Area of the 4 walls = (Rs. 400 $\div$ Re. $\frac{1}{2}$) sq. m. = 800 sq. m.

[The area of the walls is obtained by dividing the total cost of papering the walls by the cost per sq. yd.]

$\therefore$ 2(length + breadth) $\times$ height = 800 sq. m.

or 2(16 m. + 16 m.) $\times$ height = 800 sq. m.

or 64 m. $\times$ height = 800 sq. m.

$\therefore$ height $= \frac{800}{64}$ m. $= 12\frac{1}{2}$ metres.

27. A room whose height is 13 ft. and length twice its breadth takes 143 yds. of paper, 2 ft. wide, to cover its four walls. Find the area of the floor. [D. B. 1934]

Area of the four walls = area of the paper

$= 143$ yds. $\times \frac{2}{3}$ yd. $= \frac{286}{3}$ sq. yds.

or $2 \times$ height (length + breadth) $= \frac{286}{3}$ sq. yds.

or 2×13 ft. (length+breadth)$=\frac{286}{3} \times 9$ sq. ft.

or 26 ft.$\times$(length+breadth)$=286 \times 3$ sq. ft.

$\therefore$ length+breadth$=\frac{286 \times 3}{26}$ ft.$=33$ ft.

or $2 \times$breadth+breadth$=33$ ft., or, $3 \times$breadth$=33$ ft.

$\therefore$ breath$=11$ ft. and length$=11$ ft.$\times 2=22$ ft.

$\therefore$ area of the floor reqd.$=22$ ft.$\times 11$ ft.$=242$ sq. ft.

28. The length of a room is 24 metres and the height is 12 metres. The four walls are painted at a cost of 30 P. per sq. metre and the cost comes to Rs. 244. 80 P. Find the breadth of the room.

Area of the four walls

$=$(Rs. 244. 80 P.$\div$30 P.) sq. m.$=816$ sq. m.

$\therefore$ 2(length+breadth)$\times$height$=816$ sq. m.

or 2(length+breadth)$\times 12$ m.$=816$ sq. m.

or 24 m.$\times$(length+breadth)$=816$ sq. m.

or length+breadth$=816$ sq. m.$\div 24$ m.$=34$ m.

But length$=24$ m., $\therefore$ breadth$=34$ m.-24 m.$=10$ metres.

29. A room is twice as long as it is wide. The cost of carpeting its floor at 40 P. per sq. metre is Rs. 88. 20 P. and that of painting its walls at 9 P. per sq. metre is Rs. 56. 70 P. Find the dimensions of the room.

Area of the floor$=$(Rs. 88. 20 P.$\div$40 P.) sq. m.$=\frac{441}{2}$ sq. m.

$\therefore$ length$\times$breadth$=\frac{441}{2}$ sq. m.

or $2 \times$breadth$\times$breadth $=\frac{441}{2}$ sq. m.

or 2 (breadth)$^2=\frac{441}{2}$ sq. m., or, (breadth)$^2=\frac{441}{4}$ sq. m.

$\therefore$ breadth$=\sqrt{\frac{441}{4}}$ m.$=\frac{21}{2}$ m.$=10$ m. 5 dm.

$\therefore$ length$=$breadth$\times 2=\frac{21}{2}$ m.$\times 2=21$ m.

Again, the area of the four walls

$=(\text{Rs. } 56.\ 70 \text{ P.} \div 9 \text{ P.})$ sq. m. $= 630$ sq. m.

or $2(\text{length}+\text{breadth}) \times \text{height} = 630$ sq. m.

or $2(21\text{m.}+\frac{21}{2}\text{ m.}) \times \text{height} = 630$ sq. m.

or $63 \text{ m.} \times \text{height} = 630$ sq. m., $\therefore$ height $= \frac{630}{63}$ m. $= 10$ m.

$\therefore$ the dimensions required are length $= 21$ m.,

breadth $= 10$ m. 5 dm., height $= 10$ metres.

30. A room is 12 m. long, 8 m. broad and 10 metres high. Find the cost of whitewashing the four walls of the room, leaving out two doors each measuring 6 m. high and 4 m. wide, and four windows each measuring 5 m. high and 3 m. wide, if the rate is 10 P. per sq. metre.

Area of the four walls (with doors and windows)

$= 2(\text{length}+\text{breadth}) \times \text{height})$

$= 2(12 \text{ m.} + 8 \text{ m.}) \times 10 \text{ m.} = 400$ sq. m.

Area of a door $= 6 \text{ m.} \times 4$ sq. $= 24$ sq. m.

$\therefore$ area of 2 doors $= 48$ sq. m.

Again, area of 4 windows $= 5 \text{ m.} \times 3 \text{ m.} \times 4 = 60$ sq. m.

$\therefore$ area of doors and windows $= 48$ sq. m. $+ 60$ sq. m. $= 108$ sq. m.

$\therefore$ area of the portions of the walls to be whitewashed

$= 400$ sq. m. $- 108$ sq. m. $= 292$ sq. m.

$\therefore$ the reqd. cost $= 10 \text{ P.} \times 292 = 2920$ P. $=$ Rs. 29. 20 P.

31. A square field has a path 9 ft. wide within and all round it. If the area of the path is 3 acres, find the area of the field.

ABCD represents a square field which has a path 9 ft. or 3 yds. wide within and all round it. (Vide diagram of Ex. 16). Now, if we produce PQ, QR, RS and SP to the sides of the square field, the whole path is divided into 4 paths equal in area.

$\therefore$ area of each path $= \frac{3}{4}$ acre $= \frac{3}{4} \times 4840$ sq. yds. $= 3 \times 1210$ sq. yds.

and the breadth of that path $= 9$ ft. $= 3$ yds.

$\therefore$ length of each path $= \dfrac{3 \times 1210}{3}$ yds. $= 1210$ yds.

$\therefore$ each side of the square field $= 1210$ yds. $+ 3$ yds. $= 1213$ yds.

$\therefore$ area of the square field $= (1213 \times 1213)$ sq. yds.

$= 1471369$ sq. yds. $= 304$ acres 9 sq. yds.

[Otherwise] : Suppose, each side of the square field $= x$ yds.

$\therefore$ area of the path $= x \times 3 \times 2$ sq. yds.
$+ (x-6) \times 3 \times 2$ sq. yds. (Vide Ex. 16).
$= (6x + 6x - 36)$ sq. yds. $= 12(x-3)$ sq. yds.

$\therefore$ 12 yds. $(x-3)$ yds. $= 3$ acres $= 3 \times 4840$ sq. yds.

($\because$ the area of the path is 3 acres)

or, $x - 3 = \dfrac{3 \times 4840}{12}$ yds. $= 1210$ yds, $\therefore$ $x = 1213$ yds,

$\therefore$ area of the square field

$= 1213 \times 1213$ sq. yds. $= 304$ acres 9 sq. yds.

32. A rectangular garden 150 m. long and 120 metres broad has two straight paths each 4 m. wide running through the middle of its two sides to the middle of its opposite two sides. Find the cost of paving the paths at Re. $1\frac{1}{8}$ per sq. metre.

The area of the path parallel to the length
$= 150 \times 4$ sq. m. $= 600$ sq. metres.

The area of the path parallel to the breadth
$= 120 \times 4$ sq. m. $= 480$ sq. metres.

Hence, the space covered by the two paths
$= (600 + 480)$ sq. m. $-$ the area of the common square space where the two paths cross each other
$= 1080$ sq. m. $- 4$ m. $\times 4$ m. $= 1080$ sq. m. $- 16$ sq. m.
$= 1064$ sq. metres.

$\therefore$ the reqd. cost $=$ Re. $1\frac{1}{8} \times 1064 =$ Rs. 1197.

33. Walking at 3 miles per hour it takes 5 minutes to walk along one side of a rectangular field and 14 mins. to go all round. Find the area of the field. [D. B. '46]

It takes 1 hr. or 60 minutes to go 3 miles.
$\therefore$ it takes 5 mins. to go $\frac{3}{12}$ miles or $\frac{3}{12} \times 1760$ yds. or 440 yds.
And it takes 14 mins. to go $\frac{3}{60} \times 14$ miles $= 1232$ yds.
$\therefore$ length $= 440$ yds., and perimeter $= 2$(length $+$ breadth) $= 1232$ yds.
$\therefore$ length $+$ breadth $= 1232$ yds. $\div 2 = 616$ yds.
$\therefore$ breadth $= 616$ yds. $- 440$ yds. $= 176$ yds.
$\therefore$ the area of the field $= 440$ yds. $\times 176$ yds. $= 16$ acres.

Exercise 7

1. What is the width of a land whose area and length are 57·6 sq. decametres and 160 metres respectively ?

2. Find the cost of fencing a square field of 10 acres at 6 as. 8 p. per yard. [C. U. '13]

3. A rectangular field is half as long again as it is wide. The cost of levelling it at Re. $\frac{3}{8}$ per sq. m. is Rs. 1764. Find its length.

4. A railing encloses a rectangular field of 15 acres. Find the length of the railing if the length of the field is to its breadth as 3 : 2. [C. U. '39]

5. A gravel path, 5 ft. wide, runs round a rectangular garden, 105 yds. by 75 yds. ; find the cost of making it at 3 as. 6 p. per sq. yd. [D. B. '25]

6. (i) A square field contains 202·5 acres. Find the cost of running a fence round it at 5 as. 3 p. a yard. [C. U. '36]

6. (ii) The total area of two square fields is 1170 ares. The side of one square is $\frac{2}{3}$ of the other. Find the area of each square.

7. Find the cost of paving a pathway, 6 ft. wide, round and immediately outside a flower garden, 21 yds. long and 10 yds. broad, at $5\frac{4}{7}$ pies per sq. yard. [D. B. '33]

8. A rectangular garden 2000 m. long and 884 m. broad has a path of a uniform breadth of 16 m. lying all round it. Find the cost of paving it at 75 P. per sq. metre.

9. In carpeting a room, 36 ft. by 16 ft., a clear space of 3 ft. is left all round for matting. If the cost of the carpet be Re. 1. 4 as. per sq. ft. and that of matting 5 as. 6 p. per sq. yd., find the total cost. [D. B. '32]

10. A room is 27 m. 6 dm. long, 21 m. 4 dm. wide and 10 m. high. Find the cost of papering the walls at 50 P. per sq. decimetre.

11. A room 36 ft. long by 19 ft. broad is enclosed by walls 12 inches thick and a verandah 9 ft. broad runs all round enclosing the room. Find the cost of paving the verandah at 6 as. per sq. yard. [C. U. '46]

12. The walls of a room, 30 m. by 20 m., were painted at Re. $\frac{2}{3}$ per sq. m. If the length of the room were 4 m. less, the cost would have been Rs. $53\frac{1}{3}$ less. Find the height of the room.

[**Hints**—The area of the wall to be lessened on both sides of the length $=4$ m. $\times$ height $\times 2$.]

13. The cost of matting a room 16 ft. broad and 12 ft. high at 3 as. per sq. yard is Rs. 7. 9 as. 4 p. ; what will be the cost of papering the walls at the same rate, allowing for 6 doors, each 6 ft. by 3 ft. ? [C. U. 1906]

14. The expenses of carpeting a room 30 metres long was Rs. 150, but if the breadth had been 5 m. less than it was, the expenses would have been Rs. 120. Find the breadth of the room.

15. A rectangular court-yard, 76 m. by 54 m., has within it a path 3 m. wide running round it. How many square stones of largest possible size will be required to pave the path ?

16. The four walls of a room have a total area of 660 sq. m., the area of the floor is 270 sq. m. and the width of the floor is 15 m. Find the height of the room.

17. The expenses of carpeting a room thrice as long as it is broad at Rs. 7. 50 P. per sq. metre is Rs. 1102. 50 P. Find the length and breadth of the room.

18. There is a rectangular court 100 metres long and 50 metres wide. There are two paths 4 metres wide inside the court across each other running parallel to the sides. The cost of paving a sq. m. with stone is 75 P. and that of gravelling a sq. m. is Re. $\frac{3}{8}$. Find the total cost of paving the court and gravelling the path. [Vide Ex. 32] [C. U. 1887]

19. A room 10 ft. high and 20 ft. long costs £ 190 to paint its walls at 5s. per sq. ft. What is the cost of the carpet which will cover the floor at Rs. 3. 2 as. per sq. yd ? [C. U. '50]

20. The length, breadth and height of a room are 16 ft., 12 ft. and 11 ft. respectively. Find the cost of whitewashing its interior including the roof at Rs. 22. 8 as. per 100 sq. yards.

[G. U. '55]

21. A room is 18 ft. long, $16\frac{1}{2}$ ft. broad and 12 ft. high. If the cost of papering the walls with paper at 3d. per yard be £ 1. 14s. 6d., find the width of the paper used. [U. U. '47]

22. The four walls of a room have a total area of 600 sq. m. The area of the floor is 216 sq. m., the width of the floor is 18 m. What is the height of the room ?

23. The area of a rectangular field whose breadth is 500 yds. is 100 acres. Find the cost of cultivating it at Rs. 3. 2 as. 8 p. per 100 sq. yards and also the cost of fencing it round at Rs. 2. 8 as. per yd. [C. U. 1902]

CUBIC MEASURE

You know that what has length, breadth and height (or thickness or depth) is called a **solid** or a solid body.

The length, breadth and thickness (or height or depth) of a rectangular solid are called its **dimensions.**

Each face bounding a solid is called its **surface.**

The amount of space bounded by the surfaces of a solid is called its **volume.**

Parallelopiped : The solid which is bounded by six plane faces, of which each pair of opposite faces are parallel and congruent, is called a **parallelopiped.**

If the faces of a parallelopiped are all rectangles, it is called a **rectangular** parallelopiped or cuboid.

A brick, a rectangular box etc. are parallelopipeds.

Cube : If the faces are all squares, the parallelopiped is called a **cube.** A cube is, therefore, a rectangular parallelopiped whose length, breadth and height (or depth or thickness) are all equal.

In any rectangular parallelopiped or solid the measure of volume = measure of length × measure of breadth × measure of thickness, or, more briefly.

Volume = length × breadth × thickness.

∴ Length = volume ÷ (breadth × thickness),

breadth = volume ÷ (length × thickness),

and thickness or height or depth = volume ÷ (length × breadth).

Let a, b, c represent the length, breadth and thickness respectively.

Each of the six surfaces of a rectangular parallelopiped is a rectangle,

$\therefore$ the total area of the 6 surfaces of the rectangular parallelopiped

$= 2(\text{length} \times \text{breadth} + \text{length} \times \text{thickness} + \text{breadth} \times \text{thickness})$
$= 2(ab + ac + bc)$.

Diagonal of a rectangular parallelopiped

$= \sqrt{(\text{length})^2 + (\text{breadth})^2 + (\text{thickness})^2} = \sqrt{a^2 + b^2 + c^2}$.

Cubic content or **Volume of a Cube** $= (\text{edge})^3 = a^3$

[$\because$ its dimensions are all equal.]

Diagonal of a Cube $= \text{Edge} \times \sqrt{3} = \text{Side} \times \sqrt{3} = a\sqrt{3}$.

Units of Volume :

(1) English measure :

12^3 or 1728 cubic inches make 1 cubic foot.

3^3 or 27 cubic feet make 1 cubic yard.

(2) Indian measure :

24^3 or 13824 cubic fingers make 1 cubic cubit,

2^3 or 8 cubic cubits make 1 cubic yard.

8 cubic yards or 64 cubic cubits = 1 'chouk' (quarter)

1 cubic cubit = 18^3 or 5832 cubic inches.

Metric Measure

1000 cu. cm. = 1 cu. dm.

1000 cu. dm. = 1 cu. m.

1000 cu. m. = 1 cu. Dm., etc.

[**N. B.** Cubic metre ÷ metre = square metre,

Cubic m. ÷ square m. = metre,

Cubic m. ÷ cubic m. = abstract number.

Square m. ÷ m. = metre,

Square m. ÷ square m. = abstract number.

All sums on volumes should be worked out on reducing the length, breadth and thickness in term of the same unit.

Examples [8]

1. Find the cubic content of a cube whose edge is 3 ft. 4 in.

Edge of the cube $= 3$ ft. 4 in. $= \frac{10}{3}$ ft.

$\therefore$ Its volume $= (\text{side})^3 = (\frac{10}{3})^3$ cu. ft. $= \frac{1000}{27}$ cu. ft.

$= 37\frac{1}{27}$ cu. ft. $= 1$ cu. yd. 10 cu. ft. 64 cu. inch.

2. The volume of a cube is 125 cu. centimetres. Find the length of each edge and the diagonal.

The volume of the cube $= (\text{side})^3$

$\therefore$ here $(\text{side})^3 = 125$ cu. cm.

$\therefore$ each side $= \sqrt[3]{125}$ cm. $= 5$ cm.

$\therefore$ Diagonal $= a\sqrt{3} = 5\sqrt{3}$ cm.

[N. B. $\sqrt[3]{125}$ means the cube root of 125.

$5 \times 5 \times 5 = 125$. $\therefore$ the cube root of $125 = 5$.

Again, the cube root of cu. metre is metre and that of cu. ft. is ft. and so on.]

3. Find the whole surface of a cube whose edge measures 2 ft. 4 in.

Edge $= 2$ ft. 4 inch $= \frac{7}{3}$ ft.

$\therefore$ the required whole surface $= 6 \times (\text{edge})^2 = 6 \times (\frac{7}{3})^2$ sq. ft.

$= 6 \times \frac{49}{9}$ sq. ft. $= \frac{98}{3}$ sq. ft. $= 3$ sq. yds. 5 sq. ft. 96 sq. inches.

4. The length, breadth and height of a rectangular parallelopiped are 16 m., 12 m. and 4 m. 5 dm. respectively. Find its volume and the whole surface.

Here, length $= 16$ m., breadth $= 12$m., height $= 4$m. 5 dm. $= \frac{9}{2}$m.

$\therefore$ The volume of the rectangular solid $=$ length $\times$ breadth $\times$ height

$= 16 \text{ m.} \times 12 \text{ m.} \times \frac{9}{2} \text{ m.} = 864$ cu. m.

And the area of the total surface

$= 2(\text{length} \times \text{breadth} + \text{length} \times \text{height} + \text{breadth} \times \text{height})$

$= 2(16 \text{ m.} \times 12 \text{ m.} + 16 \text{ m.} \times \frac{9}{2} \text{ m.} + 12 \text{ m.} \times \frac{9}{2} \text{ m.})$

$= 636$ sq. metres.

5. A cistern 25 metres long and 12 metres broad can hold 1200 cu. metres of water. Find its depth.

Here, length of the cistern $= 25$ m., breadth $= 12$ m. and volume $= 1200$ cu. m.

$\therefore$ the depth reqd. $=$ volume $\div$ (length $\times$ breadth)

$= 1200 \text{ cu. m.} \div (25 \text{ m.} \times 12 \text{ m.}) = 1200 \text{ cu. m.} \div 300 \text{ sq. m.} = 4$ m.

6. A surface of a cube measures 16 sq. metres and its weight per cu. m. is 1 Kg. $2\frac{1}{2}$ Hg. Find the weight of the whole cube.

Area of one surface of the cube $=16$ sq. m.

$\therefore$ its edge $=\sqrt{16}$ m. $=4$ m.

$\therefore$ its volume $=(4)^3$ cu. m. $=64$ cu. m.

$\because$ the weight of 1 cu. m. $=1$ Kg. $2\frac{1}{2}$ Hg. $=\frac{5}{4}$ Kg.

$\therefore$ the total weight of the cube $=\frac{5}{4}$ Kg. $\times 64=80$ Kg.

7. A cubic copper plate whose volume is 1 cubic foot is hammered into a plate 2 ft. square ; find the thickness of the latter.

The volume of the copper cube $=1$ cu. ft.

$\therefore$ the volume of the second copper plate $=1$ cu. ft.

But the plate is 2 ft. square, i.e., each of its length and breadth is 2 ft.

$\therefore$ length $\times$ breadth $\times$ thickness of the plate $=$ its volume $=1$ cu. ft.

or, 2 ft. $\times$ 2 ft. $\times$ thickness $=1$ cu. ft.

or, 4 sq. ft. $\times$ thickness $=1$ cu. ft.

$\therefore$ the thickness reqd. $=1$ cu. ft. $\div 4$ sq. ft. $=\frac{1}{4}$ ft. $=3$ inches.

8. The volume of a cube is 37 cu. yds. 1 cu. ft., find the area of each surface.

The volume of the cube $=37$ cu. yds. 1 cu. ft. $=1000$ cu. ft.

$\therefore$ each side of the cube $=\sqrt[3]{1000}$ ft. $=10$ ft.

$\therefore$ the area of each surface $=(10)^2$ sq. ft. $=100$ sq. ft.

9. A room, 10 ft. high and half as long again as it is broad, contains 2160 cubic feet of air. Find the perimeter of the floor.

Length of the room $=\frac{3}{2}\times$ breadth, its breadth $=10$ ft. and volume $=2160$ cu. ft.

$\therefore$ length $\times$ breadth $\times$ height $=2160$ cu. ft.

or, $\frac{3}{2}\times$ breadth $\times$ breadth $\times$ 10 ft. $=2160$ cu. ft.

or, $(\text{breadth})^2\times 15$ ft. $=2160$ cu. ft.

or, $(\text{breadth})^2=\frac{2160}{15}$ sq. ft. $=144$ sq. ft.

$\therefore$ breadth $=\sqrt{144}$ ft. $=12$ ft. and

length $=\frac{3}{2}\times$ breadth $=\frac{3}{2}\times 12$ ft. $=18$ ft.

$\therefore$ the perimeter of the floor $=2$ (length $+$ breadth)

$=2$ (18 ft. $+$ 12 ft.) $=60$ ft.

10. How many bricks, each measuring $9'' \times 5'' \times 2''$, are required to construct a wall 27 ft. long, 19 ft. high and 1 ft. 3 in. thick ?

The volume of the wall $= 27$ ft. $\times 10$ ft. $\times 1$ ft. 3 in.
$= 27 \times 10 \times \frac{5}{4}$ cu. ft.,

The volume of 1 brick $= 9$ in. $\times 5$ in. $\times 2$ in.
$= \frac{3}{4}$ ft. $\times \frac{5}{12}$ ft. $\times \frac{1}{6}$ ft. $= \frac{5}{96}$ cu. ft.

$\therefore$ The number of bricks reqd. $= \dfrac{27 \times 10 \times 5}{4} \div \dfrac{5}{96}$

$$= \frac{27 \times 10 \times 5}{4} \times \frac{96}{5} = 6480.$$

11. A reservoir is 24 ft. 8 in. long and 12 ft. 9 in. broad ; find how many cu. ft. of water must be drawn off to make the surface of water sink 1 foot ?

The portion of the reservoir from which water is to be drawn off is 24 ft. 8 in. long, 12 ft. 9 in. wide and 1 foot deep.

$\therefore$ the reqd. volume of water $= 24\frac{2}{3}$ ft. $\times 12\frac{3}{4}$ ft, $\times 1$ ft.
$= \frac{74}{3} \times \frac{51}{4} \times 1$ cu. ft. $= \frac{629}{2}$ cu. ft. $= 314\frac{1}{2}$ cu. ft.

12. It costs Rs. 24 to paint the six surfaces of a cube at one anna per sq. ft. Find the volume of the cube.

Rs. $24 = 384$ as. Total cost $= 384$ as. Cost per sq. ft. $= 1$ anna,

$\therefore$ area of the six surfaces $= (384 \div 1)$ sq. ft. $= 384$ sq. ft.

$\therefore$ area of one surface $= 384$ sq. ft. $\div 6 = 64$ sq. ft.

$\therefore$ each side of the cube $= \sqrt{64}$ ft. $= 8$ ft.

$\therefore$ its volume $= (8)^3$ cu. ft. $= 512$ cu. ft.
$= 18$ cu. yds. 26 cu. ft.

13. A cistern can hold 750 gallons of water. If a gallon of water weighs 10 lbs. and 1 cubic foot of water weighs 1000 oz., how many cu. ft. of water will fill the cistern ?

750 gallons of water are needed to fill the cistern.

$\therefore$ Here we have to see how many cu. ft. of water weigh 750 gallons of water.

Weight of 750 gallons of water $= 750 \times 10$ lbs. $= 7500$ lbs.
$= 7500 \times 16$ oz. $= 120000$ oz.

Again, weight of 1 cu. ft. of water $= 1000$ oz.

$\therefore$ 120,000 oz. is the weight of $(120{,}000 \div 1000)$ or 120 cu. ft. of water.

$\therefore$ the cistern will be filled with 120 cu. ft. of water.

14. There is a rainfall of 7 inches on an acre of land. If the weight of 1 cu. ft. of water be 800 oz., find the total weight of water in tons.

Area of the land $=1$ acre $=4840$ sq. yds. $=4840\times 9$ sq. ft.

The depth of rainfall on it $=7$ in. $=\frac{7}{12}$ ft.

$\therefore$ the volume of the rain-water $=4840\times 9\times \frac{7}{12}$ cu. ft.

$=1210\times 3\times 7$ cu. ft.

$\because$ weight of 1 cu. ft. of water $=800$ oz.

$$=\frac{800}{16\times 28\times 4\times 20}\text{ tons}=\frac{5}{16\times 14}\text{ tons.}$$

$\therefore$ the weight of the whole water

$$=\frac{5}{16\times 14}\text{ tons}\times 1210\times 3\times 7=\frac{9075}{16}\text{ tons}=567\tfrac{3}{16}\text{ tons.}$$

15. The length of a room is three times its breadth and five times its height. If it contains 14400 c. ft. of air, find the area of the floor of the room.

Here, breadth $=\frac{\text{length}}{3}$, height $=\frac{\text{length}}{5}$;

Again, length $\times$ breadth $\times$ height $=14400$ cu. ft.

$\therefore$ length $\times\frac{\text{length}}{3}\times\frac{\text{length}}{5}=14400$ cu. ft.

or, $\frac{(\text{length})^3}{15}=14400$ cu. ft., or, $(\text{length})^3=14400\times 15$ cu. ft.

$=216000$ cu. ft. $=(60\text{ ft.})^3$. $\therefore$ length $=60$ ft.

$\therefore$ the area of the floor $=$ length $\times$ breadth $=60$ ft. $\times\frac{60}{3}$ ft.

$=1200$ sq. ft. $=133$ sq. yds. 3 sq. ft.

16. The inside dimensions of a rectangular cistern are length 18 m., breadth 12 m. and height 4 m. If the weight of 1 sq. m. of lead be 14 Hg. and the cost of 1 Kg. of lead be 25 P. ; find the cost of lining the four interior sides of the cistern with lead.

The area of the 4 interior sides of the cistern

$=2(18\text{ m.}+12\text{ m.})\times 4\text{ m.}=240$ sq. m.

Weight of 1 sq. m. of lead $=14$ Hg. $=\frac{7}{5}$ Kg.

$\therefore$ the total weight of lead $=\frac{7}{5}$ Kg. $\times 240=7\times 48$ Kg.

Again, 1 Kg. of lead costs 25 P. or $\frac{1}{4}$ rupee.

$\therefore$ the reqd. cost $=$ Re. $\frac{1}{4}\times 7\times 48=$ Rs. 84.

17. A cistern contains $243\frac{3}{4}$ cu. ft. of water. Find the length of the side of a second cistern 4 ft. 4 in. deep, with a square base, which contains 4 times as much water as the first. [C. U. '10]

The volume of the first cistern $=243\frac{3}{4}$ cu. ft. $=\frac{975}{4}$ cu. ft.

$\therefore$ the volume of the second cistern with square base

$$=\frac{975}{4} \text{ cu. ft.} \times 4 = 975 \text{ cu. ft.}$$

But its depth $=4$ ft. 4 in. $=\frac{13}{3}$ ft.

$\therefore$ the area of the base $=975$ cu. ft. $\div \frac{13}{3}$ ft. $=225$ sq. ft.

$\because$ the base is a square,

$\therefore$ the length of each side of the second cistern

$$=\sqrt{225} \text{ ft.} = 15 \text{ ft.}$$

18. A rectangular garden 120 metres long and 90 metres wide is to be surrounded by a wall 6 metres high and 75 centimetres thick. How many bricks, each measuring $\frac{3}{4}$ m. $\times \frac{3}{8}$ m. $\times \frac{1}{4}$ m., will be required to build the wall ?

The total length of the wall all around

$$=2\{(120 \text{ m.} + \tfrac{3}{4} \text{ m.} \times 2) + 90 \text{ m.}\}$$

$$=2 \times (121\tfrac{1}{2} \text{ m.} + 90 \text{ m.}) = 2 \times 211\tfrac{1}{2} \text{ m.} = 423 \text{ m.}$$

Height of the wall $=6$ m. and its thickness $=75$ cm. $=\frac{3}{4}$ m.

$\therefore$ the cubic content of the whole wall

$$=423 \text{ m.} \times 6 \text{ m.} \times \tfrac{3}{4} \text{ m.} = \tfrac{423 \times 9}{2} \text{ cu. m.}$$

Volume of 1 brick $=\frac{3}{4}$ m. $\times \frac{3}{8}$ m. $\times \frac{1}{4}$ m. $=\frac{9}{128}$ cu. m.

$\therefore$ the reqd. number of bricks

$$=\frac{423 \times 9}{2} \div \frac{9}{128} = \frac{423 \times 9 \times 128}{2 \times 9} = 27072.$$

19. A closed box is made of wood $\frac{1}{2}$ in thick and its external dimensions are 16″, 12″ and 10″. If 1 cu. ft. of this wood weigh $7\frac{1}{5}$ seers, find the weight of the box.

The inner length of the box $=(16-\frac{1}{2}\times 2)$ in. $=15$ in. ; breadth $=(12-\frac{1}{2}\times 2)$ in. $=11$ in. ; and height $=(10-\frac{1}{2}\times 2)$ in. $=9$ in. Now, the volume of the entire box $=16'' \times 12'' \times 10'' = 1920$ cu. in.

The internal volume of the box $=15'' \times 11'' \times 9'' = 1485$ cu. in.

$\therefore$ volume of the wood used $=1920$ cu. in. $-$ 1485 cu. in.

$$=435 \text{ cu. in.} = \frac{435}{12 \times 12 \times 12} \text{ cu. ft.}$$

$\because$ weight of 1 cu. ft. of wood $=\frac{36}{5}$ seers,

$\therefore$ the reqd. weight of the box $=\dfrac{36}{5}$ sr. $\times \dfrac{435}{12 \times 12 \times 12}$

$$=\tfrac{87}{48} \text{ seers} = 1 \text{ seer } 13 \text{ chh.}$$

20. A cistern, the inside of which is 5 ft. long, 4 ft. wide and $3\frac{2}{3}$ ft. deep, has 30 cu. ft. of water in it. Porous bricks, each measuring $9'' \times 3'' \times 2\frac{2}{3}''$ are placed under the water until the cistern is just brimful. If each brick absorbs $\frac{1}{17}$ of its own volume of water, find the number of bricks put in. [C. U. '39]

The internal volume of the cistern

$$=(5 \times 4 \times 3\tfrac{2}{3}) \text{ cu. ft.}=\tfrac{220}{3} \text{ cu. ft.}$$

There being 30 cu. ft. of water in the cistern, the volume of its empty space$=(\frac{220}{3}-30)$ or $\frac{130}{3}$ cu. ft.

Volume of one brick$=(9 \times 3 \times 2\frac{2}{3})$ cu. in.$=\frac{1}{24}$ cu. ft.

$\therefore$ each brick when placed under the water is to raise $\frac{1}{24}$ cu. ft. of water in the cistern, but it soaks water equal to $\frac{1}{17}$ part of its own volume,

$\therefore$ each brick fills only $(\frac{1}{24}-\frac{1}{24} \times \frac{1}{17})$ or $\frac{2}{51}$ cu. ft. of the empty space in the cistern

$\therefore$ ($\frac{130}{3}$ cu. ft.$\div\frac{2}{51}$ cu. ft.) or 1105 bricks are required to be dropped in the cistern.

21. A chest made of iron plate 2 inches thick, is 2 ft. 6 in. long, 2 ft. wide and 2 ft. 9 in. high and it weighs $23\frac{1}{4}$ cwt. Another chest 2 ft. 10 in. long and 2 ft. 1 in. broad is made of iron plate $2\frac{1}{2}$ in. thick and weighs $35\frac{1}{4}$ cwt. Find its height. [B. U. '22]

In the first instance, the total volume of the iron chest

$$=\tfrac{5}{2} \text{ ft.} \times 2 \text{ ft.} \times \tfrac{11}{4} \text{ ft.}=\tfrac{55}{4} \text{ cu. ft.}$$

As the iron plate is 2 in. thick, the internal dimensions will be each 4 in. less. $\therefore$ The internal length, breadth and height are 2 ft. 2 in., 1 ft. 8 in. and 2 ft. 5 in. respectively.

$\therefore$ the internal volume of the chest

$$=(\tfrac{13}{6} \times \tfrac{5}{3} \times \tfrac{29}{12}) \text{ cu. ft.}=\tfrac{1885}{216} \text{ cu. ft.}$$

$\therefore$ the volume of the iron plate$=(\frac{55}{4}-\frac{1885}{216})$ cu. ft.$=\frac{1085}{216}$ cu. ft.

$\therefore$ the weight of $\frac{1085}{216}$ cu. ft. of iron$=23\frac{1}{4}$ cwt.

$\therefore$ the weight of 1 cu. ft. of iron$=(\frac{93}{4} \times \frac{216}{1085})$ cwt.$=\frac{162}{35}$ cwt.

In the second instance, let the height of the chest be x ft. Its iron plate is $2\frac{1}{2}$ in. thick, $\therefore$ its internal dimensions will be each 5 in. less. $\therefore$ its internal length, breadth and height are 2 ft. 5 in., 1 ft. 8 in. and $(x-\frac{5}{12})$ ft. respectively.

Now, the total volume of the second chest

$=\frac{17}{6}\times\frac{25}{12}\times x$ cu. ft. $=\frac{425x}{72}$ cu. ft. and the internal volume

$=\frac{29}{12}\times\frac{5}{3}\times(x-\frac{5}{12})$ cu. ft. $=(\frac{145x}{36}-\frac{725}{432})$ cu. ft.

$\therefore$ the cubic content of the iron plate

$=\frac{425x}{72}$ cu. ft. $-(\frac{145x}{36}-\frac{725}{432})$ cu. ft. $=(\frac{135}{72}x+\frac{725}{432})$ cu. ft.

Again, the weight of the second chest $=35\frac{1}{4}$ cwt. $=\frac{141}{4}$ cwt.

But the weight of 1 cu. ft. of iron $=\frac{162}{35}$ cwt.

$\therefore$ the cubic content of the second iron plate

$=(\frac{141}{4}\div\frac{162}{35})$ cu. ft. $=\frac{1645}{216}$ cu. ft.

$\therefore$ $\frac{135}{72}x+\frac{725}{432}=\frac{1645}{216}$, or, $\frac{135}{72}x=\frac{1645}{216}-\frac{725}{432}=\frac{2565}{432}$

$\therefore$ $x=\frac{2565}{432}\times\frac{72}{135}=\frac{19}{6}=3\frac{1}{6}$

$\therefore$ the reqd. height $=3\frac{1}{6}$ ft. $=3$ ft. 2 in.

22. A cistern is filled in $3\frac{1}{2}$ hrs. by a pipe, 3 sq. inches in cross-section through which water flows at the rate of 6·4 miles per hour. What is the volume of the cistern? [R. M. A.]

Water flows at the rate of 6·4 miles per hour through the pipe whose cross-section is 3 sq. in. or, $\frac{3}{144}$ sq. ft. or, $\frac{1}{48}$ sq. ft.

$\therefore$ the volume of the water that flows through the pipe in 1 hr. into the cistern

$$=6{\cdot}4\times1760\times3\times\tfrac{1}{48}\text{ cu. ft.}=\frac{64\times176\times3}{48}\text{ cu. ft.}$$
$$=176\times4\text{ cu. ft.}$$

$\therefore$ the volume of the cistern = the volume of the water that flows through the pipe in $3\frac{1}{2}$ hrs.

$$=\frac{176\times4\times7}{2}\text{ cu. ft.}=2464\text{ cu. ft.}=91\text{ cu. yds. }7\text{ cu. ft.}$$

Exercise 8

1. Find the whole surface and the volume of a rectangular solid whose length, breadth and height are respectively 6 ft. 8 in., 4 ft. 6 in. and 5 ft. 4 in.

2. Find the volume of a cube whose edge measures 1 m. 2 dm.

3. The volume of a plate, 6 ft. 8 in. by $4\frac{1}{2}$ ft., is 70 cu ft., find its thickness.

4. How many cubes, whose edge measures 1 ft., can be made from one cubic yard of silver plate?

5. Find the edge of a cube whose volume is $91\frac{1}{8}$ cu. cm.

6. Find the width of a rectangular parallelopiped whose volume is 70 cu. ft., length 6′8″ and height 2 ft. 4 in.

7. Find the area of a surface of a cube whose volume is 49 cu. yd. 8 cu. ft.

8. How many bricks, each 1 ft. by 8 in. by 6 in., will be required to build a wall 10 ft. 6 in. long, 6 ft. 4 in. broad and 8 ft. high ?

9. A tank is 5 ft. 4 in. long and 3 ft. wide. How many cubic feet of water must be drawn off to make the water surface sink 6 inches ?

10. A cistern, $3\frac{1}{8}$ m. deep and having a square base, holds $53\frac{1}{8}$ cu. m. of water. Find its length and breadth.

11. A cubic metre of iron weighs 8 quintals. If 34 iron plates, each 10 m. by 3 m., weigh 170 quintals., find the thickness of the plate.

12. If it costs Rs. 10. 9 as. to paint the whole surface of a cube at Re. 1. 8 as. per sq. ft., find the length of each edge.

13. If a gallon of water weighs 10 lb. and one cubic foot holds 1000 oz. of water, how many gallons of water can a cistern, 12 ft. by 10 ft. by $2\frac{1}{2}$ ft., hold ?

14. The length of a room is twice its breadth and three times the height. If it holds 288 cu. metres of air, find the area of its floor.

15. A cistern of square base is 4 ft. deep and the water contained in it weighs 4 ton 1 qr. 12 lb. If 1 cu. ft. of water weighs 1000 oz., find the length and breadth of the cistern.

16. One cu. ft. of water weighs 1000 oz. If there be a rainfall of 1 inch on an acre of land, find the weight of the water.

17. The inside length, breadth and height of a rectangular vessel are 16 ft., 14 ft. and 10 ft. respectively. If one sq. ft. of lead weighs 12 oz. and the cost of one cwt. of lead be 9 s. 4 d., what should be the cost of lining the four sides of the vessel with lead ?

18. How many bricks, each $6'' \times 4'' \times 3''$, will be required to surround a rectangular garden 60 ft. long and 40 ft. wide by a wall 10 ft. high and 1 ft. 6 in. thick ?

19. A gravel path, 6 ft. wide, runs round a grass-plot 60 ft. long and 40 ft. wide. What is the cost of a coat of gravel on the path 3″ deep at 3 s. per cu. yd. ? [R. M. A.]

20. The length of a cistern $10\frac{1}{2}$ ft. deep is twice its breadth and it holds $37\frac{1}{2}$ tons of water. If 1 cu. ft. of water weighs 1000 oz., find the length and breadth of the cistern. [M.U. '26]

21. A closed box is made of wood half inch thick ; its external dimensions are 16″, 12″ and 7″. (1) Find the internal volume of the box. (2) How many sq. inches of wood are required for the box ? (3) If 1 cu. ft. of wood weighs 12 seers, find the weight of the box.

22. A box (without lid) is made of wood 1 cm. thick and its external length is 30 cm., breadth 21 cm. and height 16 cm. Find the cubic contents of the interior and the cost of painting the outside at 5 P. per sq. dm. ?

[N. B. The box being without a lid the internal height will be less than the external height by 1 cm. only ; but the external length and breadth each will be less by 2 cm. than the external ones.]

23. The internal length, breadth and depth of a cistern are 12 ft., 8 ft. and 6 ft. respectively. It contains water 4 ft. deep. How many bricks, each $6'' \times 4'' \times 3''$ are to be placed in the water so that it comes to the brim of the cistern.

[Suppose that bricks do not absorb water.]

24. A cubic foot of water weighs 1000 oz. and 4000 tons of water fall on two acres of land due to rains. Find the depth of the water in inches to 2 decimal places. [P. U. 1891]

25. A rectangular reservoir of water 100 ft long and 64 ft. wide has a pipe with a cross-section 2″ square attached to it. At what rate of speed per hour must water flow in through the pipe so that the water in the reservoir rises 2 ft. in 8 hrs. ? [B. U.]

26. The length and breadth of a room are as 4 : 3 and it contains 2304 cu. m. of air. If the cost of carpeting the floor at $33\frac{1}{3}$P. per sq. m. be Rs. 64, find the dimensions of the room.

27. The external length, breadth and height of a closed box are 3 ft., 2 ft. and $1\frac{1}{2}$ ft. respectively and it is made of wood 1″ thick. How many square feet of wood was required for it ? Find the cost of wood at Rs. 3 per cu. foot ? [M. U.]

28. A box (with lid) is made of wood $\frac{3}{8}$ inch thick. Its external length, breadth and height are $9\frac{3}{8}''$, 9″ and $7\frac{1}{3}''$ respectively. If 9 cu. in. of wood weighs 6 oz., find the weight of the box. [Cambridge]

PERCENTAGE

The term per centum or per cent. means for one hundred. I have Rs. 100 and if I spend Rs. 13, then I spend 13 per cent. of my money. The symbol % or the letters p. c. are used as an abbreviation for the words per cent. ; such as 13 per cent ; 13% or 13 p. c. Again, see that if I spend Rs. 7 out of Rs. 20, then I spend $\frac{7}{20}$th part of my money. Now, if we multiply the numerator and denominator of $\frac{7}{20}$ by 5, the fraction $=\frac{35}{100}$, which means 35 out of 100, *i. e.*, 35 per cent. So a fraction can be expressed as percentage by multiplying the fraction by 100 and conversely, a rate per cent. can be expressed in fraction by dividing it by 100 ; *e.g.*

$$\tfrac{9}{10}=\tfrac{9}{10}\times 100 \text{ or } 90 \text{ p. c. and } 15\%=\tfrac{15}{100}=\tfrac{3}{20}\text{th part.}$$

Examples [9]

1. 70 per cent of the population of a village are males ; find the number of females, if the population be 1250,

Out of 100 persons males are 70 and females 30 in number.

$\therefore$ $\frac{30}{100}$ of the population are females.

$\therefore$ number of females $=\dfrac{3\cancel{0}}{10\cancel{0}}\times 125\cancel{0}=375.$

2. There are 25600 persons in a town of whom 6400 are females, find the percentage of males.

Total number of males $=25600-6400=19200$

$\therefore$ out of 25600 persons 19200 are males

$\therefore$ ... 100 ... $\dfrac{19200\times 100}{25600}$ or 75 are males.

$\therefore$ the number of males in the town $=75$ p.c. of the population.

3. 15% of the boys of a school were absent, if 340 boys were present, find the total number of boys in the school.

15% of the boys were absent. $\therefore$ $(100-15)$ or 85% of the boys were present.

$\therefore$ 85% of total number of boys $=340$,

$\therefore$ 1% of the total number of boys $=\frac{340}{85}$

$\therefore$ 100% of the total number of boys $=\dfrac{340\times 100}{85}=400.$

$\therefore$ The required total number of boys $=400$.

[N.B. 100% of a number means the whole number itself.]

4. In an ornament the proportion of gold and silver was as 3 : 2. What per cent. of gold was there ?

The proportion of gold and silver was as 3 : 2. This means that of (3+2) or 5 parts there were 3 parts of gold *i.e.*, $\frac{3}{5}$ of the ornament is gold. $\frac{3}{5}$ part $=(\frac{3}{5}\times 100)\%=60\%$.

$\therefore$ there was 60% gold in the ornament.

5. The salary of a man increased from Rs. 250 to Rs. 300 ; find the percentage of the increment.

Rs. 300 − Rs. 250 = Rs. 50

$\therefore$ On Rs. 250 the increase is Rs. 50,

$\therefore$ the increment $=\frac{50}{250}$ or $\frac{1}{5}$ of the salary

$\therefore$ the percentage of increment $=(\frac{1}{5}\times 100)$ or 20%.

6. Being increased by 30% the income of a man became Rs. 390. What was his original income ?

If the present income be Rs. 130, the original income is Rs. 100,

$\therefore$ The original income $=\frac{100}{130}$ of the present income
$=\frac{100}{130}$ of Rs. 390 = Rs. 300.

7. The salary of an officer being increased 10% every year became Rs. 484 in the third year. What was his original salary ?

The salary of Rs. 100 being increased 10% becomes Rs. 110.

$\therefore$ the salary of a year is $\frac{100}{110}$ of that of the following year.

Here, the salary at the beginning of the 3rd year = Rs. 484.

$\therefore$ the salary at the beginning of the 2nd year $=\frac{100}{110}$ of Rs. 484

$\therefore$ the salary at the beginning of the first year
$=\frac{100}{110}\times\frac{100}{110}\times$ Rs. 484 = Rs. 400.

8. Gunpowder is composed of 65 per cent. of nitre, 20 per cent. of charcoal and 15 per cent. of sulphur. Find the quantity of each ingredient in a quintal of gunpowder.

Nitre = 65% of 1 quintal $=\frac{65}{100}$ of 100 kg. = 65 kg.

charcoal = 20% of 1 quintal $=\frac{20}{100}$ of 100 kg. = 20 kg.

and sulphur = 15% of 1 quintal $=\frac{15}{100}$ of 100 kg. = 15 kg.

9. If A's income be 25% more than B's, how much per cent is B's income less than A's ?

[**N.B.** Here Rs. 100 should be taken as the income of the man with respect to whom the other is stated to be more or less.]

Here, if B's income is Rs. 100, A's income=Rs. 125.

Now, you are to find B's income, if A's income be Rs. 100.

When A's income is Rs. 125, B's income = Rs. 100,

$\therefore$ ... ... Re. 1, ... = Re. $\frac{100}{125}$ = Re. $\frac{4}{5}$,

$\therefore$ ... ... Rs. 100, ... = Rs. $\frac{4\times100}{5}$ = Rs. 80.

$\therefore$ B's income is (100 − 80) or 20% less than A's income.

10. A man ate 10% of the eggs he had and sold $46\frac{2}{3}$% of the remaining eggs. If he had 72 eggs still left, how many eggs had he at first?

$$10\% = \frac{1}{10},\ 46\frac{2}{3}\% = \frac{140}{3\times100} = \frac{7}{15}.$$

The man ate 10% or $\frac{1}{10}$ of the number of eggs.

Then $(1 - \frac{1}{10})$ or $\frac{9}{10}$ of the eggs remained.

$\therefore$ he sold $\frac{7}{15}$ of $\frac{9}{10}$ or $\frac{21}{50}$ of the eggs.

$\therefore$ he had still left $(\frac{9}{10} - \frac{21}{50})$ or $\frac{12}{25}$ of the eggs.

$\therefore$ $\frac{12}{25}$ of the whole number of eggs = 72,

$\therefore$ the reqd. number of eggs $= \frac{72\times25}{12} = 150$.

11. The population of a country increased 7 p. c. in every 10 years. If its present population be 4007150, what was the population 20 years ago ? [M. U. 1885]

The population increases 7 p. c. in every 10 years, *i.e.*, it becomes 107 in place of 100. $\therefore$ 10 years ago the population was $\frac{100}{107}$ of the present population or $\frac{100}{107}\times4007150$ or 3745000.

$\therefore$ 20 years ago the population was $\frac{100}{107}\times3745000$ or 3500000.

[Vide Ex. 7]

12. 70 per cent. of the boys of a school are Hindu and 80% of the remainder are Mahomedan. If there be 322 more Hindus than Mahomedans, find the number of boys in the school.

Hindus=70%=$\frac{7}{10}$ of the whole number.

The remaining part $= 1 - \frac{7}{10} = \frac{3}{10}$.

$\therefore$ Mahomedans = 80% of $\frac{3}{10} = \frac{80}{100}\times\frac{3}{10} = \frac{6}{25}$,

The difference in number between the Hindus and the Mahomedans $=(\frac{7}{10}-\frac{6}{25})$ or $\frac{23}{50}$ of the whole number.

$\therefore$ $\frac{23}{50}$ of the total number $=322$,

$\therefore$ the reqd. number of boys $=322\div\frac{23}{50}$

$$=\frac{322\times50}{23}=700.$$

13. One-fifth of the candidates of an examination were girls and the rest were boys. 5% of the boys and 40% of the girls failed. If the number of candidates be 2500, find the percentage of successful candidates. [M.U.1928]

Number of girl candidates $=\frac{1}{5}\times2500=500$;

$\therefore$ the number of boys $=2500-500=2000$,

Number of unsuccessful candidates

$=5\%$ of $2000=\frac{5}{100}\times2000=100$,

Number of unsuccessful girls $=40\%$ of $500=\frac{40}{100}\times500=200$.

$\therefore$ Total number of plucked candidates $=200+100=300$,

$\therefore$ the number of successful candidates $=2500-300=2200$,

$\therefore$ of 2500 candidates 2200 candidates have passed,

$\therefore$ $(\frac{2200}{2500}\times100)\%$ or 88% of the candidates were successful.

14. In a school 60% of the students are boys and the rest are girls, 30 boys leave the school and 30 girls are admitted and it is found that the percentage of the boys is 45. Find the number of students in the school.

On the reduction of 30 boys there is a reduction of $(60-45)$ or 15 in the percentage of boys. $\therefore$ 15% of the total number $=30$.

$\therefore$ the reqd. number of students $=\dfrac{30\times100}{15}=200$.

15. A reduction of $12\frac{1}{2}\%$ in the price of salt enables a man to buy two seers more for 14 as. Find the original price of salt per seer. [D. B. '32]

$$12\tfrac{1}{2}\%=\frac{12\frac{1}{2}}{100}=\frac{1}{8}.$$

The present price is less than the original price by $\frac{1}{8}$th of it, $\frac{1}{8}$ of $14a.=\frac{7}{4}a.$ $\therefore$ the man can get 2 seers of salt for $\frac{7}{4}a.$ at the reduced price. $\therefore$ he can now get $2\times\frac{4}{7}\times14$ seers or 16 seers of salt for $14a.$ $\therefore$ $(16-2)$ or 14 seers of salt could be had for 14 $a.$ at the original price.

$\therefore$ the original price of salt per seer was 1 anna.

16. The price of wheat falls 4 per cent. How many quintals may now be bought for the money which was sufficient to buy 48 quintals at the higher price ?

Wheat worth Rs. 100 at the higher price can be had for Rs. 96 at the reduced price, *i.e.*, the present price is $\frac{96}{100}$ of the original price. ∴ for the same sum of money, we shall now get $\frac{100}{96}$ times the previous quantity of wheat.

∴ we shall get $\frac{48 \times 100}{96}$ quintals or 50 quintals of wheat for the price of 48 quintals at the higher rate.

17. In an examination 52 per cent. of the candidates fail in English and 42 per cent. fail in Mathematics. If 17 p.c. fail both in English and Mathematics, find the percentage of those who passed in both subjects. [C. U. 1917 ; P. U. 1924]

17 p. c. fail in both English and Mathematics.

∴ $(52-17)$ or 35 p.c. fail in English only ; and $(42-17)$ or 25 p.c. fail in Mathematics only. Hence, out of 100 candidates, $(17+35+25)$ or 77 candidates fail in one or both subjects.

∴ $(100-77)$ or 23% of the boys passed in both subjects.

18. 90 per cent of the boys of a school pass in spelling and 85 per cent in Arithmetic, 150 boys pass in both subjects and no boy fails in both. How many boys are there in the school ?

[D. B. 1923]

∵ No boy fails in both subjects,

∴ 10 boys who fail in spelling out of 100 boys pass in Arith.

∴ 85 boys who pass in Arith. include 10 boys who fail in spelling.

∴ $(85-10)$ or 75 boys pass in both subjects.

75 boys pass in both subjects out of 100 boys

∴ 150 boys ... ... 200 boys

∴ There were 200 boys in the school.

19. In an examination, 80 per cent. of the candidates passed in English, and 85% in Mathematics, while 75 p.c. passed in both English and Mathematics. If 45 candidates failed in both the subjects, find the total number of candidates. [C. U. 1938]

Suppose the number of candidates to be 100. Of them 80 candidates passed in English. These 80 candidates include 75

candidates who passed in both subjects and so there are $(80-75)$ or 5 candidates who failed only in Mathematics ($\because$ they passed in English). $\therefore$ 15 candidates who failed in Mathematics out of 100 candidates included 5 candidates who failed only in Mathematics. $\therefore$ the remaining $(15-5)$ or 10 candidates failed in both subjects.

Now, 10 candidates failed in both out of 100 candidates,

$\therefore$ 45 ... ... $\frac{100}{10}\times 45$

or, 450 candidates.

$\therefore$ the reqd. total number of candidates $=450$.

20. The price of cloth having been raised 75 per cent., how much per cent. must a householder reduce his consumption of that article so as not to increase his expenditure ? [C. U. '22]

Suppose the householder spends Rs. 100 for cloth.

Rs. 175 is now the price of the quantity of cloth which cost Rs. 100 previously.

$\therefore$ Rs. 175 is to be spent for cloth worth Rs. 100

$\therefore$ Re. 1 ... ... Re. $\frac{100}{175}$

$\therefore$ Rs. 100 ... ... Rs. $\frac{100}{175}\times 100$

or, Rs. $57\frac{1}{7}$.

$\therefore$ the consumption of cloth must be reduced by $(100-57\frac{1}{7})\%$

or $42\frac{6}{7}\%$ so as not to increase the expenditure.

21. The population of a town was 20,000. If the number of males increased by 10 p.c. and the number of females decreased by 6 per cent., the total population would have remained unchanged. Find the number of males and females. [C. U. 1937]

$10\%=\frac{1}{10}$, $6\%=\frac{6}{100}=\frac{3}{50}$. Let x be the number of males.

Then the number of females $=20000-x$.

$\because$ the total number remains the same,

$\therefore$ $\frac{1}{10}$ of the number of males $=\frac{3}{50}$ of the number of females

$\therefore$ $\frac{x}{10}=\frac{3}{50}(20000-x)$, or, $5x=60000-3x$,

or, $8x=60000$, $\therefore$ $x=\frac{60000}{8}=7500$.

$\therefore$ the number of males $=7500$ and that of females $=20000-7500=12500$.

22. At a Matriculation examination candidates took either Addl. Math. or History or both. If 65·3 p. c. took Addl. Math. and 61·7 p. c. took History, the total number of candidates being 20,000, find how many took up both the subjects. [C. U. '36]

Suppose the number of candidates to be 100.

Now, $65{\cdot}3+61{\cdot}7=127$, *i.e.*, the sum of the numbers of candidates taking Addl. Math. and History exceeds the total number 100 by 27. This excess of the number 27 is due to the fact that the candidates who took both subjects are included twice in these two cases.

$\therefore$ Out of 100 candidates, 27 took both subjects

$\therefore$ ... 20000 ... $\dfrac{27\times20000}{100}$ or 5400....

$\therefore$ The reqd. number$=5400$.

23. 40% of the gross receipts of a Tramway Company is taken up in meeting the working expenses, 40% of the remainder goes to reserve fund and the balance is paid away as dividends to shareholders at the rate of $3\frac{1}{3}$ per cent on their shares, the total value of which is Rs. 864000; find the amount of the gross receipts. [C. U. 1920 Addl.]

40% or $\frac{40}{100}$ or $\frac{2}{5}$ of the gross receipts meets the working expenses. $1-\frac{2}{5}=\frac{3}{5}$. $\therefore$ 40% of $\frac{3}{5}$ or $\frac{2}{5}$ of $\frac{3}{5}$ or $\frac{6}{25}$ of the receipts goes to reserve fund. Now, $\frac{2}{5}+\frac{6}{25}=\frac{16}{25}$.

$\therefore$ $(1-\frac{16}{25})$ or $\frac{9}{25}$ of the gross receipts is paid away as dividends. The total dividends$=3\frac{1}{3}\%$ of Rs. 864000

$=$Rs. $\frac{10}{3\times100}\times864000=$Rs. 28800.

$\therefore$ $\frac{9}{25}$ of the total receipts$=$Rs. 28800

$\therefore$ the gross receipts$=$Rs. $28800\times\frac{25}{9}=$Rs. 80000.

24. The population of a town, at the beginning of the year 1948 was 40000 and at the beginning of the year 1950 it was 44100. Find the rate per cent. of the yearly increase in the population. [U. U. '51]

Let r per cent. be the rate of increase in population and the number of years be n.

Then the last number$=$first number$\times\left(1+\dfrac{r}{100}\right)^n$

Here, the last number=44100, the first number=40000 and number of years=2.

$$\therefore\ 40000\times\left(1+\frac{r}{100}\right)^2=44100;$$

$$\text{or, }\left(1+\frac{r}{100}\right)^2=\frac{44100}{40000}=\frac{441}{400}\text{ or, }1+\frac{r}{100}=\frac{\sqrt{441}}{\sqrt{400}}=\frac{21}{20}.$$

$$\text{or, }\frac{r}{100}=\frac{21}{20}-1=\frac{1}{20},\quad\therefore\ r=\frac{1}{20}\times100=5.$$

$\therefore$ the rate per cent. of the yearly increase in population=5%.

Exercise 9

1. The net annual income of a man is Rs. 1900 after paying an income tax at 5 per cent. What is his total income ?

2. In an examination 34% of the candidates fail in Arithmetic and 42% in Algebra. If 20% fail both in Arithmetic and Algebra, find the percentage of those who pass in both the subjects.
[C.U. 1944]

3. 5% of the candidates for an examination were absent and 15% of those who sat for it got plucked. If 3230 candidates passed, what was the total number of the candidates ?

4. The price of cloth having been raised 65%, how much per cent. must a householder reduce his consumption of that article so as not to increase his expenditure ?

5. A reduction of $12\frac{1}{2}$% in the price of rice enables a man to buy two seers more for a rupee. What was the original price of rice ?

6. The price of petrol increased by 15%. Find how much per cent. must a man reduce his consumption of that article so as not to increase his expenditure ?

7. A man's capital increased 20% every year ; at the end of 4 years it was Rs. 5184 ; what was the capital at first ?
[C. U. '50]

SIMPLE INTEREST

Interest is the money paid for the use of money borrowed. The money borrowed or lent is called the **Principal**. The **amount** is the sum of the principal and the interest at the end of a certain time. **The rate of interest** is the money paid for the use of a certain sum for a certain time. If I borrow on the condition that for the use of every Rs. 100 in the loan for one year I shall pay an interest of Rs. 5, I am said to borrow at **the rate of 5 per cent. per annum**. Per annum means for a year. When interest is calculated simply on the original principal it is called **Simple Interest**. The total interest is obtained by subtracting the principal from the Amount.

Some formulas are given below for working out sums involving interest, but you should prefer to adopt the Unitary Method. Both the methods are shown here. In these formulas. P=principal, I=interest, T=time in years, R=rate per cent. and A=amount. In applying the formulas, principal, amount and interest should be expressed in rupees and the time in years. If the time is given in days or in days and years take 365 days for one year and if it is given in months and days take 30 days for a month and 12 months for a year for reducing the time into years.

(1) $I=\frac{P\times T\times R}{100}$,

(2) $A=P+I=P+\frac{P\times T\times R}{100}$

(3) $P=\frac{100\times I}{T\times R}$, or, $P=\frac{100\times A}{100+T\times R}$ [If the interest is given, use the first formula and if the amount is given, use the second formula for calculation of principal.]

(4) $R=\frac{100\times I}{T\times R}$, (5) $T=\frac{100\times I}{P\times R}$.

Examples [10]

1. Find the simple interest on Rs. 892 for 8 months at $6\frac{1}{4}\%$ per annum.

Here principal = Rs. 892 ; time = 8 months = $\frac{2}{3}$ year ; rate = $\frac{25}{4}\%$.

$\therefore$ the required interest $= \dfrac{\text{Principal} \times \text{Time} \times \text{Rate per cent}}{100}$

$$= \text{Rs.} \frac{892 \times 2 \times 25}{100 \times 3 \times 4} = \text{Rs.} \frac{223}{6} = \text{Rs. } 37.\ 2a.\ 8p.$$

2. Find the interest on £2187. 10*s*. for 219 days at $4\frac{1}{4}$ p. c. [C. S.]

£2187. 10*s*. $= £2187\frac{1}{2} = £\frac{4375}{2}$,

219 days $= \frac{219}{365}$ year $= \frac{3}{5}$ year,

Interest on £100 for 1 year $= £\frac{17}{4}$

$\therefore$ „ „ £ 1 „ „ $= £\frac{17}{4 \times 100}$

$\therefore$ „ „ £ 1 „ $\frac{3}{5}$ year $= £\frac{17}{4 \times 100} \times \frac{3}{5}$

$\therefore$ „ „ $£\frac{4375}{2}$ „ „ „ $= \frac{17 \times 3}{4 \times 100 \times 5} \times \frac{4375}{2}$

$$= £\frac{1785}{32} = £55.\ 15s.\ 7\tfrac{1}{2}d.$$

3. Find the interest on Rs. 525 from 8th February, to 21st April 1940 at 5%.

Here, the year 1940 being a *Leap Year* take 29 days for February.

$\therefore$ the total time = 21 days + 31 days + 21 days = 73 days
$= \frac{73}{365}$ year $= \frac{1}{5}$ year.

Here P = Rs. 525, T = $\frac{1}{5}$ year, R = 5%

$\therefore$ the reqd. interest $= \dfrac{P \times T \times R}{100}$

$$= \text{Rs.} \frac{525 \times 1 \times 5}{5 \times 100} = \text{Rs. } 5.\ 25 \text{ P.}$$

[**N. B.** Generally the first date is left out in counting the number of days from one date to another. Here the 8th February is not, therefore, taken into account, while the last date, the 21st April, is counted to find the number of days from 8th February to 21st Aprii. We have taken (29 – 8) or 21 days for February excluding the first 8 days.]

4. What will be the amount from Rs. 416. 66$\frac{2}{3}$ P. lent at 6% from the 4th January to the 17th March, 1936 ?

$$\text{Rs. } 416.\ 66\tfrac{2}{3}\text{ P} = \text{Rs. } 416\frac{2}{3} = \text{Rs. } \frac{1250}{3}$$

Time $=(27+29+17)$ days $=73$ days $=\frac{1}{5}$ year.

$$\text{Interest} = \frac{\text{Principal} \times \text{Time} \times \text{Rate per cent}}{100}$$

$$= \text{Rs. } \frac{1250 \times 1 \times 6}{4 \times 5 \times 100} = \text{Rs. } 5$$

∴ the reqd. amount $=$ Rs. 416. 66$\frac{2}{3}$ P. $+$ Rs. 5

$= $ Rs. 421. 66$\frac{2}{3}$ P.

5. What sum of money laid out at 5% per annum wili give one rupee interest a day ? [C. U. '25]

Here, interest $=$ Re. 1, time $=1$ day $=\frac{1}{365}$ year.

$$\therefore \text{ Principal} = \frac{100 \times \text{Interest}}{\text{Time} \times \text{Rate}} = \text{Rs. } \frac{100 \times 1}{\frac{1}{365} \times 5} = \text{Rs. } \frac{100}{\frac{1}{73}}$$

$= $ Rs. 7300.

6. What sum will amount to Rs. 100 in 5 years at 6$\frac{2}{3}$% simple interest ?

Int. on Rs. 100 for 1 yr. $=$ Rs. $\frac{20}{3}$

∴ " " " 5 yrs. $=$ Rs. $\frac{20}{3} \times 5 =$ Rs. $\frac{100}{3}$.

∴ When the amount is $(100+\frac{100}{3})$

or, Rs. $\frac{400}{3}$, the principal is Rs. 100,

∴ " " " Rs. 100, the principal is Rs. $\frac{100 \times 3}{400} \times 100$

$= $ Rs. 75.

7. What sum lent at 5% will amount to Rs. 5151 from June 11th to November 4th ? [P. U. '30]

Number of days from 11th June to 4th November

$=(19+31+31+30+31+4)$ days $=146$ days $=\frac{146}{365}$ year

$=\frac{2}{5}$ year.

Interest on Rs. 100 for 1 year = Rs. 5

$\therefore$,, ,, ,, $\frac{2}{5}$,, = Rs. $5\times\frac{2}{5}$ = Rs. 2

$\therefore$ Rs. 100 amounts to Rs. $(100+2)$ or Rs. 102 in $\frac{2}{5}$ years at 5%

Of the amount Rs. 102, the principal = Rs. 100

$\therefore$,, ,, Re. 1 ,, ,, = Rs. $\frac{100}{102}$

$\therefore$,, ,, Rs. 5151 ,, ,, = Rs. $\frac{100}{102}\times 5151$

= Rs. 5050.

8. A person finds that a fall of interest from 4 to $3\frac{3}{4}$ per cent. per annum diminishes his yearly income by Rs. 60. What is his capital ? [C. U. '33]

$4-3\frac{3}{4}=\frac{1}{4}$

The income falls by Re. $\frac{1}{4}$ when the capital is Rs. 100

$\therefore$,, ,, Re. 1 ,, ,, ,, Rs. 100×4

$\therefore$,, ,, Rs. 60 ,, ,, ,, Rs. $100\times 4\times 60$

or Rs. 24000

$\therefore$ the required capital = Rs. 24000.

9. If Rs. 750 amount to Rs. 873. 75 P. in 5 years and 6 months, find the simple interest per cent. per annum.

Principal = Rs. 700 ; interest = amount − principal

= Rs. 873.75 P. − Rs. 750 = Rs. 123.75 P. = Rs. $\frac{495}{4}$;

time = 5 years 6 months = $\frac{11}{2}$ years

$\therefore$ rate of interest p. c.

$$=\frac{100\times\text{Interest}}{\text{Principal}\times\text{Time}}=\frac{100\times\frac{495}{4}}{750\times\frac{11}{2}}=\frac{100\times 495\times 2}{4\times 750\times 11}=3.$$

$\therefore$ the rate of interest = 3%.

10. At what rate per cent simple interest, will £440. 6*s*. 8*d*., amount to £511 17*s*. 9*d*. in 5 years ? [D.B. '25 ; C.U. '12]

Principal = £440. 6*s*. 8*d*. = £$\frac{1321}{3}$.

Total interest $=£511.\ 17s.\ 9d. - £440.\ 6s.\ 8d.$

$$=£71.\quad 11s.\quad 1d. = £\frac{17173}{240}$$

Here interest on $£\frac{1321}{3}$ for 5 years $=£\frac{17173}{240}$,

$\therefore$ ... ... £1 ... ... $=£\frac{17173}{240}\times\frac{3}{1321}=£\frac{13}{80}$

$\therefore$ ... ... £1 ... 1 year $=£\frac{13}{80}\times\frac{1}{5}$

$\therefore$ ... ... £100... ... $=£\frac{13\times 100}{80\times 5}=£\frac{13}{4}$

$\therefore$ the reqd. rate of interest $=3\frac{1}{4}\%$.

11. At what rate per cent. per annum (simple interest) will a sum of money double itself in 10 years ? [C. U. '15]

Let the sum (*i.e.*, principal) be Rs. 100.

$\therefore$ total interest = Rs. 100

[The principal of Rs. 100 doubles itself *i.e.*. it amounts to Rs. 200. $\therefore$ interest $=(200-100)$ or 100 rupees.]

$\therefore$ The int. on Rs. 100 for 10 yrs. = Rs. 100

$\therefore$ " " " " " " 1 yr. = Rs. $\frac{100}{10}$ = Rs. 10.

$\therefore$ the reqd. rate of interest $=10\%$.

12. In what time will the interest on Rs. 250 amount to Rs. 300 at 10 p.c. ?

[N.B. Here the word "amount" may mislead you. 'Interest will amount to Rs. 300' means interest itself will be Rs. 300.]

Interest on Rs. 100 for 1 year = Rs. 10.

$\therefore$ " " Re. 1 " 1 " $=\frac{\text{Rs. } 10}{100}$ = Re. $\frac{1}{10}$

$\therefore$ " " Rs. 250 " " " = Re. $\frac{1}{10}\times 250$ = Rs. 25.

$\therefore$ the reqd. time = total interest $\div$ yearly interest of the principal = (Rs. 300 $\div$ Rs. 25) years = 12 years.

13. In what time will a sum of money be half as much again as itself at $6\frac{2}{3}$ p. c. ?

Let the capital be Rs. 100.

Here, the amount is $1\frac{1}{2}$ times the capital, *i. e.*, Rs. 150.

$\therefore$ the total interest = Rs. 150 $-$ Rs. 100 = Rs. 50.

Interest for 1 year = Rs. $\frac{20}{3}$.

$\therefore$ the reqd. time $=(50\div\frac{20}{3})$ years $=7\frac{1}{2}$ years.

14. In what time will the interest on a sum of money be $\frac{2}{5}$ths of the amount at 10 per cent. ?

Suppose, the amount = Rs. 100. ∴ interest = Rs. $100 \times \frac{2}{5}$ = Rs. 40.

∴ The capital = Rs. 100 − Rs. 40 = Rs. 60.

$$\therefore \text{ the reqd. time} = \frac{100 \times \text{Interest}}{\text{Principal} \times \text{rate of interest}}$$

$$= \frac{100 \times 40}{60 \times 10} \text{ years} = 6\tfrac{2}{3} \text{ years.}$$

15. At what rate per cent. will the interest on Rs. 800 in 4 years be the same as the interest on Rs. 625 for 8 years at 4 per cent. ? [C. U. '27]

In the second case, interest on Rs. 100 for 1 year = Rs. 4.

∴ „ „ „ „ „ Re. 1 „ = Re. $\frac{4}{100}$ = Re. $\frac{1}{25}$.

∴ „ „ „ „ „ „ „ 8 years = Re. $\frac{1}{25} \times 8$
= Re. $\frac{8}{25}$.

∴ „ „ „ „ „ Rs. 625 „ „ = Rs. $\frac{8}{25} \times 625$
= Rs. 200.

Now, in the first case,

Interest on Rs. 800 for 4 years = Rs. 200

∴ „ „ Re. 1 „ „ = Rs. $\frac{200}{800}$ = Re. $\frac{1}{4}$

∴ „ „ Re. 1 „ 1 year = Re. $\frac{1}{4 \times 4}$

∴ „ „ Rs. 100 „ „ „ = Rs. $\frac{1 \times 100}{4 \times 4}$ = Rs. $6\frac{1}{4}$.

∴ the rate of interest = $6\frac{1}{4}$%.

16. In what time will the simple interest on Rs. 900 at 6 p.c. be equal to the simple interest on Rs. 540 for 8 years at 5 p. c. ? [C. U. '28]

In the second case,

interest on Rs. 100 for 1 year = Rs. 5

∴ „ Rs. 540 „ = Rs. $\frac{5}{100} \times 540$ = Rs. 27

∴ „ „ „ 8 years = Rs. 27×8 = Rs. 216.

Now, in the first case,

interest on Rs. 100 for 1 year = Rs. 6

∴ „ Rs. 900 „ „ = Rs. $\frac{6 \times 900}{100}$ = Rs. 54.

∴ the reqd. time = total interest ÷ yearly interest of the capital
= (216 ÷ 54) years = 4 years.

17. What sum of principal money, lent out at 5 p. c., will produce in 4 years the same amount of interest as Rs. 250 lent out at 3 p. c. will produce in 6 years ? [C. U. '30]

In the second case,

$$\text{interest} = \frac{\text{Principal} \times \text{Time} \times \text{rate of interest}}{100}$$

$$= \text{Rs. } \frac{250 \times 6 \times 3}{100} = \text{Rs. } 45.$$

$\therefore$ the interest in the first case = Rs. 45.

$$\therefore \text{ the reqd. principal} = \frac{100 \times \text{Interest}}{\text{Time} \times \text{rate of interest}}$$

$$= \text{Rs. } \frac{100 \times 45}{4 \times 5} = \text{Rs. } 225.$$

18. If Rs. 450 amount to Rs. 540 in 4 years at simple interest, what sum will amount to Rs. 637. 50 P. in 5 years at the same rate ?

In the first case,

interest on Rs. 450 for 4 years = Rs. 540 − Rs. 450 = Rs. 90,

„ „ Rs. 100 for 1 year = Rs. $\frac{90 \times 100}{450 \times 4}$ = Rs. 5

In the second case,

the given amount = Rs. 637.50 P. = Rs. $\frac{1275}{2}$

Here, interest on Rs. 100 for 5 years at 5% = Rs. 5×5 = Rs. 25.

Then the amount = Rs. 100 + Rs. 25 = Rs. 125.

$\therefore$ When the amount is Rs. 125, the principal = Rs. 100

$\therefore$ „ „ „ Rs. $\frac{1275}{2}$ „ „ = Rs. $\frac{100}{125} \times \frac{1275}{2}$

= Rs. 510

$\therefore$ the reqd. principal = Rs. 510.

19. A certain sum of money at simple interest amounts to Rs. 560 in 3 years and to Rs. 600 in 5 years. Find the rate of interest and the sum. [C. U. '38]

Amount = Principal + interest

$\therefore$ Principal + interest for 5 years = Rs. 600

Again, Principal + interest for 3 years = Rs. 560

$\therefore$ (subtracting) interest for 2 years = Rs. 40

$\therefore$ interest for 1 year = Rs. 20,

$\therefore$ interest for 3 years = Rs. 60.

$\therefore$ Principal = Amount for 3 years − interest for 3 years
= Rs. 560 − Rs. 60 = Rs 500.

$\therefore$ interest on Rs. 500 for 1 year = Rs. 20

$\therefore$ „ Rs. 100 „ = Rs. $\frac{20}{5}$ = Rs. 4.

$\therefore$ the rate of interest = 4% ; the sum = Rs. 500.

20. The simple interest on Rs. 400 for 5 years together with that on Rs. 600 for 4 years came to Rs. 132, the rate being the same in both the cases. Find the rate per cent. of interest. [C. U. '39]

Interest on Rs. 400 for 5 years = Interest on Rs. 2000 for 1 year,
and interest on Rs.600 for 4years = interest on Rs. 2400 for 1 year

$\therefore$ the total interest Rs. 132 = Interest on Rs. 4400 for 1 year

$\therefore$ interest on Rs. 4400 for 1 year = Rs. 132

$\therefore$ „ Rs. 100 „ „ = Rs. $\frac{132 \times 100}{4400}$ = Rs. 3

$\therefore$ the reqd. rate of interest = 3%.

21. The principal and interest for 5 years are together Rs. 306 and the interest is $\frac{9}{25}$ of the principal. Find the principal and the rate of interest. [D. B. '36]

Let the principal be x rupees.

$\therefore$ $x + \frac{9}{25}x = 306$ ($\because$ interest $= \frac{9}{25}$ of the principal)

or, $\frac{34}{25}x = 306$, $\therefore$ $x = \frac{306 \times 25}{34} = 225.$

$\therefore$ the reqd. principal = Rs. 225.

Total interest = Rs. 306 − Rs. 225 = Rs. 81.

$\therefore$ the reqd. rate of interest $= \frac{100 \times 81}{225 \times 5}\% = 7\frac{1}{5}\%.$

22. In what time will Rs. 4000 at 3 p.c. per annum produce the same income as Rs. 5000 in 5 years at 4 p. c. simple interest ? [C. U. '40]

In the second case,
interest on Rs. 100 for 5 years = Rs. 4×5 = Rs. 20

$\therefore$ „ Rs. 5000 5 years = Rs. 20×50 = Rs. 1000

$\therefore$ in the first case also the total interest = Rs. 1000.

But interest on Rs. 100 for 1 year = Rs. 3

$\therefore$ „ Rs. 4000 „ „ = Rs. 3×40 = Rs. 120.

$\therefore$ the reqd. time = $(1000 - 120)$ years = $8\frac{1}{3}$ years.

23. In what time will the simple interest on Rs. 400 at 5 p.c. be equal to the simple interest on Rs. 500 for 4 years at 6·25% ?

Here rate of interest = $6{\cdot}25\% = \frac{625}{100}\% = \frac{25}{4}\%$

$\therefore$ From the second case,

$$\text{Total interest} = \frac{\text{Principal} \times \text{Time} \times \text{rate of interest}}{100}$$

$$= \text{Rs. } \frac{500 \times 4 \times \frac{25}{4}}{100} = \text{Rs. } 125$$

$\therefore$ total interest in the first case also = Rs. 125

$$\therefore \text{ the reqd. time} = \frac{100 \times \text{Interest}}{\text{Principal} \times \text{rate of interest}}$$

$$= \frac{100 \times 125}{400 \times 5} \text{ years} = 6\tfrac{1}{4} \text{ years.}$$

24. A sum of money amounts to Rs. 700 in 4 years at the rate at which a sum of money doubles itself in 10 years. Find the sum.

In the second case, let the principal be Rs. 100

$\therefore$ interest + principal = Rs. 200

$\therefore$ interest on Rs. 100 for 10 years = Rs. 100

$\therefore$ „ „ „ „ „ 1 year = Rs. 10.

$\therefore$ in the first case,

The int. on Rs. 100 for 4 yrs. = Rs. 40.

$\therefore$ of the amount (100 + 40) or 140 rupees, the principal = Rs. 100

$\therefore$ „ „ Re. 1 „ „ „ „ „ = Rs. $\frac{100}{140}$

$\therefore$ „ „ Rs. 700 „ „ „ Rs. $\frac{100}{140} \times 700$ = Rs. 500.

$\therefore$ The reqd. sum = Rs. 500.

25. At what rate will Rs. 500 amount to Rs. 700 in a time in which 120 dollars produce 15 dollars at 4% simple interest ?

From the second case,

The int. on 100 dollars for 1 yr. = 4 dollars

$\therefore$ „ „ „ 120 „ „ = $\frac{4}{100} \times 120$ dollars = $\frac{24}{5}$ dollars.

$\therefore$ The time = $(15 \div \frac{24}{5})$ yrs. = $\frac{25}{8}$ yrs.

Now, in the first case,

The total interest = Rs. 700 − Rs. 500 = Rs. 200

and time $= \frac{25}{8}$ years.

$\therefore$ Int. on Rs. 500 for $\frac{25}{8}$ yrs. = Rs. 200

$\therefore$ " " " 1 yr. = Rs. $\frac{200 \times 8}{25}$ = Rs. 64

$\therefore$ Int. on Rs. 100 for 1 yr. = Rs. $\frac{64}{5}$

$\therefore$ The reqd. rate of interest $= \frac{64}{5}\% = 12\frac{4}{5}\%$.

26. A sum of money amounts in 4 years at 5% to Rs. 480, in how many years will it amount to Rs. 560 ?

In the first case, interest on Rs. 100 for 4 years at 5%

= Rs. 5×4 = Rs. 20.

$\therefore$ the amount = Rs. (100 + 20) = Rs. 120.

When the amount is Rs. 120, the principal is Rs. 100

$\therefore$ " " " " Re. 1 " " Re. $\frac{100}{120}$ or Re. $\frac{5}{6}$.

$\therefore$ when amount is Rs. 480, the principal is Re. $\frac{5}{6} \times 480$ = Rs. 400

$\therefore$ in the second case, interest = Rs. 560 − Rs. 400 = Rs. 160

and principal = Rs. 400.

Now, interest on Rs. 400 for 1 year at 5% = Rs. 5×4 = Rs. 20.

$\therefore$ the reqd. time = (Rs. 160 ÷ Rs. 20) years = 8 years.

27. A sum of money increases by $\frac{1}{8}$th of itself every year and in 7 years it amounts to £ 900. Find the sum.

Suppose, the principal = £ 1. $\therefore$ interest on £ 1 for 1 year = £$\frac{1}{8}$,

$\therefore$ its interest for 7 years = £ $\frac{7}{8}$.

$\therefore$ principal + interest = £ $(1 + \frac{7}{8})$ or £ $\frac{15}{8}$.

$\therefore$ When the amount is £ $\frac{15}{8}$, the principal = £ 1.

" " " " £ 1 " " = £ $\frac{8}{15}$

" " " " £ 900 " " = £ $\frac{8 \times 900}{15}$ = £ 480.

$\therefore$ the reqd. principal = £ 480.

28. A sum of money doubles itself in 10 years. In how many years will it treble itself ?

Let the principal be Rs. 100, which amounts to Rs. 200 in 10 years.

$\therefore$ interest on Rs. 100 for 10 years = Rs. 100

$\therefore$ interest on Rs. 100 for 1 year = Rs. $\frac{100}{10}$ = Rs. 10.

Now, if Rs. 100 amounts to Rs. 300, then the total interest on Rs. 100 = Rs. 300 − Rs. 100 = Rs. 200.

∴ the reqd. time = total interest ÷ interest of the principal for 1 year = (200 ÷ 10) years = 20 years.

29. A lends Rs. 500 to B, and a certain sum to C, at the same time, at 8 per cent. simple interest. If in 4 years he altogether receives Rs. 210 as interest from the two, find the sum lent to C. [D. B. '37]

Both the sums are lent for 4 years at 8 p. c. interest.

Now, the int. on Rs. 100 for 4 yrs. at 8% = Rs. 32

∴ ,, ,, ,, Rs. 500 ,, ,, ,, = Rs. 32×5 = Rs. 160.

∴ total interest on the sum lent to C

= Rs. 210 − Rs. 160 = Rs. 50.

∴ in the second case

Rs. 32 is interest for 4 yrs. on Rs. 100

∴ Re. 1 ,, ,, ,, ,, Rs. $\frac{100}{32}$

∴ Rs. 50 ,, ,, ,, ,, Rs. $\frac{100 \times 50}{32}$.

or Rs. $\frac{625}{4}$ or Rs. 156. 25 P.

∴ The sum lent to C = Rs. 156. 25 P.

30. *A* lends Rs. 2000 to *B* and Rs. 2200 to *C* and thereby derives the total annual income of Rs. 179. If the rate of interest in the latter be $\frac{1}{2}$% higher than in the former, find the two rates of interest.

Here, interest on Rs, 100 for 1 year at $\frac{1}{2}$% = Re. $\frac{1}{2}$

∴ interest on Rs. 2200 for 1 year = Re. $\frac{1}{2} \times 22$ = Rs. 11.

∴ the total interest on the two sums (*i.e.*, Rs. 4200) for 1 yr. at the same equal rate of interest = (Rs. 179 − Rs. 11) = Rs. 168.

∴ interest on Rs. 100 = Rs. $\dfrac{168 \times 100}{4200}$ = Rs. 4.

∴ the first rate of interest = 4%

and the second rate of interest = $4\frac{1}{2}$%.

31. The interest on Rs. 400 at 4 per cent for a certain period of time and that on Rs. 500 for 2 years more at 5 p. c. are together Rs. 173. For what periods are the interests calculated ?

Here, the sum of Rs. 500 is lent for 2 years more.

Its interest for 2 years at 5% = Rs. $\dfrac{500 \times 2 \times 5}{100}$ = Rs. 50.

$\therefore$ total interest on Rs. 400 at 4% and on Rs. 500 at 5% for the same period = (Rs. 173 − Rs. 50) = Rs. 123.

Now int. on Rs. 400 for 1 year at 4% = Rs. 4×4 = Rs. 16

and „ „ Rs. 500 „ 1 „ „ 5% = Rs. 5×5 = Rs. 25

$\therefore$ total interest on the two sums for 1 year

= Rs. (16 + 25) = Rs. 41.

$\therefore$ interest of Rs. 123 is derived from the two sums in (123 ÷ 41) years or 3 years.

$\therefore$ the interest on the first sum is calculated for 3 years and that on the second sum for 5 years.

32. A person who pays 4 p. in the rupee income-tax, finds that a fall of interest from 4 to $3\frac{3}{4}$ p. c. diminishes his net yearly income by Rs. 47. What is his capital? [D. B. 1933]

In the first case, income-tax on Rs. 4 = 4 p. $\times$ 4 or 16 p.

$\therefore$ the net income after paying income-tax = (Rs. 4 − 16 p.)

= 752 p.

In the second case, income-tax on Rs. $3\frac{3}{4}$ = 4 p. $\times \frac{15}{4}$ = 15 p.

$\therefore$ net income after paying income-tax = (Rs. $3\frac{3}{4}$ − 15 p.) = 705p.

$\therefore$ the difference of the two net incomes = (752 − 705)p.

= 47 p. = Re. $\frac{47}{192}$, when the capital is Rs. 100.

$\therefore$ the income falls by Rs. $\frac{47}{192}$, when the capital is Rs. 100,

$\therefore$ ... ... Re. 1 ... ... Rs. $\frac{100 \times 192}{47}$,

$\therefore$ ... ... Rs. 47 ... ... Rs. $\frac{100 \times 192 \times 47}{47}$

or, Rs. 19200.

*33. Divide Rs. 2600 into 3 parts so that the interest on the 1st part at 4%, on the 2nd part at 6 p. c. and on the third at 8%, may be the same.

If interest on the three parts at 4%, 6% and 8% respectively be the same, then evidently the second part must be $\frac{4}{6}$ or $\frac{2}{3}$ of the first, and the third part must be $\frac{4}{8}$ or $\frac{1}{2}$ or the first part.

Now, $1 + \frac{2}{3} + \frac{1}{2} = \frac{13}{6}$: $\therefore$ $\frac{13}{6}$ of the first part = Rs. 2600,

$\therefore$ the first part = Rs. $\frac{2600 \times 6}{13}$ = Rs. 1200.

$\therefore$ the first part = Rs. 1200, the second part = Rs. $1200 \times \frac{2}{3}$ = Rs. 800 and the third part = Rs. $1200 \times \frac{1}{2}$ = Rs. 600

34. Divide Rs. 3400 into 3 parts so that interests on them at 3, 4, and 6 p. c. for 4, 6 and 10 months respectively may be all equal.

Interest for 4 months at 3% = Interest for 12 months at 1%

...	6	...	4% =	...	24	... ...
...	10	...	6% =	...	60	... ...

$\therefore$ the sums of money to be laid out at the second and third rates are $\frac{12}{24}$ or $\frac{1}{2}$ and $\frac{12}{60}$ or $\frac{1}{5}$ respectively of the sum to be laid out at the first rate of interest.

Now, $1+\frac{1}{2}+\frac{1}{5}=\frac{17}{10}$;

$\therefore$ $\frac{17}{10}$ of the first part = Rs. 3400

$\therefore$ the first part = Rs. $\dfrac{3400\times10}{17}$ = Rs. 2000,

the second part = Rs. $2000\times\frac{1}{2}$ = Rs. 1000

and the third prat = Rs. $2000\times\frac{1}{5}$ = Rs. 400.

35. The interest on a sum of money at the end of $6\frac{1}{4}$ years is $\frac{5}{16}$ of the sum itself ; what rate per cent. was charged ?

[C. U. 1946]

If Rs. 100 be the principal, interest on Rs. 100 for $6\frac{1}{4}$ or $\frac{25}{4}$ years = Rs. $100\times\frac{5}{16}$.

$\therefore$ Interest on Rs. 100 for 1 year = Rs. $100\times\frac{5}{16}\times\frac{4}{25}$ = Rs. 5.

$\therefore$ the reqd. rate of interest = 5%.

36. Two equal sums were lent out at 5% and 4% respectively and the joint interest amounted in 3 years to Rs. 405 ; find the sums. [B. C. S. '50]

Let each of the two equal sums be Rs. 100.

The first interest on Rs. 100 for 3 years at 5%

= Rs. 5×3 = Rs. 15,

The second interest on Rs 100 for 3 years at 4%

= Rs. 4×3 = Rs. 12.

$\therefore$ Rs. (15+12) or Rs. 27 is the total interest, if each sum = Rs. 100

$\therefore$ Re. 1 is the total interest, if each sum = Rs. $\frac{100}{27}$

$\therefore$ Rs. 405 ... ... = Rs. $\dfrac{100\times405}{27}$

= Rs. 1500.

$\therefore$ each sum = Rs. 1500.

37. The simple interest on a certain sum for 9 months at 5% is Rs. 125 less than the simple interest on the same sum for 15 months at 4%. Find the principal. [P. U. '20]

Let Rs. 100 be the principal.

In the first case, interest on Rs. 100 for 1 year=Rs. 5.

$\therefore$ interest on Rs. 120 for 9 months=Rs. $\frac{5\times9}{12}$=Rs. $\frac{15}{4}$.

In the second case, interest on Rs. 100 for 15 months

$$=\text{Rs. } \frac{4\times15}{12}=\text{Rs. } 5.$$

$\therefore$ Rs. $(5-\frac{15}{4})$ or Rs. $\frac{5}{4}$ is the difference of two interests when the principal is Rs. 100.

$\therefore$ Rs. 125 is the difference of two interests when the principal=Rs. $\dfrac{100\times4}{5}\times125$=Rs. 10000.

38. A Bank pays $1\frac{1}{2}$% interest for Savings Bank deposits. A man deposits Rs 350 at the beginning of the year in the Savings Bank account. After 4 months he withdraws Rs. 50. After another 3 months he deposits Rs. 100. Find the interest that he receives at the end of the year. [W.B.S.F. 1954 Compt.]

Here Rs. 350 is invested for the first 4 months, then Rs. (350 − 50) or Rs. 300 for 3 months and Rs. (300+160) or Rs. 460 for the remaining 5 months.

Now, interest on Rs. 350 for 4 months at $1\frac{1}{2}$%

$$=\text{Rs. } \frac{3\times350}{2\times100}\times\frac{4}{12}=\text{Rs. } \frac{7}{4}.$$

Interest on Rs. 300 for 3 months at the same rate

$$=\text{Rs. } \frac{3\times300}{2\times100}\times\frac{3}{12}=\text{Rs. } \frac{9}{8}.$$

and interest on Rs. 460 for 5 months=Rs. $\dfrac{3\times460}{2\times100}\times\dfrac{5}{12}$= Rs. $\dfrac{23}{8}$.

$\therefore$ the reqd. total interest=Rs. $(\frac{7}{4}+\frac{9}{8}+\frac{23}{8})$=Rs. $\frac{46}{8}$=Rs. 5.75 P.

39. A sum of Rs. 18750 is left by a father to be divided between two sons of 12 and 14 years of age, so that when they attain majority at 18, the amount (principal plus interest) received by each at 5% simple interest will be the same. Find the sum allotted at present to each son.

Here, the amount from the sum allotted to the first son in 6 years at 5% = the amount from the sum allotted to the second son in 4 years at 5%.

$\therefore$ $\frac{130}{100}$ of the principal of the first son $= \frac{120}{100}$ of the principal of the second son.

$\therefore$ Capital of first son : Capital of second son

$$= \frac{120}{100} : \frac{130}{100} = 12 : 13.$$

$\therefore$ the first son has received $\frac{12}{12+13}$ of Rs. 18750 or Rs. 9000.

$\therefore$ the second son has received Rs.$(18750 - 9000)$ or Rs. 9750.

[**Alternative Method**—Suppose the first son was given x rupees and the second son Rs. $(18750 - x)$. Now, the amount from x rupees in 6 years at 5% $= \frac{130}{100}x = \frac{13}{10}x$; and the amount from Rs. $(18750 - x)$ in 4 years at 5% $= \frac{120}{100}(18750 - x)$ rupees $= \frac{6}{5}(18750 - x)$ rupees. $\therefore$ $\dfrac{13x}{10} = \dfrac{6}{5}\left(18750 - x\right)$......]

Exercise 10

1. What sum of money must be lent out at $3\frac{3}{4}$% per annum simple interest in order to amount to Rs. 1767.50 P. in $2\frac{1}{2}$ years ?

2. What sum of money will amount to Rs. 1532.25 P. in 3 years at $4\frac{1}{2}$ p. c. simple interest ?

3. A sum of money invested at $4\frac{1}{16}$ per cent. gives Re. 1 as interest per day. Find the sum. [C. U. '35, '37]

4. What sum will amount to Rs. 500 in 5 years at 5 p. c. ? [C. U. '43]

5. A sum of rupees 425 was lent at simple interest. At the end of 9 months the debt was cancelled by the payment of Rs. 437. 75 P. What was the rate of interest ?

(Here time = 9 months $= \frac{3}{4}$ year ; Interest = Rs. 437.75 P. − Rs. 425 = Rs. $12\frac{3}{4}$ = Rs. $\frac{51}{4}$.)

6. At what rate will a sum of money treble itself in 25 years ? [C. U. '36]

[Suppose capital = Rs. 100. $\therefore$ total interest = Rs. 300 − Rs. 100 = Rs. 200]

7. I have to pay 2 P. as interest on one rupee for one month, what is the rate per cent. per annum ? [C. U. '19]

8. At what rate per cent. will Rs. 5026. 10 as. 8 p. amount to Rs. 5780. 10 as. 8 p. in 3 years 4 months ? [D. B. '26]

9. In what time will a sum of money double itself at 6 p. c. ? [C. U. 1910]

10. In what time will Rs. 2125 amount to Rs. $2943\frac{1}{3}$ at $5\frac{1}{2}$ p. c. simple interest ? [C. U. '44]

11. A sum of money amounts to Rs. 983. 14 as. in $3\frac{1}{2}$ years at $4\frac{1}{2}$%. What will it amount to in $2\frac{1}{2}$ years at $5\frac{1}{2}$% ? [P.U. '28]

[First find the principal in the first case and then the amount from that principal in the second case.]

12. A certain sum of money at simple interest amounts to Rs. 632. 50 P. in 3 years and to Rs. 673. 75 P. in 4 years 6 months. Find the sum and the rate of interest. (See Ex. 20)

13. If the principal and interest for 5 years together amount to Rs. 1100 and the interest is $\frac{3}{8}$ of the principal, find the principal and the rate per cent. per annum. (See Ex. 22) [C. U. '34]

14. A certain sum amounted to Rs. 14400 at 4% simple interest in a period of time in which Rs. 9000 amounted to Rs. 12150 at 7%. What was the sum ? [C. U. '41]

[Here first find the time from the second case and then the principal from the first case.]

15. What principal in 19 years at $7\frac{1}{2}$ p.c. simple interest will earn the same interest as Rs. 950 in 8 years at 6 p.c. ? [D. B. '34 Addl.]

16. At what rate per cent. will Rs. 300 produce the same interest in 3 years as Rs. 800 produces in 6 months at 9 p.c. ?

17. At what rate p. c. will the interest on a sum of money be $\frac{2}{5}$ of the amount in 10 years ?

18. A lent Rs. 400 to B for 3 years and Rs. 500 to C for 4 years. If he altogether received Rs. 160 as interest, find the rate per cent. (See Ex. 21)

19. What sum laid out at 4% will give 2 as. interest a day ?

20. A man invested Rs. 450 at $4\frac{1}{2}$ p. c. and Rs. 550 at 6%. How much per cent. interest did he get on his investment ?
[D. B. '34]

21. A sum of money invested at 5 p. c. amounts in 6 years to Rs. 1326, in what time will it amount to Rs. 1530 ?
[P.U. 1891]

22. Find in what time a given sum of money will quadruple itself, if lent out at simple interest at the rate of a pice per rupee per month. [P.U. '22]

23. On what capital will the interest for 219 days at 4% per annum amount to £14. 2s. 6d. ? [C.U. 1909]

24. Find the amount when a sum of Rs. 440 bears simple interest at $4\frac{1}{2}$% per annum for 3 yrs. 2 months 10 days.
[U. U. '51]

[Here take 30 days for a month and 12 months for a year]

25. Find the interest on Rs. 12,500 at $8\frac{3}{4}$% for the period from 14th April, 1953 to 26th June next. [G.U. '54]

26. What sum borrowed on 11th June will amount to Rs. 5151 on 4th November of the same year at 5% per annum simple interest ?

27. A sum of money amounts to Rs. 3576 in $4\frac{3}{8}$ years at $10\frac{1}{2}$% simple interest ; when will it double itself at the same rate ?
[U.U. '47]

[Here Rs. 100 will double itself in the same time as the given capital will take to double itself. $\therefore$ the reqd. time $=(100\div10\frac{1}{2})$ years.]

PROFIT AND LOSS

N. B. (*a*) The price at which an article is bought is called its *cost price* and the price at which it is sold is called its *selling price* or *sale price.* If a man buys goods at a certain price and sells them at a higher price, the difference between the two prices is called his *profit or gain* ; but if the cost price is greater than the selling price, the difference between them is called his *loss.* Profit or loss, is always calculated on the cost price, (that is, so much per cent. on the cost price), e.g., if an article is bought at Rs. 10 and sold at Rs. 12, there is a gain of Rs. 2 on Rs. 10. Again if it is sold at Rs. 7, there is a loss of Rs. 3 on Rs. 10. Profit = selling price − cost price. Loss = cost price − selling price.

(*b*) If we gain 5% by selling goods, it is understood that if the cost price be Rs. 100, the selling price will be Rs. 105, *i.e.*, the selling price $= \frac{105}{100}$ of the cost price.

If we incur a loss of 6%, it is understood that if the cost price be Rs. 100, the selling price will be Rs. 94, *i.e.*, selling price $= \frac{94}{100}$ of the cost price.

Examples [11]

1. By selling a book for Rs. 10. 50 P. a man gains 5%, find the cost of the book.

Here gain = 5%, ∴ if the cost price is Rs. 100, the selling price is Rs. 105.

When the selling price is Rs. 105, the cost price is Rs. 100,

∴ " " Rs. 1, " " Rs. $\frac{100}{105}$

∴ " " Rs. $10\frac{1}{2}$, " Rs. $\frac{100}{105} \times \frac{21}{2}$ = Rs. 10.

[In brief : The profit being 5%, the cost price = Rs. 100, when the selling price = Rs. 105.

Now of these two prices, the price that is to be found should be taken as numerator and the other price as denominator and the fraction is to be multiplied by the given price. Here the cost price being wanted, we take the fraction $\frac{100}{105}$, which is again multiplied by the given selling price Rs. $10\frac{1}{2}$ to find the required cost price.]

2. A man lost 6% by selling a house at Rs. 4700, find the cost price of the house.

Here, the house is sold at a loss of 6%,

$\therefore$ when the cost price is Rs. 100, the selling price is Rs. 94.

$\therefore$ the reqd. cost price$=\frac{100}{94}\times 4700=$Rs. 5000.

3. A man buys an article at Rs. 10, at what price should he sell it to gain 5 p.c. ?

To gain 5%, when the cost price is Rs. 100, the selling price should be Rs. 105.

$\therefore$ the reqd. selling price$=$Rs. $\frac{105}{100}\times 10=$Rs. 10. 50 P.

4. By selling goods at Rs. 240 a man gains 25 p. c. How much would he gain per cent. by selling them at Rs. 216 ? [C. U. '17]

First Method

In the first case the gain being 25%, when the cost price is Rs. 100, the selling price is 125.

$\therefore$ the cost price$=$Rs. $\frac{100}{125}$ of Rs. 240$=$Rs. 192.

Now, if these goods be sold for Rs. 216, the gain$=$Rs. $(216-192)$ $=$Rs. 24. $\because$ gain or loss is calculated on the cost price,

$\therefore$ the gain on Rs. 192$=$Rs. 24

$\therefore$,, ,, Re. 1 $=$Rs. $\frac{24}{192}$,

$\therefore$,, Rs. 100$=$Rs. $\frac{24}{192}\times 100=$Rs. $12\frac{1}{2}$

$\therefore$ the reqd. gain$=12\frac{1}{2}$%.

5. A house was sold for Rs. 4500 at a profit of $12\frac{1}{2}$ p. c. What per cent. would have been lost if it had been sold for Rs. 3800 ? [C. U. '24]

Second Method : By selling for Rs. 4500 the gain is $12\frac{1}{2}$%

$\therefore$ Rs. 4500$=112\frac{1}{2}$% or $\frac{225}{2}$% of the cost price

$\therefore$ Re. 1 $=\frac{225}{2\times 4500}$% ,, ,,

$\therefore$ Rs. 3800$=\frac{225\times 3800}{2\times 4500}$ or 95% ,, ,,

$\therefore$ the loss would have been$=(100-95)$% or 5%.

6. If a watch is sold for Rs. 60, the loss is 15 per cent. For how much should it be sold to make a profit of 2 per cent. ?

Here. Rs. 60=(100 - 15)% or 85% of the cost price.

Now, we are to find (100+2)% or 102% of the cost price for the reqd. selling price.

85% of the cost price=Rs. 60,

$\therefore$ 102% ,, ,, =Rs. $\dfrac{60\times 102}{85}$=Rs. 72.

$\therefore$ the reqd. sale price=Rs. 72.

7. By selling tea at Rs. 1. 21 P. a Kg. a grocer gains $\frac{1}{21}$ of his outlay ; how much did he pay for 200 Kg. of the same tea ?

Here, the gain=$\frac{1}{21}$ of the outlay.

$\therefore$ when the cost price is Re. 1, the selling price is Rs. $(1+\frac{1}{21})$ or Rs. $\frac{22}{21}$. $\therefore$ the cost price of 1 kg. of tea

$=\dfrac{1}{\frac{22}{21}}\times 121$ P.$=\frac{21}{22}\times 121$ P.$=\dfrac{21\times 11}{2}$P.

$\therefore$ the cost price of 200 kg. of tea

$=\dfrac{21\times 11}{2}\times 200$ P.=Rs. 21×11=Rs. 231.

8. Oranges were bought at 6 for 5 P. and sold at 5 for 6 P. find the gain per cent. [A. U. 1909]

The cost price of 6 oranges=5 P.

$\therefore$,, ,, 1 orange $=\frac{5}{6}$ P.

The selling price of 5 oranges=6 P.

$\therefore$,, ,, 1 orange $=\frac{6}{5}$ P.

$\therefore$ Profit$=\frac{6}{5}$ P. $-\frac{5}{6}$ P $=\frac{11}{30}$ P ;

$\therefore$ the profit on $\frac{5}{6}$ P.$=\frac{11}{30}$ P.

$\therefore$,, ,, 100 P.$=\frac{11}{30}\times\frac{6}{5}\times 100$ P.=44 P.

$\therefore$ the profit=44%.

[N. B. Here gain is to be calculated on the cost price $\frac{5}{6}$ P. but not on 1 orange. The profit or loss should always be calculated on the cost price.]

9. A certain number of plantains was bought at four for one anna and an equal number at three for an anna. The whole was sold at seven for two annas. Find the loss or gain per cent.

[C. U. '26]

The cost price of 4 plantains $=1\ a.$

$\therefore$ " " 1 plantain $=\frac{1}{4}\ a.$

Again " " 3 plantains $=1a.$

$\therefore$ " " 1 plantain $=\frac{1}{3}\ a.$

$\therefore$ " " 2 plantains of two kinds $=(\frac{1}{4}+\frac{1}{3})a=\frac{7}{12}a.$

$\therefore$ the cost price of 1 plantain on the average

$=\frac{7}{12\times 2}\ a.=\frac{7}{24}\ a.$;

But the selling price of 7 plantains $=2a.$

$\therefore$ " " 1 plantain $=\frac{2}{7}\ a.$

$\therefore$ the loss $=(\frac{7}{24}-\frac{2}{7})\ a.=\frac{1}{168}\ a.$

$\therefore$ the loss on $\frac{7}{24}\ a.=\frac{1}{168}\ a.$

$\therefore$ " 1 $a.=\frac{1}{168}\times\frac{24}{7}\ a.=\frac{1}{7\times 7}\ a.$

$\therefore$ " 100 $a.=\frac{1}{7\times 7}\times 100a.=2\frac{2}{49}\ a.$

$\therefore$ there is a loss of $=2\frac{2}{49}\%$.

10. A man bought 400 mangoes and by selling 320 of them realised the cost price of the whole, find his gain per cent.

The selling price of 320 mangoes

=the cost price of 400 or (320+80) mangoes ;

$\therefore$ On cost price of 320, the gain is the cost price of 80,

$\therefore$ his gain per cent. $=\frac{80}{320}\times 100=25.$

11. A person sold 60 metres of cloth for Rs. 28. 12 P. gaining thereby the cost price of 9 metres. Find the gain per cent.

Selling price of 60 metres=cost price of (60+9) m.

$\therefore$ the profit on the price of 60 metres=the price of 9 metres

$\therefore$ the gain p. c. $=\frac{9}{60}\times 100=15.$

12. By selling 20 mangoes a rupee a man gained 20%, find the cost price of the mangoes.

$20\%=\frac{20}{100}=\frac{1}{5}$; Now $\frac{1}{5}$ of 20 mangoes = 4 mangoes.

$\therefore$ the reqd. cost price is (20+4) or 24 mangoes per rupee.

13. By selling a house for £2576 a man gains 12 per cent. on his original outlay. How much per cent. would he have gained had the house cost him £100 less ? [C. U. '23]

In the first case, when the cost price is £100, the selling price is £112.

∴ the cost price of the house $=£\frac{100}{112}\times 2576=£2300$.

In the second case, the cost price of the house is £100 less, *i.e.*, £2200.

∴ the gain on £2200 $=(£2576-£2200)=£376$.

∴ „ „ £1 $=£\frac{376}{2200}$

∴ „ „ £100 $=£\frac{376\times 100}{2200}=£17\frac{1}{11}$

∴ the reqd. gain $=17\frac{1}{11}\%$.

14. By selling oranges at 12 a rupee there is a loss of 4 p. c., at what rate should they be sold so as to gain 44 p. c. ? [P. U. '34]

In the first case, the selling price of 12 oranges = Re. 1.

∴ the selling price of 1 orange = Re. $\frac{1}{12}$

∴ (100 − 4) or 96% of the cost price = Re. $\frac{1}{12}$,

∴ (100+44) or 144% „ „ $=\frac{1\times 144}{12\times 96}=$ Re. $\frac{1}{8}$

∴ the selling price of 1 orange = Re. $\frac{1}{8}$ *i.e.*, the oranges should be sold at 8 a rupee.

15. I mix tea purchased at Rs. 4 per Kg. with tea at Rs. 3. 50 P. per Kg. in equal quantities. At what price per Kg. should I sell the mixture to make a profit of 20 p. c. on my outlay ?

The cost price of 1 Kg. of tea of the first kind = Rs. 4

and „ „ „ „ the second kind = Rs. 3. 50 P.

∴ the cost price of 2 Kgs. of the mixture of tea

$=$ Rs. 7. 50 P. = Rs. $\frac{15}{2}$.

∴ the cost price of 1 kg. of the mixture on the average

$=$ Rs. $\frac{15}{2\times 2}=$ Rs. $\frac{15}{4}$.

Now, the profit being 20%, the selling price of 1 kg. of tea

$=\frac{120}{100}$ of Rs. $\frac{15}{4}=$ Rs. $\frac{9}{2}=$ Rs. 4. 50 P.

16. If oranges are bought at 20 for half a rupee, how many should be sold for Rs. 7 to gain 40 per cent. ? [D. B. 33]

The cost price of 20 oranges = Re. $\frac{1}{2}$.

$\therefore$,, 1 orange = Re. $\frac{1}{2 \times 20}$ = Re. $\frac{1}{40}$.

To gain 40%, the selling price of 1 orange should be

= Re. $\frac{140}{100} \times \frac{1}{40}$ or Re. $\frac{7}{200}$

$\therefore$ $(7 \div \frac{7}{200})$ or 200 oranges should be sold for Rs. 7.

17. A sold an article to B and lost 20%, B sold it to C and gained 20%. If A sold the article for the price C paid, how much per cent. would he have gained or lost ? [C. U. '42]

Suppose A bought the article for Rs. 100. Then B bought it for (100 − 20) or 80 rupees. B sold it at a gain of 20%.

$\therefore$ C bought it at Rs. $\frac{120}{100} \times 80$ or Rs. 96.

$\therefore$ if A sold the article for Rs. 96, his loss would have been (100 − 96) or 4%.

18. There was a loss of 10 per cent by selling an article, had it been sold for Re. 1. 50 P. more, 5 per cent. would have been gained. What was the cost price ?

In the first case, when the cost price is Rs. 100, the selling price = Rs. 90.

In the second case, when the cost price is Rs. 100, the selling price = Rs. 105.

$\therefore$ when the difference of the two sale prices is Rs. (105 − 90) or Rs. 15, the cost price = Rs. 100.

$\therefore$ when the difference of the two sale prices is Rs. $1\frac{1}{2}$, the cost price = Rs. $\frac{100}{15} \times \frac{3}{2}$ = Rs. 10.

$\therefore$ The reqd. cost price = Rs. 10.

19. A sells a cow at $2\frac{1}{2}$ per cent below cost price. Had he received Rs. 6 more than he did, he would have made a profit of 5 per cent. What did the cow cost ? [C. U. '34]

[This sum may be worked out as in Example 18. An alternative method is shown below.]

Second Method : If the cow be sold at Rs. 6 more, there is a gain of 5% after making up a loss of $2\frac{1}{2}$ p. c.

$\therefore$ $(2\frac{1}{2} + 5)$ or $7\frac{1}{2}$% of the cost price = Rs. 6,

i.e. $\frac{15}{2 \times 100}$ of the cost price = Rs. 6.

$\therefore$ the required cost price = Rs. $\frac{6 \times 2 \times 100}{15}$ = Rs. 80.

20. A trader marks his goods 25% above cost price, but allows his customer a commission of 10 per cent. Find his gain per cent. [U. P. '18]

Let the cost price be Rs. 100, then the marked price=Rs. 125.

The trader allows a commission of 10%, *i.e.*, if the marked price be Rs. 100, he gets Rs. 90.

∴ If the marked price be Rs. 125, he gets Rs. $\frac{90}{100} \times 125 =$ Rs. $112\frac{1}{2}$.

∴ His gain $= (112\frac{1}{2} - 100)\%$ or $12\frac{1}{2}\%$.

21. A fraudulent dealer defrauds, by false balance, to the extent of 10 per cent. in buying as well as in selling his goods. Find his gain per cent.

The dealer defrauds to the extent of 10%, *i.e.*, he spends only Rs. 100 for the goods worth Rs. 110 at the time of buying. While selling he defrauds to the extent of 10%, *i.e.*, he sells the goods worth Rs. 100 at Rs. 110.

∴ He sold goods worth Re. 1 for Rs. $\frac{110}{100}$.

∴ ... ... Rs. 110 for Rs. $\frac{110}{100} \times 110$ or Rs. 121.

∴ his gain $= (121 - 100)\%$ or 21%.

22. The manufacturer of an article makes a profit of 25 per cent., the wholesale dealer of 10 p. c. and the retail dealer of 5 p. c. What is the cost of manufacture of an article which is retailed for Rs. 231 ? [D. B. '29]

[The sums of this nature should be worked out from the end]

The retail dealer sells for Rs. 231 at a profit of 5%,

∴ his cost price=Rs. $\frac{100}{105} \times 231 =$ Rs. 220, which is the selling price of the wholesale dealer. But the latter gains 10% on selling the article for that price.

∴ his cost price $= \frac{100}{110}$ of Rs. 220 = Rs. 200, which is the selling price of the manufacturer.

Again the manufacturer sold the article at a profit of 25% ;

∴ the cost of its manufacture $= \frac{100}{125}$ of Rs. 200 = Rs. 160.

23. A trader allows a discount of 5 per cent. to his customers. What price should he mark on an article, the cost price of which is Rs. 712. 50 P., so as to make a clear profit of $33\frac{1}{3}$ p.c. on his outlay ? [C. U. 1908, D. B. '37]

The cost pice of the article = Rs. 712. 50 P. = Rs. $\frac{1425}{2}$.

The profit being $33\frac{1}{3}$%, the selling price $= \dfrac{133\frac{1}{3}}{100}$ of Rs. $\dfrac{1425}{2}$

$$= \text{Rs. } \frac{400 \times 1425}{3 \times 100 \times 2} = \text{Rs. } 950.$$

Again, the trader allows a discount of 5% to his customers.

$\therefore$ he gets Rs. 95, when his marked price = Rs. 100

$\therefore$,, ,, Re. 1, ,, ,, ,, ,, = Rs. $\frac{100}{95}$

$\therefore$,, ,, Rs. 950 ,, ,, ,, ,, = Rs. $\frac{100}{95} \times 950$

= Rs. 1000.

$\therefore$ the marked price should be Rs. 1000.

24. A speculator sells at a profit of 50 per cent., but his purchaser fails and only pays 50 P. in the rupee. How much per cent. does the speculator gain or lose by his venture ?

[A. U. 1899]

Goods worth Rs. 100 are sold for Rs. 150.

The purchaser pays Rs. $150 \times \frac{1}{2}$ or Rs. 75 at the rate of 50 P. in the rupee. $\therefore$ the speculator gets Rs. 75 only for Rs. 100.

$\therefore$ his loss is $(100 - 75)$% or 25%.

25. A man sold two horses at Rs. 1248 each. On one he gained 4 per cent. and on the other he lost 4%. Find his total gain or loss.

The first horse is sold for Rs. 1248 at a profit of 4%,

$\therefore$ its cost price = $\frac{100}{104}$ of Rs. 1248 = Rs. 1200.

The second horse is sold for Rs. 1248 at a loss of 4%,

$\therefore$ its cost price = $\frac{100}{96}$ of Rs. 1248 = Rs. 1300.

$\therefore$ the total cost price = Rs. 1200 + Rs. 1300 = Rs. 2500.

The total selling price of the two horses = Rs. 1248×2

= Rs. 2496.

$\therefore$ his total loss = (Rs. 2500 − Rs. 2496) = Rs. 4.

26. A manufacturer sells goods to a dealer and the latter to his customers each at the same rate of profit, viz., 10 per cent. How much does a customer pay above the original cost of goods purchased by him for £ 605 ? [C. U. 1931]

The customer purchased goods for £ 605.

$\because$ the dealer sells goods for £ 605 at a profit of 10%.

$\therefore$ the cost price for the dealer $= \frac{100}{110}$ of £ 605, which is again the selling price of the manufacturer at a profit of 10%.

$\therefore$ the cost of manufacture $= \frac{100}{110} \times \frac{100}{110}$ of £ 605 $=$ £ 500.

$\because$ the customer pays (£ 605 − £ 500) or £ 105 above the original cost.

27. A man purchased goods at Rs. 1950 and sold $\frac{1}{3}$ of them at a loss of $12\frac{1}{2}$%. At what rate per cent must he sell the remaining goods so as to gain 20% on his outlay ?

To gain 20% on the whole outlay his total selling price should be $\frac{120}{100}$ of Rs. 1950 or Rs. 2340. Now, $\frac{1}{3}$ of Rs. 1950 $=$ Rs. 650.

$\therefore$ Goods worth Rs. 650 are sold at a loss of $12\frac{1}{2}$%,

$\therefore$ their selling price $=$ Rs. $\dfrac{87\frac{1}{2}}{100}$ of Rs. 650 $=$ Rs. $\dfrac{175}{200} \times 650$

$=$ Rs. $\frac{2275}{4}$.

He has to get Rs. 2340, but he gets. $\frac{2275}{4}$ on selling $\frac{1}{3}$ of the goods. $\therefore$ he has to realise (Rs. 2340 − Rs. $\frac{2275}{4}$) or Rs. $\frac{7085}{4}$ on selling the remaining goods worth Rs. (1950 − 650) or Rs. 1300.

$\therefore$ selling price of the remaining goods worth Rs. 1300 $=$ Rs. $\frac{7085}{4}$

$\because$,, ,, ,, ,, Rs. 100 $=$ Rs. $\dfrac{7085 \times 100}{4 \times 1300}$

$=$ Rs. $136\frac{1}{4}$.

$\because$ the remaining goods are to be sold at the rate of $136\frac{1}{4}$%.

28. A sells a house to B for Rs. 4860, thereby losing 19%. B sells it to C at a price which would have given A a profit of 17 p. c. Find B's gain. [C. U. '29]

A sells the house for Rs. 4860 at a loss of 19%.

$\therefore$ the cost price for A $= \frac{100}{81}$ of Rs. 4860 $=$ Rs. 6000.

The selling price for A at a profit of 17% $= \frac{117}{100}$ of Rs. 6000 $=$ Rs. 7020.

$\therefore$ B bought the house for Rs. 4860 and sold it to C for Rs. 7020. $\therefore$ B's gain $=$ Rs. 7020 − Rs. 4860 $=$ Rs. 2160.

29. A man sold an article at a loss for Rs. 320. Had he sold it for Rs. 400, his gain would have been $\frac{1}{3}$ of his former loss. Find the cost price of the article.

Suppose, the cost price $= x$ rupees.

$\therefore$ in the first case, the loss$=(x-320)$ rupees and in the second case the profit$=(400-x)$ rupees. $\therefore$ $400-x=\frac{1}{3}(x-320)$,

or, $x=380$. $\therefore$ the cost price of the article$=$Rs. 380.

30. How much per cent. must a tradesman add on to the cost price of his goods so that he may make 20 p.c. profit after allowing his customers a discount of 10 p. c. on his bills ? [D. B. '40]

Let the cost price be Rs. 100. $\therefore$ the selling price at 20% profit$=$Rs. 120.

The tradesman allows his customer a discount of 10 p.c. on his bill on marked price.

$\therefore$ Rs. 90 is obtained for marked price of Rs. 100,

$\therefore$ Rs. 120 ,, ,, for Rs. $\frac{100}{90}\times 120$ or Rs. $133\frac{1}{3}$.

$\therefore$ $(133\frac{1}{3}-100)$ or $33\frac{1}{3}$% must be added on to the cost price.

31. What profit per cent. is made by selling an article at a certain price if by selling at two-thirds of that price there would be a loss of 20 per cent. ? [C. U. 1908]

On selling the article at $\frac{2}{3}$ of the fixed price there is a loss of 20%.

$\therefore$ $\frac{2}{3}$ of the fixed price$=(100-20)$ or 80% of the cost price,

$\therefore$ the total fixed price$=\frac{80\times 3}{2}$ % or 120% of the cost price.

$\therefore$ if the article be sold at the fixed price, the gain will be $(120-100)$% or 20%.

32. A man bought a horse and a carriage for Rs. 500 and sold the horse at a gain of 20 p. c. and the carriage at a loss of 10%, thus gaining 2% on his whole outlay ; for how much was the horse bought ? [D. B. '36]

$\because$ The total gain is 2% of Rs. 500,

$\therefore$ The total selling price is $\frac{102}{100}$ of Rs. 500 or Rs. 510.

Now, suppose the cost price of the horse$=x$ rupees.

$\therefore$ the cost price of the carriage$=(500-x)$ rupees.

$\therefore$ the selling price of the horse$=$Rs. $\frac{120}{100}\times x=$Rs.$\frac{6}{5}x$, and the selling price of the carriage$=\frac{90}{100}(500-x)$ rupees.

$\therefore$ the total selling price$=\frac{6}{5}x+\frac{9}{10}(500-x)$ in rupees.

$\therefore$ $\frac{6}{5}x+\frac{9}{10}(500-x)=510$, or, $12x+9(500-x)=5100$,

or, $12x-9x+4500=5100$, or, $3x=5100-4500=600$,

$\therefore$ $x=200$. $\therefore$ the cost price of the horse$=$Rs. 200.

33. A man bought 76 cows and sold 20 of them at a profit of 15 p.c., 40 at a profit of 19 p. c. and the remaining 16 at a gain of 25 p. c. If his total gain is Rs. 657, find the cost price of each cow. [P. U. 1923]

Suppose, the cost price of each cow $=x$ rupees.

$\therefore$ the cost price of 20 cows $=20x$ rupees.

On selling 20 cows the profit $=\frac{15}{100}$ of $20x$ rupees $=$ Rs. $3x$

[$\because$ the gain on Rs. 100 = Rs. 15, $\therefore$ the gain on Re. 1 = Re $\frac{15}{100}$ and the gain on $20x$ rupees $=\frac{15}{100}\times 20x$ rupees.]

Again, the cost price of 40 cows $=40x$ rupees.

$\therefore$ the profit on selling them at a gain of 19% $=\frac{19}{100}\times$ Rs. $40x$ $=$ Rs. $\frac{38}{5}x$.

The cost price of the remaining 16 cows $=16x$.

$\therefore$ the profit $=\frac{25}{100}\times$ Rs. $16x=$ Rs. $4x$.

$\therefore$ the total profit $=3x+\dfrac{38x}{5}+4x=\dfrac{73x}{5}$,

$\therefore$ $\dfrac{73x}{5}=657$, $\therefore$ $x=\dfrac{657\times 5}{73}=45$,

$\therefore$ The cost price of each cow = Rs. 45.

34. A farmer bought 18 cows and 15 lambs for Rs. 525 and sold them for Rs. 586. 50 P., thereby gaining 12 p. c. on the former and 10 p. c. on the latter. What was the cost of a cow and a lamb ? [D. B. '35]

Let the cost price of 1 cow be Rs. x and that of 1 lamb be Rs. y.

$\therefore$ from the first condition we get

$18x+15y=$ Rs. 525 (total cost price)......(1)

Now, the selling price of 1 cow $=\frac{112}{100}\times$ Rs. $x=\frac{28}{25}x$

and the selling price of 1 lamb $=\frac{110}{100}y=\frac{11}{10}y$

$\therefore$ from the second condition, we get

$\frac{28}{25}x\times 18+\frac{11}{10}y\times 15=$ Rs. $586\frac{1}{2}$ (total selling price)

or, $\frac{504}{25}x+\frac{33}{2}y=\frac{1173}{2}$, or, $1008x+825y=29325$......(2)

Now, multiplying (1) by 55 we get $990x+825y=28875$...(3)

Subtracting (3) from (2) we have $18x=450$, $\therefore$ $x=25$.

Now, from (1) we have $18\times 25+15y=525$,

or, $15y=525-450=75$, $\therefore$ $y=5$.

$\therefore$ The cost price of 1 cow is Rs. 25 and that of 1 lamb is Rs. 5.

35. A speculator invested his capital successively in four different ventures ; in the first he had his capital doubled, but in each of others he lost 20%. Did he gain or lose, and how much per cent ? [D. B. '32]

Let the capital of the speculator be Rs. 100.

After the first investment the capital=Rs. 200.

In the second investment the loss is 20%,

i.e., the capital of Rs. 100 becomes Rs. 80,

$\therefore$ the total capital$=\frac{80}{100}$ of Rs. 200=Rs. 160.

Thus the capital after the third investment$=\frac{80}{100}$ of Rs. 160, and the capital after the fourth investment

$$=\frac{80}{100} \text{ of } \frac{80}{100} \text{ of Rs. } 160=\text{Rs. } \frac{512}{5}=\text{Rs. } 102\tfrac{2}{5}.$$

$\therefore$ the speculator gets Rs. $102\frac{2}{5}$ at the end on investing Rs. 100

$\therefore$ He gains $(102\frac{2}{5}-100)\%$ or $2\frac{2}{5}\%$.

36. A man bought two cows for Rs. 206. By selling one of them at a gain of 10 p. c. and the other at a loss of 4 p. c., he got the same money in each case. Find the cost price of the cows.

Let Rs. x be the cost price of the first cow.

Then the cost price of the second cow$=(206-x)$ rupees.

Now, there being 10% profit, the selling price of the first cow $=\frac{110}{100}x=\frac{11}{10}x$.

Again, there being 4% loss, the selling price of the second cow $=\frac{96}{100}(206-x)=\frac{24}{25}(206-x)$.

$\therefore$ $\frac{11}{10}x=\frac{24}{25}(206-x)$, or, $55x=48(206-x)$

or $55x+48x=48\times 206$, or, $103x=48\times 206$

$$\therefore \quad x=\frac{48\times 206}{103}=96.$$

$\therefore$ the cost prices of the two cows are Rs. 96 and Rs. 110 respectively.

37. A man buys an article and sells it at a profit of 20%. If he had bought it at 10 p. c. less and sold it for 10s. less he would have made a profit of 25 p. c. Find the cost price. [P. U. '14]

Let the cost price of the article be 100 s.

$\therefore$ the selling price=120 s., gain being 20%.

Had he bought it at 10 p. c. less, its cost price would have been 90s.

The selling price of the article worth 90s. at 25% profit $=\frac{125}{100}$ of 90s. $=\frac{225}{2}$s.

Now, the first selling price $=120$ s. and the second selling price $=\frac{225}{2}$ s.

$\therefore$ the difference of the two selling prices $=120$s. $-\frac{225}{2}$s. $=\frac{15}{2}$s.

But in the problem the difference of the two selling prices is given to be 10s.

The difference is $\frac{15}{2}$s., when the cost price is 100s.

$\therefore$ " 1 s. " " " $\frac{100\times 2}{15}$ s.

$\therefore$ " 10 s. " " " $\frac{100\times 2\times 10}{15}$ s.

or $\frac{400}{3}$ s.

$\therefore$ the reqd. cost price $=\frac{400}{3}$s. $=$ £ 6. 13 s. 4d.

38. A house was sold for Rs. 800 thereby gaining $\frac{1}{5}$ of the sellling price. At what price would he have sold it to lose $\frac{1}{3}$ of the selling price ?

If the selling price be Rs. 800, the profit $=$ Rs. $800\times\frac{1}{5}$ or Rs. 160

$\therefore$ cost price $=$ Rs. 800 $-$ Rs. 160 $=$ Rs. 640.

In the second case let the selling price be x rupees.

$\therefore$ the loss $=(640-x)$ rupees and it is $\frac{1}{3}$ of the selling price.

$\therefore$ $\frac{1}{3}x=640-x$, or, $\frac{4}{3}x=640$, $\therefore$ $x=\frac{640\times 3}{4}=480$.

$\therefore$ the reqd. selling price $=$ Rs. 480.

39. A merchant sells 90 quintals of wheat at a profit of 8 p.c. and 50 quintals at a profit of 10% ; if he had sold the whole at a profit of 9%, he would have received Rs. 3 more than he actually did. How much per quintal did he pay for the whole ?

If the merchant would sell 90 quintals at a profit of 9% instead of 8%, he would get 1% more on the cost price of 90 qu. Again, if he would sell 50 qu. at a profit of 9% instead of 10% he would get 1% less on the cost price of 50 qu.

$\therefore$ by the given condition of the problem 1% of the cost price of 90 qu. $-$ 1% of the cost price of 50 qu. $=$ Rs. 3.

or, 1% or $\frac{1}{100}$ of the cost price of 40 qu. $=$ Rs. 3.

$\therefore$ The cost price of 40 qu. $=$ Rs. 3×100

$\therefore$ the reqd. cost price of 1 quintal $=$ Rs. $\frac{3\times 100}{40}=$ Rs. 7.50 P.

Exercise 11

1. If by selling a house for Rs. 490 there be a loss of $12\frac{1}{2}$ p.c., what per cent. is gained or lost by selling it for Rs. 596. 40 P. ?

2. At what price should a horse be sold to gain 12%, if there be a loss of 12% by selling it at Rs. 550 ? [U. P. '19]

3. If there be a loss of 5 p. c. by selling tea at 19 P. per Decagram, at what price per Dg. should it be sold to gain $17\frac{1}{2}$ per cent. ?

4. There was a gain of twenty per cent. by selling an article ; if it had been sold for Rs. 3 less, there would have been a loss of 4 per cent. ; find its cost price.

5. A man sold a cow at a loss of 10%. Had he charged Rs. 9 more, he would have gained $12\frac{1}{2}$%. Find the cost of the cow. [D. B. '31]

6. A sold an article to B at a profit of $22\frac{1}{2}$ p. c. and B sold it to C at a profit of $7\frac{1}{2}$ p. c. If C paid Rs. 5267. 50 P. for it, how much did it cost A ?

7. A man sold a horse at Rs. 50 and found that his loss amounted to 5 per cent. of the sale price ; find the cost price. [D. B. '35]

8. A person gains 5 p. c. by selling an article for Rs. $13\frac{1}{8}$. How much does he gain or lose per cent. if he sells it for Rs. 12 ? [C. U. 1945]

9. A cow and a horse were bought for Rs. 1600. The cow was sold at a loss of 25 p. c. and the horse at a gain of 15 p.c., thus making a profit of 5 p. c. on the whole ; what was the cost price of the horse ? [See. Ex. 32]

10. A bought a cycle for Rs. 275 and sold it to B at a gain of 6 P. in the rupee. B sold it to C at a loss of 4 P. in the rupee. How much did C pay for it ? [See P. U. 1904]

11. A man gained 35 p. c. by selling an article for Rs. 6. 75 P. What per cent. would he have gained if he had sold it for Rs. 8. 50 P. ?

12. A boy buys eggs at 9 for 4d. and sells them at 11 for 5d. What does he gain or lose per cent. ? [P. U. '12]

13. A man sells a house at a loss for Rs. 4000 ; had he sold it for Rs. 5000 his gain would have been $\frac{3}{5}$ of his former loss ; find the cost price of the house. [D. B. '24

14. A man sold a horse and a carriage for Rs. 800, gaining 10% on the horse and 20% on the carriage. Had he sold them so as to receive 15% on the horse and 25% on the carriage he would have got Rs. 35 more than before. Find the original cost of each. [I. P. S. '40]

[**Hints :** Suppose that the cost price of a horse$=x$ rupees and the cost price of a carriage$=y$ rupees. We get from the two selling prices

$$\tfrac{110x}{100}+\tfrac{120y}{100}=800\ldots(1)$$

and $$\tfrac{115x}{100}+\tfrac{125y}{100}=835\cdots(2)\ ]$$

15. If 34 kg. of tea at Rs. $1\frac{1}{8}$ per kg. be mixed with 29 kg. of tea at 75 P. per kg., at what price per kg. must the mixture be sold to gain 5% on the whole outlay ?

16. A man bought a horse and a cow for Rs. 370 and sold them for Rs. 412, thereby gaining 20% on the former and losing 15% on the latter. What was the cost price of the horse ? [C. U. '51]

17. A man having bought a quantity of goods for Rs. 1500 sells $\frac{1}{8}$ at a loss of 4% ; by what increase per cent must he raise the selling price that by selling the remainder at the increased rate he may gain 4% on the whole transaction ? [D. B. '45]

18. A dealer reduces the sale price by 10% on all his goods when payment is made in cash, and the price at which he buys is only 60% of that at which he sells. What profit does he make on his cash transaction ? [W. B. S. B. '55 Addl.]

WORK PROBLEM

N. B. (1) If a piece of work is performed in 5 days, $(1 \div 5)$ or $\frac{1}{5}$ of the work is performed in 1 day.

(2) If $\frac{1}{12}$ of a piece of work is done in 1 day, the whole work is done in $(1 \div \frac{1}{12})$ or 12 days.

(3) The whole work is denoted by 1.

Examples 12

1. A can do a piece of work in 10 days, B in 12 days and C in 15 days. In what time can they do it together?

A can do $\frac{1}{10}$th., B $\frac{1}{12}$th. and C $\frac{1}{15}$th. of the work in 1 day.

$\therefore$ they together can do $(\frac{1}{10}+\frac{1}{12}+\frac{1}{15})$ or $\frac{1}{4}$ of the work in 1 day.

$\therefore$ they can do the whole work in $(1 \div \frac{1}{4})$ or 4 days.

2. A man can do a piece of work in $5\frac{1}{2}$ days, but with the help of his son it can be done in 3 days. In what time can the son do it alone?

They together can do the work in 3 days,

$\therefore$ they can do $\frac{1}{3}$ of the work in 1 day.

Again, the man can do the work in $\frac{11}{2}$ days,

$\therefore$ he can do $\frac{2}{11}$ of the work in 1 day.

$\therefore$ the son alone can do $(\frac{1}{3}-\frac{2}{11})$ or $\frac{5}{33}$ of the work in 1 day,

$\therefore$ The son alone can do the whole work in $(1 \div \frac{5}{33})$ days or $\frac{33}{5}$ days or $6\frac{3}{5}$ days.

3. A alone can do a piece of work in 12 days and B alone can do it in 6 days; they work together for 2 days after which B leaves. In how many days will A finish the work? [C.U. 1931]

A and B can do $(\frac{1}{12}+\frac{1}{6})$ or $\frac{1}{4}$ of the work in 1 day.

$\therefore$ they do $\frac{1}{4} \times 2$ or $\frac{1}{2}$ of it in 2 days.

The remaining $(1-\frac{1}{2})$ or $\frac{1}{2}$ of the work will be done by A alone.

A can alone do the work in 12 days. $\therefore$ A will do $\frac{1}{2}$ of the work in $(12 \times \frac{1}{2})$ or 6 days.

4. A and B together can do a piece of work in 6 days, B and C in 9 days, and A and C in 12 days. In how many days can each do it separately ?

A and B together can do $\frac{1}{6}$ of the work in 1 day

B and C ,, ,, $\frac{1}{9}$,, ,,

C and A ,, ,, $\frac{1}{12}$,, ,,

$\therefore$ Twice the work of A, B and C done in 1 day

$=(\frac{1}{6}+\frac{1}{9}+\frac{1}{12})$ or $\frac{13}{36}$ of the work.

$\therefore$ A, B and C can do $(\frac{13}{36}\div 2)$ or $\frac{13}{72}$ of the work in 1 day,

$\therefore$ A can do $(\frac{13}{72}-\frac{1}{9})$ or $\frac{5}{72}$ of the work in 1 day.

[A's work is obtained by subtracting the works of B and C from the work of A, B and C.]

$\therefore$ A will do the whole work in $(1\div\frac{5}{72})$ or $\frac{72}{5}$ or $14\frac{2}{5}$ days.

Similarly, B can do $(\frac{13}{72}-\frac{1}{12})$ or $\frac{7}{72}$ of the work in 1 day.

$\therefore$ B can do the whole work in $\frac{72}{7}$ or $10\frac{2}{7}$ days.

C can do $(\frac{13}{72}-\frac{1}{6})$ or $\frac{1}{72}$ of the work in 1 day,

$\therefore$ C can do the whole work in 72 days.

5. A can do a piece of work in 9 days and B in 18 days. They begin together but A goes away 3 days before the work is finished. How long does the work last ? [C. U. '34]

A goes away 3 days before the work is finished, $\therefore$ B alone works for the last 3 days. B does $\frac{1}{18}$ of the work in 1 day.

$\therefore$ in the last three days $\frac{1}{18}\times 3$ or $\frac{1}{6}$ of the work has been done by B.

$\therefore$ Prior to this A and B together do $(1-\frac{1}{6})$ or $\frac{5}{6}$ of the work.

$\because$ A and B do $(\frac{1}{9}+\frac{1}{18})$ or $\frac{1}{6}$ of the work in 1 day,

$\therefore$ they together do $\frac{5}{6}$ of the work in $(\frac{5}{6}\div\frac{1}{6})$ or 5 days.

$\therefore$ the whole work is finished in $(5+3)$ or 8 days.

6. A can do a piece of work in 12 days and B in 16 days. They worked together for 4 days. C then finished the work in 10 days. How long will C take to do the work ?

A and B together do $(\frac{1}{12}+\frac{1}{16})$ or $\frac{7}{48}$ of the work in 1 day.

$\therefore$ they do $\frac{7}{48}\times 4$ or $\frac{7}{12}$ of the work in 4 days.

$\therefore$ C does the remaining $(1-\frac{7}{12})$ or $\frac{5}{12}$ of the work in 10 days.

$\therefore$ C does $(\frac{5}{12}\div 10)$ or $\frac{1}{24}$ of the work in 1 day.

$\therefore$ C can do the whole work in $(1\div\frac{1}{24})$ or 24 days.

7. A does $\frac{7}{10}$ of a piece of work in 14 days; he then calls in B, and they finish the work in 2 days. How long would B take to do the work by himself? P. U '18]

A does $\frac{7}{10}$ of the work in 14 days, $\therefore$ A does $\frac{7}{10} \div 14$ or $\frac{1}{20}$ of the work in 1 day.

$\therefore$ A and B together do the remaining $(1 - \frac{7}{10})$ or $\frac{3}{10}$ of the work in 2 days.

$\therefore$ A and B do $(\frac{3}{10} \div 2)$ or $\frac{3}{20}$ of the work in 1 day; but A does $\frac{1}{20}$ of the work in 1 day. $\therefore$ B does $(\frac{3}{20} - \frac{1}{20})$ or $\frac{1}{10}$ of the work in 1 day. $\therefore$ B can do the whole work in $(1 \div \frac{1}{10})$ or 10 days.

8. A does $\frac{1}{2}$ of a piece of work in $3\frac{1}{2}$ hours; B does $\frac{1}{4}$ of the remainder in $1\frac{1}{2}$ hours, and C finishes it in $5\frac{1}{4}$ hours. How long would it have taken the three working together to do the work?

A does $\frac{1}{2}$ of the work in $3\frac{1}{2}$ hours. $\therefore$ A does $(\frac{1}{2} \div 3\frac{1}{2})$ or $\frac{1}{7}$ of the work in 1 hour. Then remains $(1 - \frac{1}{2})$ or $\frac{1}{2}$ of the work.

$\therefore$ B does $\frac{1}{4}$ of $\frac{1}{2}$ or $\frac{1}{8}$ of the work in $1\frac{1}{2}$ hours.

$\therefore$ B does $(\frac{1}{8} \div \frac{3}{2})$ or $\frac{1}{12}$ of the work in 1 hour.

$(\frac{1}{2} + \frac{1}{8})$ or $\frac{5}{8}$ of the work being finished there still remains $(1 - \frac{5}{8})$ or $\frac{3}{8}$ of the work to be done by C in $5\frac{1}{4}$ hours.

$\therefore$ C does $(\frac{3}{8} \div \frac{21}{4})$ or $\frac{1}{14}$ of the work in 1 hour.

$\therefore$ A, B and C together can do $(\frac{1}{7} + \frac{1}{12} + \frac{1}{14})$ or $\frac{25}{84}$ of the work in 1 hour. $\therefore$ they can do the whole work in $(1 \div \frac{25}{84})$ or $\frac{84}{25}$ or $3\frac{9}{25}$ hours.

9. A can do a piece of work in 15 hours, B in 12 hours and C in 10 hours. They begin together, but A leaves after 3 hours, and B leaves 2 hours before the work is done. How long does the work last?

In 1 hr. A, B and C together do $(\frac{1}{15} + \frac{1}{12} + \frac{1}{10})$ or $\frac{1}{4}$ of the work.

$\therefore$ they do $\frac{1}{4} \times 3$ or $\frac{3}{4}$ of the work in 3 hours.

$\therefore$ A goes away when $\frac{1}{4}$ of the work remains undone.

Again, B leaves 2 hours before the work is done.

$\therefore$ C works alone for the last two hours.

C can do $(\frac{1}{10} \times 2)$ or $\frac{1}{5}$ of the work in 2 hours. C has done $\frac{1}{5}$ of the work in the last two hours out of the remaining $\frac{1}{4}$ of the work.

$\therefore$ B and C together do $(\frac{1}{4} - \frac{1}{5})$ or $\frac{1}{20}$ of the work. But they together do $(\frac{1}{12} + \frac{1}{10})$ or $\frac{11}{60}$ of the work in 1 hour.

$\therefore$ they do $\frac{1}{20}$ of the work in $(\frac{1}{20} \div \frac{11}{60})$ or $\frac{3}{11}$ hour.

$\therefore$ it takes $(3 + \frac{3}{11} + 2)$ hrs. or $5\frac{3}{11}$ hours to finish the whole work.

10. A can do a piece of work in 20 days ; A and B together can do it in $11\frac{1}{9}$ days. A works alone for 8 days, A and C together for 6 days and B finishes it in 3 days. Find in what time B and C together could do it ? [D. B. '35]

A does $\frac{1}{20}\times8$ or $\frac{2}{5}$ of the work in the first 8 days.

A and B together can do the work in $11\frac{1}{9}$ or $\frac{100}{9}$ days,

$\therefore$ A and B do $\frac{9}{100}$ of the work in 1 day.

$\therefore$ B does $(\frac{9}{100}-\frac{1}{20})$ or $\frac{1}{25}$ of the work in 1 day.

$\therefore$ B does $\frac{1}{25}\times3$ or $\frac{3}{25}$ of the work in the last 3 days.

$\therefore$ $(\frac{2}{5}+\frac{3}{25})$ or $\frac{13}{25}$ of the work is finished in the first 8 days and last 3 days.

$\therefore$ A and C have done $(1-\frac{13}{25})$ or $\frac{12}{25}$ of the work in 6 days

$\therefore$ A and C do $(\frac{12}{25}\div6)$ or $\frac{2}{25}$ of the work in 1 day,

$\therefore$ C does $(\frac{2}{25}-\frac{1}{20})$ or $\frac{3}{100}$ of the work in 1 day,

$\therefore$ B and C do $(\frac{1}{25}+\frac{3}{100})$ or $\frac{7}{100}$ of the work in 1 day.

$\therefore$ B and C could do the whole work in $(1\div\frac{7}{100})$ or $\frac{100}{7}$ or, $14\frac{2}{7}$ days.

11. A and B undertake to do a piece of work for Rs. 8. 40 P. A can do it alone in 8 days and B in 6 days. With the assistance of C it is finished in 3 days. How should the money be divided ?

A does $\frac{1}{8}$ of the work in 1 day. $\therefore$ he does $\frac{3}{8}$ of the work in 3 days. $\therefore$ he will get $\frac{3}{8}$ of the whole sum of money.

$\therefore$ A will get $\frac{3}{8}$ of Rs. 8. 40 P. or Rs. 3. 15 P.

B does $\frac{1}{6}\times3$ or $\frac{1}{2}$ of the works in 3 days,

$\therefore$ B will get $\frac{1}{2}$ of Rs. 8. 40 P. or Rs. 4. 20 P.

$\therefore$ C will get (Rs. 8. 40 P. – Rs. 3. 15 P. – Rs. 4. 20 P.) or, Re. 1. 5 P.

12. A man and a boy can do a piece of work in 36 days. If the man works alone for the last 10 days, it is completed in 40 days. How long would the boy take to do it alone ? [D. B. '31]

If the man works alone for the last 10 days, it is completed in 40 days, $\therefore$ the man and the boy together work for $(40-10)$ or 30 days. They do $\frac{1}{36}$ of the work in 1 day.

$\therefore$ they do $\frac{1}{36}\times30$ or $\frac{5}{6}$ of the work in 30 days.

$\therefore$ the man alone does the remaining $(1-\frac{5}{6})$ or $\frac{1}{6}$ of the work in the last 10 days.

$\therefore$ the man does $(\frac{1}{6}\div10)$ or $\frac{1}{60}$ of the work in 1 day.

$\therefore$ the boy does $(\frac{1}{36}-\frac{1}{60})$ or $\frac{1}{90}$ of the work in 1 day.

$\therefore$ the boy can do the whole work in 90 days.

13. Two men can do in 5 days as much work as 5 women can do in 4 days or as 6 boys in 10 days. In how many days can 1 man, 4 women and 5 boys together do the same work ?

The work done by 2 men in 5 days = The work done by 1 man in 10 days.

The work done by 5 women in 4 days = The work done by 1 woman in 20 days.

The work done by 6 boys in 10 days = The work done by 1 boy in 60 days.

$\therefore$ 1 man does $\frac{1}{10}$ of the work in 1 day.

1 woman does $\frac{1}{20}$ of the work in 1 day.

$\therefore$ 4 women do $\frac{1}{20}\times 4$ or $\frac{1}{5}$ of the work in 1 day.

5 boys do $\frac{1}{60}\times 5$ or $\frac{1}{12}$ of the work in 1 day.

$\therefore$ 1 man, 4 women and 5 boys together do $(\frac{1}{10}+\frac{1}{5}+\frac{1}{12})$ or $\frac{23}{60}$ of that work in 1 day.

$\therefore$ they can do that whole work in $(1\div\frac{23}{60})$ days or $2\frac{14}{23}$ days.

14. In 8 days B can do thrice and in 12 days C can do 5 times the work done by A in 3 days. If they have to work 9 hours a day in how many hours would they together do the work ? [M. U. 1865, P. U. 1927]

A does the work in 3 days, $\therefore$ he does $\frac{1}{3}$ of the work in 1 day. B does 3 times the work in 8 days, $\therefore$ B does the work in $\frac{8}{3}$ days. $\therefore$ B does $\frac{3}{8}$ of the work in 1 day.

Again, C does 5 times the work in 12 days,

$\therefore$ C does the work in $\frac{12}{5}$ days ; $\therefore$ C does $\frac{5}{12}$ of the work in 1 day. $\therefore$ A, B and C do $(\frac{1}{3}+\frac{3}{8}+\frac{5}{12})$ or $\frac{27}{24}$ of the work in 1 day.

$\therefore$ they do the whole work in $(1\div\frac{27}{24})$ or $\frac{24}{27}$ day. But they work for 9 hours in 1 day.

$\therefore$ they will do the whole work in $\frac{24}{27}\times 9$ or 8 hours.

15. A can do as much work as B and C together. A and B can together do a piece of work in 9 hours 36 min. and C can do it in 48 hours. In what time can B alone do it ? [P. U. '26]

A can do as much work as B and C together.

$\therefore$ the work of A and B = the work of 2B and C.

$\therefore$ 2B and C together do the work in 9 hrs. 36 min. or $\frac{48}{5}$ hrs.

$\therefore$ 2B and C do $\frac{5}{48}$ of the work in 1 hour ; but C does $\frac{1}{48}$ of the work in 1 hour. $\therefore$ 2B do $(\frac{5}{48}-\frac{1}{48})$ or $\frac{1}{12}$ of the work in 1 hour. $\therefore$ B does $(\frac{1}{12}\div 2)$ or $\frac{1}{24}$ of the work in 1 hour.

$\therefore$ B does the whole work in 24 hours.

16. A can do a piece of work in 20 days, B in 30 days and C in 60 days. In how many days can A do the work, if he is assisted by B and C on every third day ? [P.U. '30]

A does alone $\frac{1}{20}\times 2$ or $\frac{1}{10}$ of the work in the first 2 days. A, B and C do $(\frac{1}{20}+\frac{1}{30}+\frac{1}{60})$ or $\frac{1}{10}$ of the work on the third day.

$\therefore$ $(\frac{1}{10}+\frac{1}{10})$ or $\frac{1}{5}$ of the work is done in every 3 days.

$\because$ $\frac{1}{5}$ of the work is done in 3 days.

$\therefore$ the whole work is done in $(3\div\frac{1}{5})$ days or 15 days.

*17. A can do in 2 days as much work as B can do in 3 days, and C can do in 5 days as much as B can do in 4 days. In how many days can A, B and C together do the work which A can do in 11 days ? [P. U. '25]

A does in 2 days as much work as B does in 3 days.

$\therefore$ A ... 1 day ... ... B ... $\frac{3}{2}$ days.

$\therefore$ A ... 11 days ... ... B ... $\frac{3}{2}\times 11$ or $\frac{33}{2}$ days.

Again, B ... 4 days ... ... C ... 5 days.

$\therefore$ B ... 1 day ... ... C ... $\frac{5}{4}$ days.

$\therefore$ B ... $\frac{33}{2}$ days ... ... C...$\frac{5}{4}\times\frac{33}{2}$ or $\frac{165}{8}$ days.

$\therefore$ A can do the work in 11 days, B in $\frac{33}{2}$ days and C in $\frac{165}{8}$ days. $\therefore$ they together can do $(\frac{1}{11}+\frac{2}{33}+\frac{8}{165})$ or $\frac{1}{5}$ of the work in 1 day. $\therefore$ they can do the whole work in 5 days.

18. A and B can do a piece of work in 10 days, B and C in 15 days, and A and C in 25 days ; they all work at it together for 4 days ; A then leaves, and B and C go on together for 5 days and then B leaves ; in how many days will C complete the work ? [C. U. '41]

A and B do $\frac{1}{10}$ of the work in 1 day, B and C do $\frac{1}{15}$ of the work in 1 day and A and C do $\frac{1}{25}$ of the work in 1 day.

$\therefore$ $2(A+B+C)$ do $(\frac{1}{10}+\frac{1}{15}+\frac{1}{25})$ or $\frac{31}{150}$ of the work in 1 day.

$\therefore$ $A+B+C$ do $(\frac{31}{150}\div 2)$ or $\frac{31}{300}$ of the work in 1 day.

They do $\frac{31}{300}\times 4$ or $\frac{31}{75}$ of the work in 4 days, A then leaves ; and then B and C do $\frac{1}{15}\times 5$ or $\frac{1}{3}$ of the work in 5 days.

$\therefore$ $(\frac{31}{75}+\frac{1}{3})$ or $\frac{56}{75}$ of the work is done and C has to do the remaining $(1-\frac{56}{75})$ or $\frac{19}{75}$ of the work.

C does $(\frac{31}{300}-\frac{1}{10})$ or $\frac{1}{300}$ of the work in 1 day.

[1 day's work of C is found by subtracting 1 day's work of A and B from 1 day's work of A, B and C].

$\therefore$ C does $\frac{19}{75}$ of the work in $(\frac{19}{75}\div\frac{1}{300})$ or 76 days.

19. If 12 men and 10 boys can do $\frac{2}{9}$ of a piece of work in 3 days and if 4 men and 5 boys can do $\frac{1}{27}$ of the same piece of work in 7 days, find the time in which 7 men can do the whole work.

[C. U. '42]

12 men and 10 boys do $(\frac{2}{9}\div 3)$ or $\frac{2}{27}$ of the work in 1 day...(1)

Again, 4 men and 5 boys do $(\frac{1}{27}\div 7)$ or $\frac{1}{189}$ of the work in 1 day.

$\therefore$ (by doubling it) 8 men and 10 boys do $(\frac{1}{189}\times 2)$ or $\frac{2}{189}$ of the work in 1 day......(2)

Now, subtracting (2) from (1) we get,

4 men do in 1 day $(\frac{2}{27}-\frac{2}{189})$ or $\frac{14-2}{189}$ or $\frac{12}{189}$ of the work

$\therefore$ 1 man does in 1 day $\frac{12}{189\times 4}$ or $\frac{1}{63}$ of the work

$\therefore$ 7 men do in 1 day $\frac{1}{63}\times 7$ or $\frac{1}{9}$ of the work,

$\therefore$ 7 men do the whole work in 9 days.

***20.** Forty men finish a piece of work in 40 days ; if 5 men leave the work after every tenth day, in what time will the whole work be completed ?

[D. B. '40 addl., A.U. 1892]

40 men finish the work in 40 days,

$\therefore$ 40 men do $\frac{1}{40}$ of the work in 1 day,

$\therefore$ 1 man does $(\frac{1}{40}\div 40)$ or $\frac{1}{1600}$ of the work in 1 day.

40 men do $(\frac{1}{40}\times 10)$ or $\frac{1}{4}$ of the work in the first 10 days.

Then 35 men work for 10 days. $\because$ 1 man does $\frac{1}{1600}$ of the work in 1 day,

$\therefore$ 35 men do $\frac{1}{1600}\times 35\times 10$ or $\frac{7}{32}$ of the work in 10 days.

$\therefore$ $(\frac{1}{4}+\frac{7}{32})$ or $\frac{15}{32}$ of the work is finished in 20 days.

Then 30 men do $\frac{1}{1600}\times 30\times 10$ or $\frac{3}{16}$ of the work in 10 days.

$\therefore$ $(\frac{15}{32}+\frac{3}{16})$ or $\frac{21}{32}$ of the work is done in 30 days.

Again, 25 men do $\frac{1}{1600}\times 25\times 10$ or $\frac{5}{32}$ of the work in 10 days.

$\therefore$ $(\frac{21}{32}+\frac{5}{32})$ or $\frac{26}{32}$ of the work is done in 40 days.

Again, 20 men do $\frac{1}{1600}\times 20\times 10$ or $\frac{1}{8}$ of the work in 10 days.

$\therefore$ $(\frac{26}{32}+\frac{1}{8})$ or $\frac{15}{16}$ of the work is done in 50 days.

Now we have to find the time 15 men will take to do the remaining $(1-\frac{15}{16})$ or $\frac{1}{16}$ of the work.

15 men do $\frac{1}{1600}\times 15$ or $\frac{3}{320}$ of the work in 1 day.

$\therefore$ they do $\frac{1}{16}$ of the work in $(\frac{1}{16}\div\frac{3}{320})$ days or $6\frac{2}{3}$ days.

$\therefore$ the whole work is done in $(50+6\frac{2}{3})$ or $56\frac{2}{3}$ days.

21. A, B and C are set to do a piece of work. A alone can do it in $12\frac{1}{2}$ days, B in 10 days and C in 12 days. They work together for one day and then A and C leave; B, however, continues to work and after $3\frac{1}{2}$ days C comes and brings D along with him and these three finish the remainder of the work in 2 days more. In what time would D alone have done the work? [C. U. 1943]

A does $\frac{2}{25}$, B $\frac{1}{10}$ and C $\frac{1}{12}$ of the work in 1 day.

$\therefore$ A, B and C do $\left(\frac{2}{25}+\frac{1}{10}+\frac{1}{12}\right)$ or $\frac{24+30+25}{300}$ or $\frac{79}{300}$ of the work in 1 day.

Then A and C leave and B does $\frac{1}{10}\times 3\frac{1}{2}$ or $\frac{7}{20}$ of the work in $3\frac{1}{2}$ days.

Now, $(\frac{79}{300}+\frac{7}{20})$ or $\frac{46}{75}$ of the work is done, and B, C and D finish the remaining $(1-\frac{46}{75})$ or $\frac{29}{75}$ of the work in 2 days.

$\therefore$ B, C and D do $(\frac{29}{75}\div 2)$ or $\frac{29}{150}$ of the work in 1 day.

But B and C do $(\frac{1}{10}+\frac{1}{12})$ or $\frac{11}{60}$ of the work in 1 day.

$\therefore$ D does $(\frac{29}{150}-\frac{11}{60})$ or $\frac{1}{100}$ of the work in 1 day.

$\therefore$ D can do the whole work in $(1\div\frac{1}{100})$ or 100 days.

22. A can do in one day three times the work done by B in one day. They together finish $\frac{2}{3}$ of a work in 9 days. In how many days will it be done by each separately? [C. U. '46]

A and B together do $\frac{2}{3}$ of the work in 9 days.

$\therefore$ they do $(\frac{2}{3}\div 9)$ or $\frac{2}{27}$ of the work in 1 day.

A can do three times the work done by B,

$\therefore$ the work of A and B together is equal to the work of 4B.

$\therefore$ 4B do $\frac{2}{27}$ of the work in 1 day,

$\therefore$ B does $(\frac{2}{27}\div 4)$ or $\frac{1}{90}$ of the work in 1 day.

$\therefore$ B can do the whole work in 90 days,

$\therefore$ A can do the whole work in $(90\div 3)$ or 30 days.

23. Two taps can fill a cistern in 10 and 15 hours respectively. In what time can they together fill it?

The two taps together can fill $(\frac{1}{10}+\frac{1}{15})$ or $\frac{1}{6}$ of the cistern in 1 hour.

$\therefore$ they can fill the whole cistern in $(1\div\frac{1}{6})$ hrs. or 6 hours.

24. A supply-pipe fills a cistern in 12 hours and a waste-pipe can empty it in 18 hours. In what time will the cistern be filled, if both the pipes be opened ?

The first pipe fills $\frac{1}{12}$ of the cistern and the second pipe empties $\frac{1}{18}$ of it in 1 hour.

$\therefore$ if both the pipes be opened, $(\frac{1}{12}-\frac{1}{18})$ or $\frac{1}{36}$ of the cistern is filled in 1 hour.

$\therefore$ it will take $(1\div\frac{1}{36})$ or 36 hours to fill the whole cistern.

25. Two taps can separately fill a cistern, when the waste-pipe is closed, in 12 and 16 minutes respectively, and when the waste-pipe is opened they can together fill it in 15 minutes. How long does it take the waste-pipe to empty the cistern when the taps are not running. [C. U. 1938, '51]

The two taps together can fill $(\frac{1}{12}+\frac{1}{16})$ or $\frac{7}{48}$ of the cistern in 1 minute.

But when the two taps and the pipe are opened, the cistern is filled in 15 minutes.

$\therefore$ $\frac{1}{15}$ of the cistern is filled in 1 minute.

$\therefore$ the waste-pipe empties $(\frac{7}{48}-\frac{1}{15})$ or $\frac{35-16}{240}$ or $\frac{19}{240}$ of the cistern in 1 minute.

$\therefore$ it will empty the whole cistern in $(1\div\frac{19}{240})$ or $12\frac{12}{19}$ minutes.

26. Two pipes can fill a cistern in 20 and 30 minutes respectively. Both the pipes being opened, find when the first pipe must be turned off so that the cistern may be filled in 10 minutes more. [C. U. 1926]

The first pipe having been turned off, the cistern is filled in 10 minutes more.

$\therefore$ in these 10 minutes the second pipe fills $\frac{1}{30}\times10$ or $\frac{1}{3}$ of the cistern.

$\therefore$ the two pipes have already filled $(1-\frac{1}{3})$ or $\frac{2}{3}$ of the cistern. They together fill $(\frac{1}{20}+\frac{1}{30})$ or $\frac{1}{12}$ of the cistern in 1 minute.

$\therefore$ they fill $\frac{2}{3}$ of the cistern in $(\frac{2}{3}\div\frac{1}{12})$ or 8 minutes.

$\therefore$ the first pipe must be turned off after 8 minutes.

27. A cistern can be filled by two pipes A and B in 20 and 30 minutes respectively. Both the pipes are opened together, when the cistern is empty, but after some time A is stopped and the cistern is filled in 18 minutes in all. When was A stopped ? [D. B. 1927]

B remains open for 18 minutes.

$\therefore$ B fills $(\frac{1}{30} \times 18)$ or $\frac{3}{5}$ of the cistern in 18 minutes.

$\therefore$ A fills the remaining $(1-\frac{3}{5})$ or $\frac{2}{5}$ of the cistern.

A fills $\frac{1}{20}$ of the cistern in 1 minute.

$\therefore$ A fills $\frac{2}{5}$ of it in $(\frac{2}{5} \div \frac{1}{20})$ or 8 minutes.

$\therefore$ A was stopped after 8 minutes.

28. A cistern has three pipes, the first two can fill it in 3 hours and 3 hours 45 minutes respectively, while the third can empty it in 1 hour. If these pipes be opened in order at 1, 2 and 3 o'clock, when will the cistern be empty ? [P. U. '29]

3 hr. 45 min. $= \frac{15}{4}$ hrs.

The first pipe fills $\frac{1}{3}$ of the cistern in 1 hour and the second fills $\frac{4}{15}$ of it in 1 hour. The first pipe fills $\frac{2}{3}$ of the cistern in 2 hours from 1 to 3 o'clock.

The second pipe fills $\frac{4}{15}$ of it in 1 hour from 2 to 3 o'clock. $\therefore$ $(\frac{2}{3}+\frac{4}{15})$ or $\frac{14}{15}$ of the cistern in filled at 3 o'clock. The third pipe is opened at 3 o'clock and it can empty the cistern in 1 hour. $\therefore$ When the three pipes work together $(1-\frac{1}{3}-\frac{4}{15})$ or $\frac{2}{5}$ of the cistern is emptied in 1 hour. $\therefore$ $\frac{14}{15}$ of the cistern is emptied in $(\frac{14}{15} \div \frac{2}{5})$ of $\frac{7}{3}$ hours or 2 hrs. 20 mins.

$\therefore$ the cistern will be emptied at 20 minutes past 5.

29. A cistern has two pipes. The first pipe can fill it in 40 minutes and the second can empty it in 1 hour. If the pipes be opened one at a time in alternate minutes, in what time will the cistern be filled ? [P. U. 1931]

The first pipe fills $\frac{1}{40}$ of the cistern in the first minute. In the next minute the first pipe being stopped the second pipe is opened and it empties $\frac{1}{60}$ of the cistern.

$\therefore$ in every two successive minutes $(\frac{1}{40}-\frac{1}{60})$ or $\frac{1}{120}$ of the cistern is filled. This process will continue until there remains a portion of the cistern that can be filled by the first pipe alone

within 1 minute, there being no necessity of opening the second pipe after that. $\therefore$ we have to find the time when $(1-\frac{1}{40})$ or $\frac{39}{40}$ of the cistern is filled up as before.

$\frac{1}{120}$ of the cistern is filled in 2 minutes

$\therefore$ 1 (whole) „ „ „ 2×120 mins.

$\therefore$ $\frac{39}{40}$ „ „ „ „ $\frac{2 \times 120 \times 39}{40}$ mins.

or 234 minutes.

The first pipe fills the remaining $\frac{1}{40}$ of the cistern in 1 min.

$\therefore$ The cistern will be filled in 235 mins. or 3 hrs. 55 mins.

30. A monkey in climbing up a greased pole 33 feet high ascends 7 feet and slips down 4 feet in alternate minutes. How long will it take him to go to the top? [O. U. 1939 Sup.]

The monkey ascends 7 feet in 1 min. and slips down 4 feet in the next minute. $\therefore$ he ascends $(7-4)$ or 3 feet in all in every 2 minutes. Thus ascending and descending in alternate minutes the monkey will reach a certain height whence the top of the pole is 7 feet or less than 7 feet. He will ascend this remaining portion wholly within the next minute at the rate of 7 feet per minute without slipping down.

Now, 33 ft. $-$ 7 ft. $=$ 26 ft. and 26 is not divisible by 3. $\therefore$ We have to find out the next higher number which is divisible by 3. Evidently it is 27. He will take 9×2 or 18 minutes to ascend 27 feet in climbing and slipping 9 times.

He will take $\frac{6}{7}$ minutes to ascend the remaining $(33-27)$ or 6 feet at the rate of 7 feet per minute.

$\therefore$ the monkey will take $(18+\frac{6}{7})$ or $18\frac{6}{7}$ minutes to reach the top of the pole.

31. A boy and a girl begin to fill a cistern. The boy brings 4 kilo-litres of water every 3 minutes, and the girl 3 kilo-litres every 4 minutes. If the cistern holds 84 kilo-litres of water, in what time will it be filled?

The L. C. M. of 3 mins. and 4 mins. $=$ 12 mins.

In 12 minutes the boy brings water 4 times and the girl 3 times. Thus the boy brings 4×4 or 16 kl. of water and the girl 3×3 or 9 kl. of water in 12 mins.

Thus (16+9) or 25 kl. of water is poured in every 12 mins.

$\therefore$ in 12 mins.$\times$3 or 36 mins., 25 kl.$\times$3 or 75 kl. of water will be poured. Now there remain 9 kl. of water to be poured. After 36 minutes the boy brings 4 kl. of water at the end of the 3rd minute and the girl 3 kl. of water at the end of the fourth minute. Thus (75+4+3) or 82 kl. of water are filled. Then the boy brings 4 kl. of water at the end of the 6th minute and pours only 2 kl. of water to fill the whole cistern.

Thus it will take (36+6) or 42 minutes to fill the cistern.

32. A snail on the average creeps 1 ft. $7\frac{1}{2}$ inches during 12 hours in the night and slips down 11 inches during 12 hours in the day. How many hours will he be in getting at the top of a pole 93 ft. high ? [C.U. 1943]

The snail creeps 1 ft. $7\frac{1}{2}$ ins. during 12 hrs. in the night and slips down 11 ins. during 12 hrs. in the day.

$\therefore$ it ascends $(19\frac{1}{2}-11)$ or $8\frac{1}{2}$ inches in 24 hrs.

93 ft.$=93\times12$ ins.$=1116$ ins.,

1116 ins.$-19\frac{1}{2}$ ins.$=1096\frac{1}{2}$ ins.

$1096\frac{1}{2}\div8\frac{1}{2}=129$. Thus the snail takes 129×24 hrs. or 3096 hrs. to ascend $1096\frac{1}{2}$ inches by creeping and slipping 129 times.

It will take 12 hrs. to ascend the remaining $13\frac{1}{2}$ ins.

$\therefore$ The snail will take (3096+12) or 3108 hrs. to get at the top of the pole.

33. If 40 men, 60 women or 80 children can do a piece of work in 6 months, in what time will 10 men, 10 women and 10 children do half the work ? [B. U.]

40 men can do the work in 6 months,

$\therefore$ 1 man ... ... in 6×40...

$\therefore$ 10 ... ... in $\frac{6\times40}{10}$ or 24...

Similarly, 10 women can do the work in $\frac{6\times60}{10}$ or 36 months,

and 10 boys $\frac{6\times80}{10}$ or 48 months.

$\therefore$ 10 men, 10 women and 10 boys can do $(\frac{1}{24}+\frac{1}{36}+\frac{1}{48})$ or $\frac{13}{144}$ of the work in 1 month.

$\therefore$ they do the whole work in $\frac{144}{13}$ months.

$\therefore$ they do half of the work in $(\frac{144}{13}\times\frac{1}{2})$ or $5\frac{7}{13}$ months.

34. If 3 men and 5 women do a piece of work in 8 days which 2 men and 7 boys can do in 12 days, find how long 13 men, 14 boys and 15 women will take to do it ? [B. U.]

3 men and 5 women can do $\frac{1}{8}$ of the work in 1 day.

$\therefore$ 9 men and 15 women can do $\frac{1}{8}\times 3$ or $\frac{3}{8}$ of the work in 1 day [multiplying by 3]...(1)

Again, 2 men and 7 boys can do $\frac{1}{12}$ of the work in 1 day.

$\therefore$ 4 men and 14 boys can do $\frac{1}{12}\times 2$ or $\frac{1}{6}$ of the work in 1 day [multiplying by 2]...(2)

Now, adding (1) and (2) we find that 13 men, 14 boys and 15 wonen together can do $(\frac{3}{8}+\frac{1}{6})$ or $\frac{13}{24}$ of the work in 1 day.

$\therefore$ They do the whole werk in $(1\div\frac{13}{24})$ or $\frac{24}{13}$ or $1\frac{11}{13}$ days.

35. 3 men and 8 women can construct a road in 24 days, 5 men and 14 women can do it in 14 days ; 7 men and 10 women worked on it for 3 days and then they left the work. Find in how many days 8 men and 6 women will finish it. [D.B. '34]

3 men and 8 women do $\frac{1}{24}$ of the work in 1 day...(1)

$\therefore$ 15 ... ... 40 ... ... $\frac{1}{24}\times 5$ or $\frac{5}{24}$ ... ...(2)

Again, 5 ... ... 14 ... ... $\frac{1}{14}$ of the work ... ...(3)

$\therefore$ 15 ... ... 42 ... ... $\frac{3}{14}$ ... ...(4)

Subtracting (2) and (4) we have

2 women do $(\frac{3}{14}-\frac{5}{24})$ or $\frac{1}{168}$ of the work in 1 day,

$\therefore$ 1 woman can do $\frac{1}{168\times 2}$ or $\frac{1}{336}$ of the work in 1 day.

Now, from (1), 3 men do $(\frac{1}{24}-\frac{8}{336})$or $\frac{3}{168}$ of the work in 1 day.

$\therefore$ 1 man does $\frac{1}{168}$ of the work in 1 day. $\therefore$ 1 man=2 women.

$\therefore$ 7 men and 10 women=(14+10) women=24 women and they do $\frac{1}{336}\times 24\times 3$ or $\frac{3}{14}$ of the work in 3 days.

Now, there remains $(1-\frac{3}{14})$ or $\frac{11}{14}$ of the work.

8 men and 6 women or 22 women do $\frac{1}{336}\times 22$ or $\frac{11}{168}$ of the work in 1 day.

$\therefore$ They will do $\frac{11}{14}$ of the work in $(\frac{11}{14}\div\frac{11}{168})$ days or 12 days.

36. Seven boys and 5 men can do a piece of work in 3 days and 2 men and 7 women can do it in 4 days. In what time can a boy, a woman and a man together do the work ?

7 boys and 5 men do $\frac{1}{3}$ of the work in 1 day...(1)

and 7 women and 2 men do $\frac{1}{4}$ of the work in 1 day...(2)

Adding (1) and (2) we have,

7 boys, 7 women and 7 men do $(\frac{1}{3}+\frac{1}{4})$ or $\frac{7}{12}$ of the work in 1 day.

$\therefore$ 1 boy, 1 woman and 1 man do $(\frac{7}{12}\div 7)$ or $\frac{1}{12}$ of the work in 1 day.

$\therefore$ they can do the whole work in $(1\div\frac{1}{12})$ or 12 days.

37. A can do a piece of work in 12 days and B in 8 days. A begins the work and after working for an exact number of days leaves the work and B then finishes it. If the work lasts 9 days, find how many days each of them worked.

[First Method]

If A works for all the 9 days, $\frac{1}{12}\times 9$ or $\frac{3}{4}$ of the work is done and $\frac{1}{4}$ of it is left undone.

$\therefore$ It is evident that what B does during his period of work is more than what A does during his period of work by $\frac{1}{4}$ of the whole work,

B does $(\frac{1}{8}-\frac{1}{12})$ or $\frac{1}{24}$ of the work more than A in 1 day.

$\therefore$ B does $\frac{1}{4}$ of the work more than A in $(\frac{1}{4}\div\frac{1}{24})$ days or 6 days.

$\therefore$ B works for 6 days and A for 3 days.

[Alternative Method]

Suppose A works for x days. Then B works for $(9-x)$ days.

A does $\dfrac{x}{12}$ of the work in x days and B does $\dfrac{9-x}{8}$ of the work in $(9-x)$ days. $\therefore$ $\dfrac{x}{12}+\dfrac{9-x}{8}=1$ (the whole work)

or, $2x+27-3x=24$, $\therefore$ $x=3$.

$\therefore$ A works for 3 days and B for 6 days.

Exercise 12

1. A and B can together do a piece of work in 12 days, B and C together in 15 days, and C and A together in 20 days. Find in how many days A can do the work himself ? [C.U. 1939]

2. A, B and C can do a piece of work in 10, 12 and 15 days respectively. In how many days will they together do it, and how much of the work will be done by each ? [P.U. 1922]

3. A and B complete a piece of work in 8 days, B and C do the same in 12 days, and A, B and C finish it in 6 days. In how many days will A and C complete the work ? [A. U. 1889]

4. Two men undertake to do a piece of work for Rs. 30. One can do it in 10 days and the other in 12 days. With the help of a boy they finish it in 4 days. How should the money be divided ?

5. A can do a piece of work in 6 days and B in 8 days, each working 7 hours a day. In what time will they finish it together, working 8 hours a day ?

[C. U. 1930]

6. A and B together can do a piece of work in 25 days. If B works alone for the last 10 days, it is completed in 30 days. In how many days A alone can do it ?

7. A and B can do a piece of work in 12 days ; after working 2 days they are assisted by C, who works at the same rate as A, and the work is finished in $6\frac{1}{4}$ days more ; in how many days would B alone do the work ?

[A.U. 1903]

8. A can do a piece of work is 80 hours, B in 60 hours and C in 40 hours. They begin together, but B leaves 10 hours and A 15 hours before the work is finished. How long does the work last ?

9. A cistern can be filled by a tap in 5 hours, while it can be emptied by another tap in 6 hours. If both the taps are turned on together when the cistern is empty, in what time will it be filled ?

[C. U. 1928]

10. A cistern has a supply-pipe which can fill it in 3 hours and also a waste-pipe which can empty it in 4 hours. If both the pipes are opened when the cistern is empty, in what time will the cistern be filled ?

[C.U. 1933]

11. A cistern has 3 pipes. The first and the second can fill it in 10 and 12 minutes respectively while the third is a waste-pipe. When all the pipes are open the cistern is filled in 15 minutes. In what time can the third pipe empty the whole cistern ?

[Civil Service]

12. A cistern has 3 pipes A, B and C; A and B can fill it in 3 and 4 hours respectively, and C can empty it in 1 hour; if the pipes be opened in order at 3, 4 and 5 P. M., when will the cistern be empty?

13. A alone takes as much time as B and C together take to do a piece of work. If A and B together take 10 days and C alone 50 days to do it, how long will B alone take to do it? [M. E. '43]

14. Three boys begin to fill a cistern; one of them brings a pint every 5 minutes, another a quart every 6 mins., and the third a gallon every 8 minutes. If the cistern holds $50\frac{1}{2}$ gallons, in what time will it be filled? [C. U. 1941]

(Hints: 2 pints = 1 quart, 8 pints = 1 gallon)

15. A cistern, which is filled in 10 hours, takes 2 hours more to be filled owing to a leak at the bottom. In what time will the leak empty the whole cistern?

16. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one man or one boy to do it? [P. U. 1931]

17. A can do as much work in one day as B can do in 2 days or as C can do in 3 days or as D can do in 4 days. They together finish a piece of work in 8 days. How many days would C take to do it singly? [G. U. '48]

18. 3 men and 2 boys can do a piece of work in 15 days, and 2 men and 3 boys can do the same in 18 days, in what time will a man and a boy jointly do the work? [C. U. '50]

19. A and B have to do a piece of work in 16 days. A alone can do it in 30 days and B in 45 days. They work together for 10 days and then with the help of C finish it in time. If they are paid Rs. 22. 50 P. for the whole work, what are their respective shares?

TIME AND DISTANCE

(Race and games of skill)

N. B. (1) If two persons run in opposite directions at the rates of 5 and 3 miles per hour respectively from the same place, then the distance between them in 1 hour $=(5+3)$ or 8 miles.

(2) If they run in the same direction at the above rates, then the distance between them in 1 hour $=(5-3)$ or 2 miles.

(3) If the distance between A and B is 50 kilometres and if A runs at the rate of 6 Km. and B at the rate of 4 Km. per. hr. towards each other, then they will meet when they together will cover the distance of 50 kilometres. $\therefore$ A and B meet each other after $(50 \div 10)$ or 5 hours.

(4) A runs 10 Km. and B 6 Km. per hour. If A is 24 Km. behind B, how long will A take to overtake B ? Here, the distance between A and B is 24 Km. When this distance between them vanishes, A will overtake B. A runs $(10-6)$ or 4 Km. more than B in every hour, i.e., the distance between them decreases by 4 Km. per hour. $\therefore$ A will overtake B after $(24 \div 4)$ or 6 hours.

(5) The time that a train takes to pass a telegraph post is the same as it takes to travel a distance equal to its own length. The time that a train takes to pass a platform is the same as it takes to travel a distance equal to the sum of the lengths of the train and the platform. This rule holds good in the case of a train crossing a bridge or another train.

(6) When a man rows down a river, the rate at which he advances $=$ (rate of his own pull) $+$ (rate of the current). When a man rows up a river, the rate at which he advances $=$ (rate of his own pull) $-$ (rate of the current).

Examples [13]

1. The distance between Howrah and Burdwan is 60 kilometres. A travels from Howrah to Burdwan at $12\frac{1}{2}$ kilometres per hour and B from Burdwan to Howrah at $7\frac{1}{2}$ kilometres per hour. When and where will they meet ?

Here A and B are travelling in opposite directions. $\therefore$ The distance between them is reduced by $(12\frac{1}{2}+7\frac{1}{2})$ or 20 Km. in

1 hour. $\therefore$ the whole distance of 60 Km. is reduced in $(60 \div 20)$ or 3 hrs. $\therefore$ they will meet after 3 hrs.

Now, A travels ($12\frac{1}{2}$ Km. $\times 3$) or $37\frac{1}{2}$ Km. in 3 hrs. $\therefore$ A and B meet each other at a distance of $37\frac{1}{2}$ Km. from Howrah.

2. A man runs after a thief 15 minutes after the latter's escape. They run at the rates of 8 and 6 Km. per hour respectively. When and where will the thief be overtaken?

The thief has started 15 mins. or $\frac{1}{4}$ hr. before the man, and is therefore 6 Km. $\times \frac{1}{4}$ or $1\frac{1}{2}$ Km. away when the man starts. So the man has to gain $1\frac{1}{2}$ Km. over the thief at the rate of $(8-6)$ or 2 Km. per hour. This he does in $(1\frac{1}{2} \div 2)$ or $\frac{3}{4}$ hr. or 45 mins. $\therefore$ the man overtakes the thief in 45 mins. after he starts and at a distance of (8 Km. $\times \frac{3}{4}$) or 6 Km.

3. A man rides at the rate of 352 yards per minute and stops 6 minutes to change horses at the end of every sixth mile; how long will he take to go a distance of 108 miles? [C. U. 1925]

The man rides 352 yards or $\frac{352}{1760}$ mile or $\frac{1}{5}$ mile in 1 min.

$\therefore$ he will take $(108 \div \frac{1}{5})$ or 540 mins. to go 108 miles.

Again, he changes horses at the end of every sixth mile. 108 miles $\div$ 6 miles $= 18$. $\therefore$ he has to change horses $(18-1)$ or 17 times and takes (17×6) or 102 mins. to do it. His stoppage at the last time (*i.e.*, 18th stoppage) when he reaches the destination is not to be taken into account.

$\therefore$ the man takes $(540+102)$ or 642 mins. or 10 hrs. 42 mins. to go 108 miles.

4. A greyhound pursues a hare and takes 4 leaps for every 5 leaps of the hare, but 3 leaps of the hound are equal to 4 of the hare; compare the speeds of the hound and the hare. [C.U. 1935]

The greyhound takes 4 leaps for every 5 leaps of the hare and 3 leaps of the hound are equal to 4 of the hare. $\therefore$ 1 leap of the hound $= \frac{4}{3}$ leaps of the hare. $\therefore$ 4 leaps of the hound $= \frac{4}{3} \times 4$ or $\frac{16}{3}$ leaps of the hare. $\therefore$ it is seen that when the hound goes a distance equal to $\frac{16}{3}$ leaps of the hare, the hare goes a distance equal to 5 leaps of its own.

$\therefore$ the ratio of the speeds of the hound and the hare $= \frac{16}{3} : 5$

$$= \frac{\frac{16}{3}}{\frac{5}{1}} = \frac{16}{15} = 16 : 15.$$

5. A hare pursued by a greyhound was 60 of her own leaps before him ; while the hare takes 5 leaps, the greyhound takes 4 leaps ; in one leap the hare goes 2 metres and the greyhound 3 metres. In how many leaps will the greyhound overtake the hare ?

While the hound goes 4×3 or 12 metres in 4 leaps, the hare goes 5×2 or 10 metres in 5 leaps.

∴ the hound goes 2 metres more than the hare in his 4 leaps.

60 leaps of the hare $=60\times2$ m. or 120 m.

∴ the hound was 120 metres behind the hare *i.e.* the distance between them was 120 metres.

The hound gains 2 m. over the hare in his 4 leaps,

∴ " " 120 m. " " " $\frac{4}{2}\times120$ or 240 leaps.

∴ the hound overtakes the hare in his 240 leaps.

6. A man has to reach a certain place at a certain time. If he walks 4 kilometres an hour he is 10 minutes too late and if 5 kilometres an hour he is 5 minutes too early. Find the distance he has to walk.

If the man walks 4 Km. an hour, he takes $\frac{1}{4}$ hr. or 15 mins. to go 1 Km. Again, if he walks 5 Km. an hour, he takes $\frac{1}{5}$ hr. or 12 mins. to go 1 Km.

∴ The difference of the times taken to go 1 Km. at the two rates is $(15-12)$ or 3 mins. But here the total difference of the two times is $(10+5)$ or 15 mins. (∵ in the first case he is 10 mins. late, but in the second case he makes up these 10 mins. and is 5 mins. too early).

The difference of time is 3 mins. for a distance of 1 Km.

∴ " " " 15 mins. " " $\frac{1}{3}\times15$ or 5 Km.

∴ the required distance $=5$ Km.

7. Two trains start at the same time from London and Edinburgh and proceed towards each other at the rates of 30 and 50 kilometres per hour respectively. When they meet, the quicker train has run 100 Km. more than the other. What is the distance between London and Edinburgh ? [Cf. D. B. 1938]

The second train runs $(50-30)$ or 20 kilometres more than the first train in 1 hour. ∴ it runs 100 Km. more in $(100\div20)$ or 5 hours. ∴ the two trains meet each other in 5 hours after they start.

They together run (30+50) or 80 Km. in 1 hour.

$\therefore$ they run 80 Km.$\times$5 or 400 Km. in 5 hours.

$\therefore$ the distance between the two places=400 kilometres.

8. A train starts from Howrah at 8 A. M. and reaches Burdwan at 12 A. M., another train leaves Burdwan at 9 A. M. and reaches Howrah at 12 A. M. When do they meet ?

The first train runs the whole distance in 4 hrs., $\therefore$ it runs $\frac{1}{4}$ of the distance in 1 hr. The second train runs the whole distance in 3 hours, $\therefore$ it runs $\frac{1}{3}$ of the distance in 1 hr.

The first train starts 1 hr. beforehand and has gone $\frac{1}{4}$ of the distance in that one hour. $\therefore$ at 9 A. M. the distance between the two trains is $(1-\frac{1}{4})$ or $\frac{3}{4}$ of the whole distance.

They together cover $(\frac{1}{4}+\frac{1}{3})$ or $\frac{7}{12}$ of the distance in 1 hr.

$\therefore$ the time required to cover $\frac{3}{4}$ of the distance is $(\frac{3}{4}\div\frac{7}{12})$ hrs. or $1\frac{2}{7}$ hrs. or 1 hr. $17\frac{1}{7}$ mins. $\therefore$ the two trains meet 1 hr. $17\frac{1}{7}$ mins. after 9 A. M. *i. e.*, at $17\frac{1}{7}$ minutes past 10 A. M.

9. A train 88 metres long is running at the rate of 35 Km. 200 metres an hour. How long will it take to pass a telegraph post ?

The train has to run its own length to pass a post.

Here we have to find the time the train takes to run 88 metres.

The train runs 35 Km. 200 m. or 35200 m. in 1 hr. or 60$\times$60 secs.,

$$\therefore \text{ it runs 1 metre in } \frac{60\times60}{35200} \text{ seconds,}$$

$$\therefore \text{ it runs 88 m. in } \frac{60\times60\times88}{35200} \text{ secs. or 9 seconds.}$$

$\therefore$ the reqd. time=9 seconds.

10. Two trains 110 yds. and 88 yds. long are running at the rates of 20 miles an hour and 25 miles an hour respectively. How long will they take to cross each other (*a*) when they are running in the same direction, (*b*) when they are running in the opposite directions ? [C. U. 1947]

(*a*) The total length of the two trains=(110+88) or 198 yards. When they run in the same direction, they pass each

other in the time in which 198 yards are passed over at the relative speed of (25 − 20) or 5 miles an hour.

$$\therefore \text{ the reqd. time} = \frac{198}{5 \times 1760} \text{ hr.} = \frac{198 \times 60 \times 60}{5 \times 1760} \text{ seconds}$$
$$= 81 \text{ seconds.} \quad \text{(Ans.)}$$

(b) When the two trains run in opposite directions, they pass each other in the time in which 198 yards *i.e.*, the sum of the lengths of the two trains are passed over at the relative speed of (20 + 25) or 45 miles per hour. $\therefore$ the required time $= \frac{198}{45 \times 1760}$ hr.

$$= \frac{198 \times 60 \times 60}{45 \times 1760} \text{ seconds} = 9 \text{ seconds.} \quad \text{(Ans.)}$$

11. A man standing on the platform of a station notices that a train whose speed is 36 miles an hour passes the platform in 20 secs. If the length of the platform is 200 yards, what is the length of the train?

[P. U. 1925]

The train passes the platform in 20 seconds. $\therefore$ it goes in 20 seconds a distance equal to the sum of its own length and the length of the platform.

In 1 hr. or 60×60 seconds the train runs 36×1760 yds.,

$\therefore$ in 1 second it runs $\frac{36 \times 1760}{60 \times 60}$ yds.

$\therefore$ in 20 seconds it runs $\frac{36 \times 1760 \times 20}{60 \times 60}$ yds. or 352 yards.

$\therefore$ The total length of the train and the platform = 352 yds.

$\therefore$ the length of the train = 352 yds. − 200 yds. = 152 yards.

12. A train passes a bridge 177 metres long in 20 seconds and another bridge 100 metres long in 13 seconds. Find the length of the train and the rate at which it runs.

The train passes its own length + 177 m. in 20 seconds. ...(1)

Again, " " " " " + 100 m. in 13 seconds...(2)

Subtracting (2) from (1) we have, the train passes 77 metres n 7 seconds. $\therefore$ the train passes $\frac{77 \times 20}{7}$ or 220 m. in 20 seconds.

$\therefore$ from (1) we get, the length of the train + 177 m. = 220 m.

$\therefore$ the length of the train = 220 m. − 177 m. = 43 metres.

Again, the train runs 77 m. in 7 seconds.

$\therefore$ " " " 11 m. in 1 second.

$\therefore$ " " " 11 m. $\times 60 \times 60$ or 39600 m. in 1 hour.

$\therefore$ the speed of the train = 39·6 kilometres per hour.

13. A train 110 yds. long overtook a person walking along the line at the rate of 3 miles an hour, and passed him completely in 9 seconds; afterwards it overtook another person and passed him in $9\frac{3}{8}$ seconds. At what rate was the second person walking? [W.B.S.F. '52]

The first man walks $\frac{3\times9}{60\times60}$ or $\frac{3}{400}$ miles in 9 seconds.

The length of the train $=110$ yards$=\frac{1}{16}$ mile.

The train runs in 9 seconds its own length+the distance the man walks in 9 seconds *i.e.*, ($\frac{1}{16}$ mile$+\frac{3}{400}$ mile) or $\frac{7}{100}$ mile.

$\therefore$ the train runs in $9\frac{3}{8}$ seconds $\frac{7}{100\times9}\times\frac{75}{8}$ mile or $\frac{7}{96}$ mile.

$\therefore$ the second person walks in $9\frac{3}{8}$ seconds a distance

$=(\frac{7}{96}$ mile $-$ the length of the train$)=\frac{7}{96}$ mile $-\frac{1}{16}$ mile

$=\frac{1}{96}$ mile,

$\therefore$ the second person walks $\frac{1}{96}\times\frac{8}{75}\times60\times60$ m. or 4 miles per hour.

14. A boat rows 1 kilometre in 6 minutes down the river and 6 kilometres in 1 hour up the river. Find the speed of the current per hour.

[With the current, a boat goes its own speed+the speed of the current in 1 hr.; against the current it goes its own speed $-$ the speed of the current.]

The boat goes 1 kilometre in 6 minutes

i.e., 10 Km. in 60 mins. or 1 hour down the river.

$\therefore$ the speed of the boat+the speed of the current
$=10$ Km. per hour...(1)

Again, the boat advances 6 Km. in 1 hour up the river,

$\therefore$ the speed of the boat $-$ the speed of the current
$=6$ Km. in 1 hour...(2)

Now subtracting (2) from (1) we get

$2\times$the speed of the current$=4$ Km.,

$\therefore$ the current flows at the rate of 2 Km. per hour.

15. A man can row 9 miles in 1 hour in still water, but takes thrice as much time in going against the current. Find the velocity of the current per hour. [C. S.]

The boat goes 9 miles in 1 hour in still water and 9 miles in 3 hours against the current. $\therefore$ it goes 3 miles in 1 hr. against the current.

$\therefore$ the speed of the boat – the speed of the current $=3$ miles per hour. But the speed of the boat is 9 miles per hr.

$\therefore$ the reqd. speed of the current $=(9-3)$ or 6 miles per hour.

*16. Guns are fired at intervals of 5 minutes at a certain place towards which a train is approaching. A man in the train hears two successive reports at intervals of 4 minutes 49 secs.; if sound travels 1156 feet per second, find the speed of the train per hour.

Suppose, the guns are fired at the place marked A and the man hears the first and second reports of the guns at the places marked B and C respectively. If the man remained at B, he would have heard the second report at an interval of 5 mins. But he hears two successive reports at an interval of 4 mins. 49 secs. $\therefore$ it is evident that as the man proceeds the distance BC towards the guns, the second report has not to travel the distance CB to reach him and consequently the second report reaches him in (5 mins. – 4 mins. 49 secs.) or 11 secs. less time.

$\therefore$ the train takes 4 mins. 49 seconds and the sound 11 secs. to go the distance BC. $\therefore$ BC $=1156$ ft. $\times 11$.

$\therefore$ in 4 mins. 49 secs. or 289 secs., the train goes 1156×11 ft.,

$\therefore$ in 1 sec. the train goes $\frac{1156 \times 11}{289}$ ft.

$\therefore$ in 1 hour the train goes $\frac{1156 \times 11 \times 60 \times 60}{289 \times 3 \times 1760}$ miles or 30 miles.

17. Two men start together from the same place to go round a circular path. One takes 10 minutes and the other 12 minutes to make one complete round. When will they meet again (1) if they walk in the same direction, (2) if they walk in opposite directions?

The first man goes $\frac{1}{10}$ and the second man $\frac{1}{12}$ of the circular path in 1 minute.

(1) $\therefore$ if they walk in the same direction, the distance between them is $(\frac{1}{10}-\frac{1}{12})$ or $\frac{1}{60}$ of the complete circuit in 1 min. They meet each other when the distance between them is the complete circuit. $\therefore$ they are together again at the end of $(1\div\frac{1}{60})$ or 60 mins. or 1 hr.

(2) If the two men walk in opposite directions, the distance between them is $(\frac{1}{10}+\frac{1}{12})$ or $\frac{11}{60}$ of the circuit in 1 min.

$\therefore$ they are together again in $(1\div\frac{11}{60})$ or $\frac{60}{11}$ or $5\frac{5}{11}$ mins.

18. A, B and C start together from the same place to walk round a circular path 5 miles round. A walks at the rate of $2\frac{1}{2}$ miles, B 3 miles and C 2 miles per hour. When will they next meet together at the starting place? [M. V. 1927]

A takes $(5\div2\frac{1}{2})$ or 2 hrs., B $(5\div3)$ or $\frac{5}{3}$ hrs. and C $(5\div2)$ or $\frac{5}{2}$ hrs. to go a distance of 5 miles. $\therefore$ they come back to the starting place at intervals of 2 hrs., $\frac{5}{3}$ hrs. and $\frac{5}{2}$ hrs. respectively. The L.C.M. of 2 hrs., $\frac{5}{3}$ hrs. and $\frac{5}{2}$ hrs.=10 hrs.

$\therefore$ A, B and C will next meet together at the starting place in 10 hours.

19. A, B and C start from the same place of a circular path 60 Km. round. A and B walk in the same direction and C in the opposite direction at the rate of 2, 5 and 3 Km. per hour respectively. When will they next meet together?

B gains $(5-2)$ or 3 Km. over A in 1 hour.

$\therefore$ he gains 60 Km. or a complete circuit in $\frac{60}{3}$ or 20 hrs.

$\therefore$ A and B meet together at the end of every 20 hrs.

Again, as A and C go in opposite directions (*i. e.*, approach each other),

$\therefore$ The distance betweem them is reduced by $(2+3)$ or 5 Km. in 1 hour. $\therefore$ 60 Km. is reduced in $\frac{60}{5}$ or 12 hrs.

$\therefore$ A and C meet in every 12 hrs.

The L.C.M. of 20 hrs. and 12 hours=60 hrs.

$\therefore$ A, B and C will next meet after 60 hours.

20. A circular track is 984 yds. in circumference. Two men start to run round in opposite directions from the same point; one runs at the rate of 10 miles and the other at $10\frac{1}{2}$ miles an hour. Find when and where they will meet (i) for the first time, (ii) for the second time. [C. U. 1934]

The two men approach each other from opposite directions;

$\therefore$ the distance between them is reduced by $(10+10\frac{1}{2})$ or $20\frac{1}{2}$ miles per hour. $\therefore$ they together pass over the whole track in $(\frac{984}{1760}\div20\frac{1}{2})$ hrs. or $\frac{984\times2\times60}{1760\times41}$ mins. or $1\frac{7}{11}$ mins.

$\therefore$ they meet for the first time after $1\frac{7}{11}$ mins. and for the second time after $1\frac{7}{11}$ mins.$\times2$ or $3\frac{3}{11}$ mins.

Again, the first man goes 10×1760 yards in 60 minutes.

$\therefore$ he goes $\frac{10\times1760}{60}\times\frac{18}{11}$ or 480 yds. in $1\frac{7}{11}$ mins. and 480 yds.$\times2$ or 960 yards in $3\frac{3}{11}$ mins.

$\therefore$ (1) they will meet for the first time after $1\frac{7}{11}$ mins., when the first man has run 480 yds. and (ii) they will meet for the second time after $3\frac{3}{11}$ mins. when the first man has run 960 yards.

*21. In the game of skill A can give B and B can give C 10 points out of a game of 50 points. How many should A give C?

[C. U. 1879]

A makes 50 points while B makes 40 points.

Again B makes 50 points while C makes 40 „

$\therefore$ B „ 1 point „ C makes $\frac{40}{50}$ „

$\therefore$ B „ 40 points „ C makes $\frac{40}{50}\times40$ points

or 32 points

$\therefore$ A can give C $(50-32)$ or 18 points in a game of 50 points.

*22. In a mile race A beats B by 30 yds. and C by 50 yards, and B beats C by 2 secs. in the same race. How long does each take to run the race?

While A runs 1760 yards, B runs $(1760-30)$ or 1730 yards; and while A runs 1760 yards, C runs $(1760-50)$ or 1710 yds.

$\therefore$ while B runs 1730 yards, C runs 1710 yards.

$\therefore$ while B runs 1760 yards, C runs $\frac{1710\times1760}{1730}$

or $1739\frac{113}{173}$ yards

$\therefore$ in a mile race B beats C by $(1760-1739\frac{113}{173})$ or $20\frac{60}{173}$ yards.

$\therefore$ from the given condition, C runs $20\frac{60}{173}$ yards in 2 seconds.

$\therefore$ C runs 1 yard in $\frac{2\times173}{3520}$ second.

$\therefore$ C runs 1760 yards or 1 mile in $\frac{2\times173\times1760}{3520}$ or 173 seconds.

In a mile race B beats C by 2 seconds,

$\therefore$ B runs 1 mile in $(173-2)$ or 171 seconds.

Again B runs 1760 yards in 171 seconds,

$\therefore$ B runs 1730 yards in $\dfrac{171\times1730}{1760}$ or $168\frac{15}{176}$ seconds.

But while B runs 1730 yards, A runs 1760 yards.

$\therefore$ A runs 1 mile in $168\frac{15}{176}$ seconds.

$\therefore$ A takes $168\frac{15}{176}$ secs, B 171 secs.

and C 173 secs. to run 1 mile.

23. Messengers travelling 15 kilometres an hour are sent out every 10 minutes to meet a person approaching at the rate of 10 Km. an hour; at what intervals will they meet him? [Cf. P. U. '26]

Messengers are sent out every 10 minutes.

A messenger travels 15 Km. in 60 minutes,

$\therefore$ he travels $\dfrac{15\times10}{60}$Km. or $\dfrac{5}{2}$ Km. in 10 minutes.

$\therefore$ the second messenger is $\frac{5}{2}$ Km. behind the first messenger.

$\therefore$ When the man meets the first messenger, the distance between the man and the second messenger is $\frac{5}{2}$ Km. The man and the second messenger are approaching each other from opposite directions. They together cover $(15+10)$ or 25 Km. per hour. Therefore $\frac{5}{2}$ Km. are covered in $(\frac{5}{2}\div25)$ hrs. or $\frac{1}{10}$ hr. or 6 minutes.

$\therefore$ the messengers meet the man at intervals of 6 minutes.

24. If a train maintains an average speed of 42 kilometres an hour, it arrives at its destination punctually; if, however, the average speed is 40 Km. an hour, it arrives 15 minutes late. Find the length of the journey. [Cf. D. B. 1927]

If the train runs 42 Km. per hour, it takes $\frac{1}{42}$ hr. to run 1 Km. If it runs 40 Km. per hour, it takes $\frac{1}{40}$ hr. to run 1 Km. $\therefore$ in running 1 Km. it takes $(\frac{1}{40}-\frac{1}{42})$ or $\frac{1}{840}$ hr. more at the latter rate than at the former. $\therefore$ the train is $\frac{1}{840}$ hr. late in 1 Km.

$\therefore$ it is 15 mins. or $\frac{1}{4}$ hr. late in $(\frac{1}{4}\div\frac{1}{840})$ Km. or 210 Km.

$\therefore$ The required length $=$ 210 kilometres.

25. A railway train having travelled at $\frac{3}{5}$ of its proper speed reaches its journey's end $2\frac{1}{2}$ hours behind time. In what time would the journey have been done ? [P. U. 1883]

Suppose, the train takes x hours to make the journey.

$\therefore$ if it travels at $\frac{3}{5}$ of its proper speed, it takes $x \times \frac{5}{3}$ hrs.

$\therefore$ $(\frac{5}{3}x - x) = 2\frac{1}{2}$, or, $\frac{2}{3}x = \frac{5}{2}$, $\therefore$ $x = \frac{5}{2} \times \frac{3}{2} = \frac{15}{4}$.

$\therefore$ the reqd. time $= \frac{15}{4}$ hrs. $=$ 3 hrs. 45 mins.

*26. Guns are fired at intervals of 10 minutes in a town towards which a train is approaching at the rate of 30 miles an hour ; if sound travels 1156 ft. per sec., at what interval will a man in the train hear two successive reports ?

A B C

Suppose the guns are fired at the place A and the man hears the first report at C and the second report at B.

Also suppose the man hears two reports at an interval of x minutes. $\therefore$ the man goes the distance CB in x minutes and sound travels the distance BC in $(10 - x)$ minutes. (vide Exp. 16)

$$\therefore \quad \frac{x \times 30}{60} = \frac{(10 - x) \times 60 \times 1156}{3 \times 1760}$$

[$(10 - x)$ minutes are reduced to seconds and 1156 ft. to yards.]

$$\therefore \quad x = \frac{289}{30} = 9\tfrac{19}{30}.$$

$\therefore$ the man will hear two successive reports at intervals of $9\frac{19}{30}$ mins. or 9 mins. 38 secs.

27. A man shooting at a target, distant 561 yds., hears the bullet strike the target $3\frac{1}{2}$ secs. after he fires. A spectator on the other side of the target and at a distance of 187 yds. from it, hears the shot strike the target $\frac{1}{2}$ sec. after he heard the report. Find the velocity of the bullet and sound.

A 561 yds. B 187 yds. C

Suppose, A is the place of shooting, B and C are the positions of the target and the spectator respectively.

$\because$ the man at A hears the bullet strike the target $3\frac{1}{2}$ secs. after he fires, $\therefore$ the time of the bullet going the distance AB + the time of the sound coming the distance BA $= 3\frac{1}{2}$ secs...(1)

Again, the spectator at C has first heard the report and $\frac{1}{2}$ sec. thereafter he hears the shot strike the target.

$\therefore$ (the time of the bullet going the distance AB+the time of the sound going the distance BC) − the time of the sound going the distance AC$=\frac{1}{2}$ sec...(2)

Now, from (2) we have, the time of the bullet going the distance AB+the time of the souud going the distance BC − the time of the sound going the distance AB − the time of the sound going the distance BC$=\frac{1}{2}$ sec.,

Or, the time for the bullet going the distance AB − the time for the sound going the distance AB$=\frac{1}{2}$ sec...(3)

$\therefore$ from (1)+(2) we have, $2\times$the time of the bullet going the distance AB=4 sec. $\therefore$ the bullet goes 561 yds. (*i.e.*, the distance AB) in 2 secs, $\therefore$ it goes $\frac{561}{2}$ yds. or $841\frac{1}{2}$ feet per second.
$\therefore$ from (1), the time of the sound going the distance AB or 561 yards$=3\frac{1}{2}$ secs. − 2 secs.$=\frac{3}{2}$ secs.

$\therefore$ the sound travels (561 yds.$\div\frac{3}{2}$) or 1122 feet per second.

28. In running a four-mile race on a course half a mile round, A overlaps B at the middle of the sixth round. Assuming that each competitor runs at a constant speed, by what distance will A win ? [Gau. U. 1948]

It requires 8 rounds to complete a four-mile race. In a race in a circular path one overtakes or meets another when one gains one complete round over the other. (1) Here if A overtakes B at the middle of the sixth round, it is evident that A gains 1 round or $\frac{1}{2}$ mile over B while making $5\frac{1}{2}$ rounds.

$\therefore$ A goes $\frac{1}{2}$ mile more than B in $5\frac{1}{2}$ rounds.

$\therefore$ A goes $\frac{1}{2}\times\frac{2}{11}\times8$ or $\frac{8}{11}$ mile more than B in 8 rounds.

$\therefore$ A wins by $\frac{8}{11}$ mile.

(2) Again, if A meets B while B makes $5\frac{1}{2}$ rounds, then A makes $6\frac{1}{2}$ rounds by that time.

$\therefore$ A wins by $\frac{1}{2}$ mile on making $6\frac{1}{2}$ rounds.

$\therefore$ A wins by $\frac{1}{2}\times\frac{2}{13}\times8$ or $\frac{8}{13}$ mile in making 8 rounds.

*29. Buses run along a certain route at intervals of 10 minutes at a speed of 8 miles an hour. A man walking steadily in the same direction is overtaken by a bus every 15 mins. (a) What is the rate of walking? (b) If he walked at the same rate in the opposite direction, how often would he meet a bus?

[D. B. '45]

(a) Suppose that when the man is at the point B, a bus overtakes him and then the next bus is behind him at A at a distance of 10 minute's run at its own rate.

The man goes from B to C in 15 minutes and the second bus goes from A to C to overtake him at C in the same period of 15 minutes.

$\therefore$ the bus takes 10 mins. to go from A to B, $\therefore$ it takes $(15-10)$ or 5 mins. to go from B to C.

$\therefore$ the distance BC=the distance the bus runs in 5 minutes

$=\frac{8}{60}\times 5$ mile$=\frac{2}{3}$ mile.

$\therefore$ the man goes $\frac{2}{3}$ mile in 15 mins.

$\therefore$ the man goes $\frac{2}{3}\times 4$ or $2\frac{2}{3}$ miles in an hour.

(b) Vide Ex. 23 to solve this part.

[Ans. The man will meet a bus at an interval of $7\frac{1}{2}$ minutes, if he walks in the opposite direction.]

30. A person, who can walk down a hill at the rate of $3\frac{1}{2}$ kilometres an hour and up at the rate of $2\frac{1}{2}$ Km. an hour, ascends and comes down to his starting point after walking 4 hrs. 30 mins. How far did he ascend?

The man walks up a hill at the rate of $2\frac{1}{2}$ Km. per hour. He walks down it at the rate of $3\frac{1}{2}$ Km. an hour.

$\therefore$ the time of walking up a distance : the time of walking down the same distance$=\frac{7}{2}:\frac{5}{2}=7:5$.

Now, if we divide the given time $4\frac{1}{2}$ hours in the ratio of 7 : 5 we get the times of walking up and walking down the hill respectively.

$\therefore$ The man walks up the hill for $4\frac{1}{2}\times\frac{7}{7+5}$ or $\frac{21}{8}$ hours.

$\therefore$ the man ascends $\frac{5}{2}$ miles$\times\frac{21}{8}$ or $6\frac{9}{16}$ miles.

Exercise 13

1. A train running at the rate of 30 kilometres an hour stops half an hour after every 75 Km. How long will it take to make a journey of 375 kilometres ?

2. A constable is 100 yds. behind a thief, the former takes 6 mins. and the latter 10 minutes to run a mile. At what distance will the constable overtake the thief ?

3. A starts 3 hours after B who rides 12 kilometres an hour and overtakes him after 5 hours ; at what rate is A travelling ?

4. Two trains start at the same time from Mirzapore and Delhi and proceed towards each other at the rates of 16 and 21 Km. per hour respectively. When they meet, it is found that one train has run 60 Km. more than the other. Find the distance between the two stations.

5. A train leaves Burdwan at 8 A. M. and reaches Howrah at 10 A. M. and another train leaves Howrah at 8-30 A. M. and arrives at Burdwan at 11 A.M. When did they meet ?

6. Find how many seconds it will take a train, 130 yds. long travelling at $33\frac{3}{4}$ miles per hour, to pass completely through a station 200 yds. long. [C. U. '51 ; D.B. 1936]

7. A man walks a certain distance, and rides back in 3 hours 45 minutes ; he could ride both ways in $2\frac{1}{2}$ hours. How long would he take to walk both ways ? [P. U. 1949]

8. A launch travels down stream to a point 42 Km. away in 3 hours. The return journey takes 7 hours. Calculate the speed of the launch and of the current. [B. C. S. '39]

9. The 4 P. M. passenger train from Delhi to Tundla stops first at Ghariabad $12\frac{3}{4}$ miles distant at 4-30 P. M., the whole journey is $127\frac{1}{2}$ miles and 20% of the time is expended in stoppages ; at what time is the train due at Tundla ? [A. U.]

10. A and B are at a distance of 95 miles and start at 7 A.M., cycling towarns each other at the rate of 8 and 10 miles per hour respectively. After an hour A has an accident which detains him for half an hour after which he continues as before. Find when they meet. [C. U. '32]

11. A race course is $2\frac{1}{4}$ kilometres round. Four men start together to walk round it. They walk at the rates of $3\frac{1}{4}$, $3\frac{3}{4}$, $4\frac{1}{2}$ and 5 Km. per hr. respectively. Show that they will next meet at the starting point after 9 hours.

12. In a two-mile race A wins, B being 22 yds. behind, and C 106 yds. behind B. By how much would B beat C in a three-mile race ? [D. B. 1932]

13. A person has to walk a certain distance in a certain time. If he walks at the rate of 5 Km. an hour he is 5 minutes late, if he increases his speed to 6 Km. an hour he is 10 minutes too early. What is the distance he has to walk ?

14. A and B run a race. A has a start of 40 yds. and sets off 5 mins. before B at the rate of 10 miles an hour. How soon will B overtake him, if he runs 12 miles per hour ? [D. B. 1926]

15. A train having to run 250 Kilometres is obliged after 103 Km. to reduce its speed by $\frac{1}{5}$, and consequently reaches its destination 1 hr. 10 mins. behind time. What is its ordinary speed ?

16. Messengers travelling 10 kilometres an hour are sent out every 12 minutes to meet an officer approaching at the rate of 5 kilometres an hour. At what intervals of time will they meet him ?

17. A gives B 290 yds. start in a mile race, and is beaten by 80 yds. If A and B start together, who will win and by how many yards ? [D.B. 1940]

18. In a race A beats B by 44 yds., and C by 83 yds. When B and C run over the course together, B wins by 40 yards ; find the length of the course. [D. B. 1939]

19. Two guns are fired from a fort and a man riding towards it at the rate of 14 miles an hour hears the reports at an interval of 10 mins. ; if sound travels 1120 ft. per second, at what interval are the guns fired ?

[**Hints :** Find the time the sound takes to go the distance travelled by the man in 10 minutes. The reqd. time=10 minutes +that time.]

20. A person shooting at a target, 550 yds. distant, hears the bullet strike the target 4 seconds after he fires. A spectator, equally distant from the target and the shooter, hears the shot strike the target $2\frac{1}{2}$ seconds after he heard the report. Find the velocity of sound. [D. B. 1933]

21. A train running at $\frac{4}{7}$ of its own speed reached a place in 14 hours. In what time could it reach there running at its own speed ?

22. A and B started at the same time for a certain place. B walking at $\frac{4}{7}$ of A's rate reached the place 3 hrs. 15 mins. after A. In what time did each reach the place ?

23. If a train runs 30 kilometres per hr., it takes 1 hour more to travel a distance than if it runs at 40 kilometres per hr. Find the distance. [See Examp. No 23]

24. On a stream, B is intermediate to and equidistant from A and C ; a boat can go from A to B and back in 5 hrs. 15 mins. and from A to C in 7 hrs. How long would it take to go from C to A ? [B. U. 1892]

25. A, B and C can walk at the rates of 3, 4 and 5 kilometres an hour ; they start from Poona at 1, 2, 3 o'clock respectively ; when B catches A, B sends him back with a message to C ; when will C get the message ?

26. A snail creeps 9 decimetres up a pole and slips down 2 decimetres in alternate minutes. If the pole be 6 metres high, in what time will it get to the top ?

27. The path from A to B is 3 Km. uphill, then 8 Km. on level ground and then 6 Km. downhill. A man walks 1 Km. an hour uphill, 4 Km. an hr. on level path and 6 Km. an hr. downhill. How long will he take to make the journey from A to B and back ?

28. Two trains start at the same time, one from Calcutta to Madhupur, the other from Madhupur to Calcutta. If they arrive at Madhupur and Calcutta respectively 1 hour and 4 hours after they pass each other, show that one travels twice as fast as the other. [C. U. 1946]

[**Hints :** Suppose, the first and the second train travel x and y kilometres per hour respectively, and they meet each other a

hours aftea they start. $\therefore$ the first train travels ax miles and the second train ay miles when they meet. $\therefore$ after they pass each other, the first train has gone ay miles in 1 hour and the second train ax miles in 4 hours.

$\therefore$ We have $x=ay$...(1) and $4y=ax$...(2)

Dividing (1) by (2) we get $\dfrac{x}{4y}=\dfrac{y}{x}$ or $x^2=4y^2$, $\therefore$ $x=2y$.

$\therefore$ the first train travels twice as fast as the second train.]

29. A is at Calcutta and B at Ichapore ($21\frac{1}{2}$ miles apart). A leaves for Ichapore at 9 A.M. and B for Calcutta at 9-30 A. M. They meet at 12-8 P. M. and find that B has walked at a rate $\frac{1}{2}$ mile per hour more than A. How many miles has each travelled ?

[B. C. S. 1946]

[**Hints** : B travels for $\frac{79}{30}$ hrs. and A for $\frac{47}{15}$ hrs. before they meet. B travels $\frac{1}{2}\times\frac{79}{30}$ or $\frac{79}{60}$ miles in $\frac{79}{30}$ hrs. at a rate of $\frac{1}{2}$ mile per hour more than A.

$\therefore$ A and B have gone ($21\frac{1}{2}$ miles $-\frac{79}{60}$ miles) or $\frac{1211}{60}$ miles in $\frac{47}{15}$ and $\frac{71}{30}$ hours respectively at equal speed.

The ratio of their times$=\frac{47}{15}:\frac{79}{30}=94:79$.

Now, divide $\frac{1211}{60}$ miles in the ratio of $94:79$ and you will find the distances travelled by A and B.]

CLOCK PROBLEMS

[**N. B.** (1) The dial of a clock is divided into 60 equal parts. Each part is called a *minute-division* and in one minute the minute-hand passes over one such division. So, the minute-hand makes a complete round of the whole dial (circumference), that is, passes over 60 minute-divisions in 60 minutes. By that time the hour-hand passes over 5 minute-divisions.

$\therefore$ In a given time the hour-hand moves $\frac{1}{12}$ as much as the minute-hand.

(2) In 60 minutes the minute-hand gains $(60-5)$ of 55 minute-divisions over the hour-hand.

(3) When the hands are coincident, the number of minute-divisions between them is zero.

(4) The hour-hand and the minute-hand are at right-angles to each other, when they are 15 minute-divisions apart.

(5) When the hands are opposite to each other, the number of minute-divisions between them is 30.

(6) When the two hands are either coincident or 30 minute-divisions apart they are said to lie in the same straight line.

[Easy clock problems have been shown in the chapter of Algebraical problems. First work out the sums of that chapter.]

Examples [14]

1. At what time between 1 and 2 o'clock will the hour and minute hands of a clock be together ? [W. B. S. F. '35]

At 1 o'clock the minute-hand is 5 minute-divisions behind the hour-hand. ∴ the minute-hand has to gain 5 minute-divisions over the hour-hand to be together between 1 and 2 o'clock.
Now, the minute-hand gains 55 divisions on hour-hand in 60 mins.,

∴ " " " 5 " in $\frac{60 \times 5}{55}$ or $5\frac{5}{11}$ mins.

∴ the time required is $5\frac{5}{11}$ mins. past 1.

2. A clock which gains 48 seconds a day, is set right at noon ; what time does it indicate at 4 P. M. the following day ? Also what is the true time when the clock indicates 4 P. M. ?

The time from the noon (12 o'clock) of one day to 4 P. M. of the following day=28 hours.

In 24 hrs. the clock goes 48 secs. fast,

∴ In 28 hrs. " " " $\frac{48}{24} \times 28$ or 56 secs. fast.

∴ the clock will indicate 56 secs. past 4 at 4 P. M. the following day.

Again, when the clock shows 24 hrs. 48 secs., the true time is 24 hrs.

i.e., when it shows $\frac{1801}{75}$ hrs., the true time is 24 hrs.

∴ " " 28 hrs. " " $\frac{28 \times 75 \times 24}{1801}$ hrs.

$=27$ hrs. $59\frac{121}{1801}$ minutes.

∴ when the clock indicates 4 P. M. the following day, the true time is $59\frac{121}{1801}$ minutes past 3.

3. The hands of a clock coincide after every 64 minutes of correct time. How much is the clock fast or slow in 24 hours ?

[D B. 1935]

After the two hands of a clock once coincide, they will again coincide when they are 60 minute-divisions apart.

Now, they are 55 minute-divisions apart in 60 minutes

$\therefore$ „ „ 60 „ „ in $\frac{60 \times 60}{55}$ or $65\frac{5}{11}$ mins.

$\therefore$ the two hands of the clock coincide after every $65\frac{5}{11}$ minutes of its own time.

$\therefore$ for $65\frac{5}{11}$ minutes of the clock, the correct time is 64 minutes, $\therefore$ the clock goes fast by $(65\frac{5}{11} - 64)$ or $1\frac{5}{11}$ minutes in every 64 minutes of correct time.

$\therefore$ the clock is fast by $\dfrac{16}{11 \times 64}$ min. or $\dfrac{1}{44}$ minute in 1 minute.

$\therefore$ it is $\dfrac{1 \times 60 \times 24}{44}$ mins. or $32\frac{8}{11}$ minutes fast in 24 hours.

4. Two clocks strike 9 A. M. at right time on Sunday. One of them gains 30 seconds and the other loses 20 seconds in one hour. (1) When will the faster clock gain half an hour over the slower ? (2) What hour will each then indicate ?

(1) One clock goes 30 secs. fast and the other 20 secs. slow in 1 hour.

$\therefore$ One clock gains on the other $(30+20)$ secs. or 50 secs. or $\frac{5}{6}$ min. in 1 hour of correct time.

The first clock gains $\frac{5}{6}$ min. in 1 hour,

$\therefore$ „ „ „ 1 „ $\frac{6}{5}$ „

$\therefore$ „ „ „ 30 „ $\frac{6}{5} \times 30$ or 36 hours.

$\therefore$ after 36 hours from 9 A. M. on Sunday (*i.e.*, at 9 P. M. on Monday) the first clock will gain half an hour over the second clock.

(2) The second clock loses (or is slow) 20 secs. or $\frac{1}{3}$ min. in one hour, $\therefore$ it loses $\frac{1}{3} \times 36$ or 12 minutes in 36 hours.

$\therefore$ the second clock indicates 48 mins. past 8 at 9 P. M. on Monday. And by that time the first clock gains 30 minutes over the second clock, $\therefore$ the first clock will then indicate 18 minutes past 9.

5. It is between 2 and 3 o'clock; but a person looking at the clock and mistaking the hour-hand for the minute-hand fancies that the time of the day is 57 mins. earlier than the reality. What is the true time? [P.U. 1912]

The time fancied by the man through mistake is 57 minutes earlier than the correct time.

Evidently the minute-hand is behind the hour-hand.

∴ it is $(60-57)$ or 3 minute-divisions behind the hour-hand. Now, just at 2 o'clock the minute-hand was 10 minute-divisions behind the hour-hand and now it is 3 minute-divisions behind the hour-hand.

∴ the minute-hand has gained $(10-3)$ or 7 minute-divisions over the hour-hand.

Now, the minute-hand gains 55 minute-divisions over the hour-hand in 60 minutes,

∴ It gains 7 minute-divisions in $\frac{60}{55}\times 7$ or $7\frac{7}{11}$ mins.

∴ the true time is $7\frac{7}{11}$ mins. past 2.

6. A clock gains 2 mins. per day. It shows the correct time at 1 P. M. on January 1st. 1936. On what date and at what time will it again show the correct time? [B. C. S. 1937]

The clock shows the correct time at 1 P. M. on the 1st January and from that time it is 2 mins. fast per day. The rule is that in going fast when the clock is 12 hours fast it will again show the correct time and then at 1 A. M. correct time the clock will also show 1 A.M.

Now, the clock is 2 mins. fast in 1 day

∴ " " " 1 hr. fast in 30 days

∴ " " " 12 hrs. fast in 30×12 or 360 days.

∴ The clock will show the correct time 360 days after 1st January. The year 1936 being a leap-year has 366 days. Leaving 1st January and the last 5 days of December we have 360 days left. ∴ The required time is 1 P.M. on 26th December, 1936.

7. A clock was 12 mins. too slow at noon on Sunday; on Thursday at the same hour it is 10 mins. too fast. When will it show correct time again?

The time from noon on Sunday to noon on Thursday next is 4 days and the clock is $(10+10)$ or 20 mins. fast in those 4 days.

$\therefore$ it is 5 mins. fast in 1 day. Now we know that the clock indicates 10 mins. past 12 at 12 noon on Thursday. $\therefore$ When the clock will be (12 hrs. − 10 mins.) or 710 mins. fast it will show the correct time. The clock is 5 mins. fast in 1 day.

$\therefore$ it will be 710 mins. fast in $(710 \div 5)$ or 142 days.

$\therefore$ after 142 days the clock will show correct time again.

Exercise 14

1. A clock is 10 mins. too fast at noon ; it loses 2 mins. in an hour ; find the true time when the hands are (i) at right angles, (ii) opposite to each other, and (iii) coincident, between 4 and 5 o'clock.

2. How many times will the hands of a clock pass each other in 24 hours ? [P. U. 1905]

3. The hands of a clock coincide after every 66 minutes of correct time. How much is the clock fast or slow in 24 hours ? [C. U. 1908]

4. Two clocks indicate 12 at the same instant ; one of them gains 7 secs. and the other loses 8 secs. in 12 hours ; after what interval will one have gained half an hour on the other and what o'clock will each then show ?

5. One clock gains 25 secs. an hr. when another loses 1 min. an hr. They are both set at the right time at 8 A.M. on August 15. On what day and at what time will they differ by 1 hour ? [A. U. 1918]

6. A man, who went out between 4 and 5 o'clock and returned between 5 and 6 o'clock, found that the hands of the clock exactly changed places. When did he go out ? [C. U. 1930]

7. A clock in the kitchen loses at the rate of 6·5 seconds an hour when the fire is alight, and gains at the rate of 3·9 secs. an hour when the fire is not burning ; but in the whole day it neither gains nor loses. How long in the 24 hours is the fire burning ? [C. U. 1920]

8. Two clocks points to 2 o'clock at the same instant on the afternoon of 25th April ; one loses 7 secs. and the other gains 8 secs. in 24 hours ; when will one be half an hour before the other, and what time will each clock then show ? [B. U. 1893]

MIXTURE OR ALLIGATION

Examples [15]

1. Two equal glasses are respectively $\frac{1}{3}$ and $\frac{1}{4}$ full of milk; they are then filled up with water, and the contents mixed in a tumbler; find the ratio of milk and water in the tumbler.

$\frac{1}{3}$ of the first glass is filled with milk,

$\therefore$ $(1-\frac{1}{3})$ or $\frac{2}{3}$ of the first glass is filled with water.

$\frac{1}{4}$ of the second glass is filled with milk,

$\therefore$ $(1-\frac{1}{4})$ or $\frac{3}{4}$ „ „ „ water.

$\therefore$ In the tumbler, $(\frac{1}{3}+\frac{1}{4})$ or $\frac{7}{12}$ part is milk and $(\frac{2}{3}+\frac{3}{4})$ or $\frac{17}{12}$ part is water.

$\therefore$ the reqd. ratio of milk and water in the tumbler

$$=\frac{7}{12}:\frac{17}{12}=7:17.$$

2. A vessel is filled with liquid, 3 parts of which are water and 5 parts syrup. How much of the mixture must be drawn off and replaced with water so that the mixture may be half water and half syrup? [M.U., B.U.]

If the mixture be divided into (3+5) or 8 equal parts, 3 of the parts will be water and 5 parts syrup.

$\therefore$ $\frac{3}{8}$ of the mixture is water and $\frac{5}{8}$ of it is syrup.

Afterwards the quantity of syrup will be $\frac{1}{2}$ instead of $\frac{5}{8}$ part.

$\therefore$ the quantity of syrup should be decreased by $(\frac{5}{8}-\frac{1}{2})$ or $\frac{1}{8}$ part.

Now, we are to see how much mixture is to be drawn off to decrease the quantity of syrup by $\frac{1}{8}$ part.

$\frac{5}{8}$ part of the mixture is syrup in 1 part of the mixture

$\therefore$ 1 „ „ „ „ $\frac{8}{5}$ „ „ „

$\therefore$ $\frac{1}{8}$ „ „ „ „ $\frac{8}{5}\times\frac{1}{8}$ or $\frac{1}{5}$ „

$\therefore$ $\frac{1}{5}$ of the mixture must be drawn off and replaced with water.

3. A cask contains 65 litres of a mixture of milk and water mixed in the ratio of 10 : 3 ; how much water must be added to it so that the ratio of milk and water may be 8 : 5 ?

If the mixture of 65 litres be divided into (10+3) or 13 equal parts, 10 of the parts will be milk and 3 parts water.

$\therefore$ the quantity of milk in the mixture $= \frac{10}{13} \times 65$ litres $= 50$ litres and the quantity of water $= (65-50)$ or 15 litres.

Now, suppose that x litres of water are added to the mixture so that the new mixture now contains $(15+x)$ litres of water and the same 50 litres of milk.

$\therefore$ by the given condition,

$\frac{50}{15+x} = \frac{8}{5}$, or, $8x + 120 = 250$, or, $8x = 130$, $\therefore x = 16\frac{1}{4}$.

$\therefore$ $16\frac{1}{4}$ litres of water should be added to the mixture.

4. Three equal glasses are filled with mixture of milk and water. The proportions of milk and water in each glass are as follows—in the first glass as $3:1$, in the 2nd glass as $5:3$, and in the 3rd glass as $9:7$. The contents of the glasses are emptied into a single vessel. Show that the mixture in the vessel will contain milk and water in the ratio of $31:17$.

[D. B. 1929]

$\frac{3}{4}$ part of the first glass is milk and $\frac{1}{4}$ of it is water
$\frac{5}{8}$,, ,, second ,, ,, ,, $\frac{3}{8}$,, ,, ,,
$\frac{9}{16}$,, ,, third ,, ,, ,, $\frac{7}{16}$,, ,, ,,

All the glasses being equal the total quantity of milk $= (\frac{3}{4} + \frac{5}{8} + \frac{9}{16})$ or $\frac{31}{16}$ parts of one glass and the total quantity of water $= (\frac{1}{4} + \frac{3}{8} + \frac{7}{16})$ or $\frac{17}{16}$ parts of one glass.

$\therefore$ the ratio of milk and water in the new mixture $= \frac{31}{16} : \frac{17}{16} = 31 : 17$.

5. Water and milk are mixed in the proportion of $2:7$ in one vessel. In another vessel they are mixed in the proportion of $2:9$. In what proportion would you take mixtures from the two vessels in order to produce a mixture which would contain water and milk in the proportion of $1:4$.

[C. U. 1944]

Let $x : y$ be the proportion in which mixtures have been taken from the two vessels, that is, x parts of the first vessel and y parts of the second vessel have been taken. Now, of the $(2+7)$ or 9 equal parts of the mixture in the first vessel 2 parts are water and 7 parts milk.

$\therefore$ x parts of the first vessel contain $\frac{2x}{9}$ parts of water and $\frac{7x}{9}$ parts of milk. Again, in the second vessel of 11 equal parts there are 2 parts of water and 9 parts of milk, $\therefore$ y parts of the second vessel contain $\frac{2y}{11}$ parts of water and $\frac{9y}{11}$ parts of milk ;

$\therefore$ in the new mixture the quantity of water is $\left(\frac{2x}{9}+\frac{2y}{11}\right)$ parts and that of milk is $\left(\frac{7y}{9}+\frac{9y}{11}\right)$ parts.

$\therefore$ by the given condition,

$$\frac{\frac{2x}{9}+\frac{2y}{11}}{\frac{7x}{9}+\frac{9y}{11}}=\frac{1}{4}, \text{ or, } \frac{22x+18y}{77x+81y}=\frac{1}{4}, \text{ or, } 11x=9y, \therefore \frac{x}{y}=\frac{9}{11}.$$

$\therefore$ The required proportion is 9 : 11.

6. A cask contains 18 kilolitres of spirits ; 2 kilolitres are drawn off and the cask is filled with water ; 2 kilolitres are again drawn off and the cask is filled up as before. This is done a third time. Compare the quantities of spirits and water remaining in the cask.

The cask contains 18 kilolitres of spirit. 2 kilolitres *i.e.* $\frac{2}{18}$ or $\frac{1}{9}$ part of the contents of the cask is being drawn off every time from it. $\therefore$ $(1-\frac{1}{9})$ or $\frac{8}{9}$ of the previous spirit is being left in the cask every time.

Thus, in the first time 2 kl. being drawn off from 18 kl. there remain 16 kl. of spirit. In the second time there remain $16\times\frac{8}{9}$ or $\frac{128}{9}$ kl. of spirit.

In the third time there remain $\frac{128}{9}\times\frac{8}{9}$ or $\frac{1024}{81}$ kl of spirit.

$\therefore$ there remain $(18-\frac{1024}{81})$ or $\frac{434}{81}$ kl. of water in the cask.

$\therefore$ After the third time the spirit and water are in the proportion of $\frac{1024}{81}:\frac{434}{81}=1024:434=512:217$.

7. In what ratio must tea worth 2*s*. 5*d*. per lb. be mixed with tea worth 3*s*. 4*d*. per lb. to make a mixture worth 2*s*. 9*d*. per lb. ?

[D. B. 1930]

[**First Method**.] The price of 1 lb. of tea of the first kind is less than that of 1 lb. of the mixture by (2*s*.9*d*. – 2*s*. 5*d*.) or 4*d*.

Again, the price of 1 lb. of tea of the second kind is greater than that of 1 lb. of the mixture by (3*s*. 4*d*. – 2*s*. 9*d*.) or 7*d*.

Now, the two kinds oi tea should be so mixed that the gain at the first price is equal to the loss at the second price.

∴ the required ratio=7 : 4.

[**N. B**. The first is less by 4*d*. and the second greater by 7*d*. ∴ the ratio should be inverse to this *i.e.*, 7 : 4.

8. In what proportion should milk at 48 P. per kilolitre be mixed with water so that the mixture may be worth 45 P. per kilolitre ?

[**Second Method**.] In the first case loss per 1 kl. of milk is (48 – 45) or 3 P., ∴ the loss on $\frac{1}{3}$ kl. of milk is 1 P.

In the second case, the price of water being nil or 0, there is a gain of 45 P. on mixing 1 kl. of water,

∴ the gain is 1 P. on mixing $\frac{1}{45}$ kl. of water.

∴ if we mix $\frac{1}{3}$ kl. of milk with $\frac{1}{45}$ kl. of water and sell them at 45 P. per kl., there will be neither gain nor loss.

∴ the reqd. ratio of milk and water$=\frac{1}{3}:\frac{1}{45}=15:1$.

9. In what ratio must a grocer mix sugar at 6 as. per seer with sugar at 4 as. per seer so that by selling the mixture at 5 as. 3 p. per seer he may gain $16\frac{2}{3}$% ?

If the mixture be sold at 5 as. 3 p. per seer, the gain is $16\frac{2}{3}$% ; ∴ the cost price of 1 seer of the mixture

$$=\frac{100}{116\frac{2}{3}}\times 5 \text{ as. } 3 \text{ p.}=\frac{100\times 3}{350}\times\frac{21}{4}\text{as.}=4 \text{ as. } 6 \text{ p.}$$

∴ the price of 1 seer of sugar of the first kind is greater than that of 1 seer of the mixture by (6 as. – 4 as. 6 p.) or 18 p.

And the price of 1 seer of sugar of the second kind is less than that of 1 seer of the mixture by (4 as. 6 p. – 4 as.) or 6 p.

∴ the reqd. ratio=6 : 18=1 : 3.

10. A man buys milk at a certain rate per seer and after mixing it with water sells again at the same rate. Find how many chhataks of water there are in every seer of the mixture if the man makes a profit of 20 per cent. [C. U. '32]

Here the man gains 20% on selling the mixture of the milk and water at cost price. $\therefore$ 120 parts of the mixture of milk and water contain 100 parts of milk and 20 parts of water, *i.e.* 120 seers of the mixture contain 20 seers of water. $\therefore$ 1 seer of the mixture contains $\frac{20}{120}$ seer or $\frac{1}{6}$ seer or $2\frac{2}{3}$ chhataks of water.

Exercise 15

1. In a mixture of 72 kilolitres of milk and water mixed in the ratio of 11 : 1, how much water must be added so that milk and water may be in the ratio of 9 : 1 ?

2. A cup contains 3 parts pure milk and 1 part water. How much of the mixture must be withdrawn and water substituted in order that the resulting mixture may be half milk and half water ? [U. P. '11]

3. Two equal glasses are respectively $\frac{1}{3}$ and $\frac{1}{6}$ full of milk ; they are then filled up with water and the contents mixed in a tumbler. Find the ratio of milk to water in the tumbler.

4. How much water must be added to a quintal of milk so that by selling at cost price there may be $12\frac{1}{2}$% profits ?

5. Three equal glasses are filled with a mixture of spirit and water. The proportion of spirit to water in each glass is as follows :—in the first glass 2 : 3, in the 2nd 3 : 4, and in the third 4 : 5. The contents of three glasses are poured into a single vessel ; what is the proportion of spirit to water in it ? [C. U. '29]

6. I buy some tea at Re. 1. 2 as. per lb., and some at Rs. 1. 12 as. per lb. ; in what ratio must they be mixed so that by selling them at Rs. 2. 2 as. 8 p. per lb., I may gain 30 per cent ? [C. U. '39]

7. I mix tea purchased at 4 s. per lb. with tea at 3s. 6d. per lb. in equal quantities. At what rate per lb. should I sell the mixt ure to make a profit of 20 per cent on my outlay ? [C.U.'30]

8. A shopkeeper buys two kinds of sugar one at 42 P. per kilogram, and the other at 63 P. per kilogram. How should he mix them so that he may gain 3 P. on every kilogram, selling the mixture at 60 P. per kilogram ?
[Cf. U. '32]

9. Pure gold at Rs. 50 per tola is mixed with a metal at Rs. 24 per tola in a certain ratio. Find the ratio if by selling the mixture at Rs. $39\frac{3}{5}$ per tola, the seller makes a profit of 10% on the cost price.
[C. U. '43]

10. A man buys milk at $2\frac{1}{2}$d. per quart, dilutes it with water and sells the mixture at 3d. per quart. How much water is added to each quart if his profit is 60 per cent ?
[A.U. '17]

11. How many pounds of tea at Re. 1. 2 as. a pound should be added to 25 lb. at Re. 1. 14 as. a lb. so that by selling the mixture at Re. 1. 9 as. a lb. a gain of 25% may be made ?
[D. B. '29]

12. A man buys wine at 25 P. per litre, he mixes it with water and by selling the mixture at 20 P. per litre gains $12\frac{1}{2}$ per cent on his outlay. How much water did each litre of the mixture contain ?
[Cf. C. U. '03]

*13. A cask contains 16 gallons of spirits ; 2 gallons are drawn off and the cask is filled with water ; 2 gallons are again drawn off and the cask is filled up as before. This is done a third time. Compare the quantities of spirits and water remaining in the cask.
[M. U.]

*14. Three bottles whose capacities are 5 : 3 : 2 are filled with milk mixed with water. The ratios of milk and water in the mixtures of the bottles are as 3 : 2, 2 : 1 and 3 : 1 respectively. Find the percentage of milk and water in the new mixture obtained when $\frac{1}{3}$ of the first bottle, $\frac{1}{2}$ of the second and $\frac{2}{3}$ of the third are taken out and mixed together.
[D. B. '34]

15. Nine gallons are drawn from a cask full of wine, it is then filled with water. Nine gallons of the mixture are drawn and the cask is again filled with water. The ratio of wine to water now in the cask is 16 : 9. How much wine does the cask hold ?
[Pu. U. 1891]

[Suppose that the cask contains x gallons of wine. 9 gallons of wine are drawn off from the cask and it is filled with water.

$\therefore$ the quantity of water is $\frac{9}{x}$ of the total quantity of the mixture. In the second time 9 gallons of the mixture are drawn off and $\therefore$ $\frac{9\times 9}{x}$ or $\frac{81}{x}$ gallons of water are drawn off along with it, $\therefore$ there remain $(9-\frac{81}{x})$ gallons of water. In the second time the cask is again filled with 9 gallons of water.

$\therefore$ the total quantity of water $=\left(9+9-\dfrac{81}{x}\right)$ or $\dfrac{18x-81}{x}$ gallons.

This is $\frac{9}{25}$ of the whole quantity. $\therefore$ $\dfrac{18x-81}{x}=\dfrac{9}{25}x.$]

16. In what ratio must tea worth 2s. 5d. per lb. be mixed with tea worth 3s. 9d. per lb. to make a mixture worth 2s. 9d. per lb. ? [C. U. 1951]

17. Some sugar purchased at Rs. 28 a maund is mixed in a certain ratio with another kind purchased at Rs. 40 per md. Find this ratio if by selling the mixture at Rs. 36 a md. the seller makes a profit of 20%. [G. U. '55]

RATIO

(*a*) Ratio is the relation between two quantities of the same kind, *i.e.*, the relation that is found by comparing their magnitudes—one being a multiple or a part of the other.

So it is understood that the ratio of one quantity to another (of the same kind) is determined by the fraction whose numerator is the measure of the first quantity and whose denominator is the measure of the second quantity, both the quantities being expressed in terms of the same unit.

(*b*) The first of the two quantities forming a ratio is called the **antecedent** and the second is called the **consequent** of the ratio ; the two together are called the terms of the ratio.

How to write and read a ratio :

The ratio 3 to 5 is written as 3 : 5 (: being the shortened form of the sign of division ÷) and 3 : 5 is read as 'three is to five'.

Mark the following examples :—

(1) The ratio of 6 to 11 $= \frac{6}{11} = 6 : 11$.

(2) The ratio of Rs. 3 to Rs. 5 $= \frac{\text{Rs. } 3}{\text{Rs. } 5} = \frac{3}{5} = 3 : 5$.

(3) The ratio of 2 yards to 5 feet $= \frac{6 \text{ ft.}}{5 \text{ ft.}} = \frac{6}{5} = 6 : 5$.

(4) The ratio of 5 as. 4 p. to Re. 1 $= \frac{5 \text{ as. } 4 \text{ p.}}{\text{Re. } 1} = \frac{\text{Re. } \frac{1}{3}}{\text{Re. } 1} = \frac{1}{3} = 1 : 3$.

(*c*) If we divide a quantity by another of the same kind the quotient is an abstract number. ∴ the ratio of two concrete or mixed quantities of the same kind is an abstract fraction, *e.g.*, Rs. 3 : Rs. 4 = 3 yards : 4 yards = 3 metres : 4 metres = $\frac{3}{4}$ (an abstract fraction). Thus the value of a ratio does not depend upon the nature of the quantities involved.

The ratio of two quantities of different kinds is not possible. ∴ 4 metres : Rs. 6 is inadmissible, as the comparison is not possible.

(*d*) **A ratio of greater equality and a ratio of less inequality :**

A ratio is of greater inequality when the antecedent is greater than the consequent ; and it is of less inequality when the antecedent is less than the consequent. A ratio of greater inequality is, therefore, greater than unity and is expressed by an improper fraction ; while a ratio of less inequality is less than unity and is expressed by a proper fraction. A ratio is of equality when the antecedent is equal to the consequent. *e.g.*, 1 : 1, Rs. 2 : 32 as., etc are ratios of equality ; 4 : 2, 5 kg : 4 kg., etc. are ratios of greater inequality and 5 : 6, 3 yards : 7 yards, etc. are ratios of less inequality.

(*e*) **Inverse ratio :** If of two ratios, the antecedent and the consequent of the one are respectively the consequent and antecedent of the other, they are said to be inverse or reciprocal to one another ; *e.g.*, the inverse ratio of 3 : 4 is 4 : 3 and the inverse ratio of 4 : 3 is 3 : 4. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ and the reciprocal of 3 is $\frac{1}{3}$.

(*f*) The ratios are of two kinds, simple and compound. Rs. 4 : Rs. 5 is a simple ratio. Ratios are **compounded** by taking the product of the antecedents for a new antecedent and the product of the consequents for a new consequent.

Thus the ratio compounded of the ratios 4 : 5, 6 : 7 and 5 : 6 is $(4\times6\times5) : (5\times7\times6)$ or 4 : 7.

(*g*) **A few important properties.**

(1) $\text{Ratio}=\dfrac{\text{Antecedent}}{\text{Consequent}}$.

(2) As ratios are fractions, the properties of the latter apply equally to the former. Therefore the value of a ratio is not altered by multiplying or dividing both its terms by the same number. Thus 2 : 3, $2\times2 : 3\times2$ or 4 : 6—these two ratios are equal. Similarly Rs. 3 : Rs. 4 = Rs. 15 : Rs. 20.

(3) Fractions can be arranged in order of magnitude. On following just the same method a few ratios can be arranged in ascending or descending order of magnitude.

(4) The product of a ratio and its inverse ratio is 1.

Examples [16]

1. Find the ratio of Rs. 2. 3. as. to Rs. 2, 10 as.

Rs. 2. 3 as.=35 as., Rs. 2. 10 as.=42 as.

∴ Rs. 2. 3 as. : Rs. 2. 10 as.=35 as. ; 42 as.$=\frac{35}{42}=\frac{5}{6}=5 : 6$.

2. Reduce $1\frac{1}{8}$ of 8 kg. : $1\frac{7}{25}$ of 12 Kg. 5 Hg. to the lowest terms.

8 Kg.=80 Hg., 12 Kg. 5 Hg.=125 Hg,

$1\frac{1}{8}$ of 8 Kg. $=\frac{9}{8}$ of 80 Hg. =90 Hg.

$1\frac{7}{25}$ of 12 Kg. 5 Hg. $=\frac{32}{25}$ of 125 Hg.=160 Hg.

∴ the ratio $=\frac{90}{160}=\frac{9}{16}=9 : 16$.

3. Find the compound ratio of the following :—

5 : 7, 14 : 15 and 9 : 20.

The reqd. compound ratio$=\dfrac{5\times14\times9}{7\times15\times20}=\dfrac{3}{10}=3 : 10$.

4. Which is greater, 4 : 7 or 10 : 11 ?

Here L. C. M. of 7 and 11=77.

Now, $4 : 7=\frac{4}{7}=\frac{4\times11}{7\times11}=\frac{44}{77}$, and $10 : 11=\frac{10}{11}=\frac{10\times7}{11\times7}=\frac{70}{77}$,

∵ $\frac{70}{77}>\frac{44}{77}$, ∴ 10 : 11 is greater.

5. The ratio of two quantities is 4 : 5. If the consequent be 65 metres, find the antecedent.

$$\frac{\text{Antecedent}}{65 \text{ m.}}=\frac{4}{5}=\frac{4\times 13}{5\times 13}=\frac{52}{65}=\frac{52 \text{ m.}}{65 \text{ m.}}$$

$\therefore$ the reqd. antecedent = 52 metres.

6. Express $\frac{3}{5} : \frac{4}{7}$ as the ratio of integers.

The value of a ratio does not change, if its two terms are multiplied by the same number. $\therefore$ here two terms are to be multiplied by the L. C. M. of 5 and 7, i.e., by 35.

$\therefore$ $\frac{3}{5} : \frac{4}{7} = \frac{3}{5}\times 35 : \frac{4}{7}\times 35 = 21 : 20.$

7. A earns Rs. 80 in 7 days and B Rs. 90 in 12 days. Find the ratio of their earnings.

The earning of A for 7 days = Rs. 80.

$\therefore$ " " " 1 day = Rs. $\frac{80}{7}$

Again, the earning of B for 12 days = Rs. 90,

$\therefore$ " " B for 1 day $=\frac{\text{Rs. } 90}{12}=\text{Rs. } \frac{15}{2}.$

$\therefore$ the ratio of their earnings

$=\frac{80}{7} : \frac{15}{2} = \frac{80}{7} \div \frac{15}{2} = \frac{80}{7}\times\frac{2}{15}=\frac{32}{21}=32 : 21.$

Exercise 16

Arrange the following ratios in order of magnitude :—

1. 6 : 14, 5 : 25, 8 : 12 2. $\frac{1}{3} : \frac{2}{9}$, 3 : 5, $2\frac{1}{2} : 3\frac{1}{3}$

3. $3\cdot 2 : 4\cdot 8$, $\cdot\dot{3} : \cdot\dot{5}$, $\cdot\dot{1}\dot{5} : \cdot\dot{8}\dot{5}$

4. Rs. 6 : Rs. 10, 12 mds : 18 mds., 3 yds. 2 ft. : 4 yds. 1ft.

5. The ratio of two quantities is 3 : 4. If the antecedent be 15, find the consequent.

6. The value of a ratio is $\frac{2}{9}$ and the consequent is 81. Find the antecedent.

7. The ratio of two quantities is 4 : 5. If the antecedent be Rs. 3. 12 as., find the consequent.

8. The value of a ratio is $\frac{5}{7}$; if the consequent be 4 yds. 2 ft., find the antecedent.

9. A's money : B's money = 10 : 11. If A has Rs. 120, find what B has.

10. The age of Ram : the age of Hari $=3:4$. If the age of Hari be 28 years, find the age of Ram.

11. Express $\frac{3}{4} : \frac{5}{6}$ as the ratio of two integers.

12. Divide Rs. 324 between A and B in the ratio of $11:7$.

13. A travels 150 kilometres in 12 days and B travels 87 km. in 9 doys. Find the ratio of their rates of speed.

14. A ship travels 2760 kilometres in 9 days 14 hours and another ship 405 kilometres in 18 hours. Compare the rates of their speed.

15. A greyhound pursues a hare and takes 5 leaps for every 6 leaps of the hare, but 3 leaps of the hound are equal to 5 of the hare. Compare the rates of speed of the hound and hare.

PROPORTION

When two ratios are equal, they are said to form a proportion. Four quantities are said to be in proportion or are proportionals when the ratio of the first to the second is equal to the ratio of the third to the fourth. *e.g.*,

(1) $4:6$, $10:15$ are equal ratios, for $4:6=\frac{4}{6}=\frac{2}{3}$, and $10:15=\frac{10}{15}=\frac{2}{3}$.

The value of the two ratios being $\frac{2}{3}$, *i.e.*, equal, 4, 6, 10 and 15 are in proportion.

(2) Rs. 2, Rs. 5, 12 metres, 30 metres—these four quantities are in proportion. $\because$ Rs. 2 : Rs. 5 $=\frac{2}{5}$ and 12 m. : 30 m. $=\frac{12}{30}=\frac{2}{5}$. Here the ratio of the first quantity to the second is equal to the ratio of the third to the fourth, $\therefore$ the four quantities are in proportion.

Writing and reading a proportion

We have $4:6=10:15$. In writing a proportion the sign $::$ is put for the sign of equality ($=$).

Thus we write $4:6::10:15$; and it is read "4 is to 6 as 10 is to 15".

Two terms of a ratio must be of the same kind. But when four quantities are in proportion, it is not necessary that all of them should be of the same kind ; the first two should be of the same kind, and the other two may be of another same kind. (Vide the examples given above).

In the proportion the first and the fourth quantities are called the **Extremes**, the second and the third are called the means and the fourth quantity is said to be the fourth proportional to the first, second and third.

Thus, when 2 : 3 : : 4 : 6, 2 and 6 are called the extremes, and 3 and 4 the means, 6 is called a fourth proportional to 2, 3 and 4.

3. Continued Proportion

Three quantities of the same kind are said to be in **Continued Proportion** when the ratio of the first to the second is equal to the ratio of the second to the third. The second quantity is called a **mean proportional** between the first and third, and the third quantity is called a **third proportional** to the first and second.

Thus, 2, 4, 8 are in continued proportion ; for 2 : 4 : : 4 : 8.

Here 4 is a mean proportional between 2 and 8 and 8 is a third proportional to 2 and 4.

There may be continued proportion in case of more than three quantities of the same kind. If of the 5 quantities the first : the second=the second : the third=the third : the fourth=the fourth : the fifth, then the 5 quantities are said to be in continued proportion. Thus,

1 : 2=2 : 4=4 : 8=8 : 16, ∴ 1, 2, 4, 8, 16 are in continued proportion.

4. Continuous ratio

The mutual ratios of many quantities of the same kind can be written continuously with the sign of ratio. Thus,

2 yds : 4 yds : 10 yds : 12 yds : 14 yds.
=2 : 4 : 10 : 12 : 14=1 : 2 : 5 : 6 : 7.

5. A few important things to be noted here :—

(1) When four abstract numbers are in proportion, the product of the extremes is equal to the product of the means. (*i.e.*, the first × the fourth = the second × the third).

$\because$ 3 : 5 : : 12 : 20, $\therefore$ $3\times20=5\times12$.

Proof : Here $\frac{3}{5}=\frac{12}{20}$; multiplying both the ratios by 5×20 (or the product of the two consequents) we get

$\frac{3}{5}\times5\times20=\frac{12}{20}\times5\times20$, or, $3\times20=12\times5$.

This sort of multiplication is known as **Cross Multiplication.**

Corollary : (*a*) To test if four quantities are in proportion, just see whether the product of the first and the fourth is equal to the product of the second and the third. If they are equal, the four quantities are in proportion, otherwise not.

(*b*) If any three terms of a proportion of 4 terms are known, the remaining term can be found by this rule. Thus,

an extreme = product of the means ÷ the other extreme

and a mean = product of the extremes ÷ the other mean,

(2) If three quantities be in continued proportion, the product of the first and third is eqnal to the square of the second. It can be proved by the previous corollary.

$\because$ the first : the second : : the second : the third,

$\therefore$ the first × the third = the second × the second = (the second)^2.

Thus 4, 12, 36 are in continued proportion. Here $4\times36=(12)^2$.

Corollary : If you are to find the mean proportional between two quantities, the square root of their product will be the mean proportional. Thus, the mean proportional between 9 and 16

$=\sqrt{9\times16}=\sqrt{144}=12$.

(3) When four quantities are in proportion (*a*) the first quantity : the second quantity : : the third quantity : the fourth quantity.

(*b*) The second quantity : the first quantity : : the fourth quantity : the third quantity, because the inverse ratios of two equal ratios are also equal.

Thus $\because$ 4 : 7 : : 12 : 21, $\therefore$ 7 : 4 : : 21 : 12.

Proof : $\because$ $\frac{4}{7}=\frac{12}{21}$, $\therefore$ $1\div\frac{4}{7}=1\div\frac{12}{21}$.

or, $\frac{7}{4}=\frac{21}{12}$, i.e. 7 : 4 = 21 : 12.

(4) If four abstract numbers or quantities of the same kind be in proportion, then the first : the third : : the second : the fourth.

Thus, $\because$ $2 : 3 :: 8 : 12$, $\therefore$ $2 : 8 :: 3 : 12$.

Proof : $\frac{2}{3}=\frac{8}{12}$, $\therefore$ $\frac{2}{3}\times\frac{3}{8}=\frac{8}{12}\times\frac{3}{8}$ (multiplying both sides by $\frac{3}{8}$) or, $\frac{2}{8}=\frac{3}{12}$, or, $2 : 8 :: 3 : 12$.

(5) If the terms of a ratio be multiplied or divided by the same number, the value of the ratio does not change. Thus,

(*a*) $3 : 4 = 3\times5 : 4\times5 = 15 : 20$, for $\frac{15}{20}=\frac{3}{4}$.

(*b*) $4 : 10 = \dfrac{4\div2}{10\div2}=\dfrac{2}{5}=2 : 5$.

(*c*) $2 : 3 = 4 : 6$, $\therefore$ $2\times5 : 3\times5 :: 4\times3 : 6\times3$

$$\because \frac{2\times5}{3\times5}=\frac{2}{3} \text{ and } \frac{4\times3}{6\times3}=\frac{2}{3}.$$

Examples [17]

1. Find the fourth proportional to 5, 15 and 8.

Here, $\dfrac{5}{15}=\dfrac{8}{\text{the reqd. number}}$

$\therefore$ $5\times$the number required $=15\times8$

$\therefore$ the number required $=\dfrac{15\times8}{5}=24$.

2. Find a third proportional to $\frac{2}{3}$ and $\frac{4}{5}$.

Here $\frac{2}{3} : \frac{4}{5} = \frac{4}{5}$: the number required,

$\therefore$ $\frac{2}{3}\times$the number required $=\frac{4}{5}\times\frac{4}{5}$

$\therefore$ the number required $=\frac{4}{5}\times\frac{4}{5}\times\frac{3}{2}=\frac{24}{25}$.

3. Find a mean proportional between 4 and 64.

The required mean proportional

$=\sqrt{4\times64}=\sqrt{256}=16$,

4. Of the four numbers the first : the second $=2 : 3$, the second : the third $=4 : 5$, the third : the fourth $=6 : 7$. Find the ratio of the first to the fourth.

$$\because \frac{\text{first}}{\text{second}}=\frac{2}{3}, \frac{\text{second}}{\text{third}}=\frac{4}{5}, \frac{\text{third}}{\text{fourth}}=\frac{6}{7},$$

$$\therefore \frac{\text{first}}{\text{second}}\times\frac{\text{second}}{\text{third}}\times\frac{\text{third}}{\text{fourth}}=\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}, \quad \therefore \frac{\text{first}}{\text{fourth}}=\frac{16}{35}.$$

$\therefore$ the ratio of the first to the fourth $=$ **16 : 35.**

5. A's money is $\frac{3}{4}$ of B's money and B's money is $1\frac{1}{5}$ of C's money. Find the ratio of A's money to C's money.

$$\because \text{ A's money} = \tfrac{3}{4} \text{ of B's money.} \quad \therefore \frac{\text{A's money}}{\text{B's money}} = \frac{3}{4}.$$

Again, $\because$ B's money $= 1\frac{1}{5}$ of C's money, $\therefore \dfrac{\text{B's money}}{\text{C's money}} = \dfrac{6}{5}.$

$$\therefore \frac{\text{A's money}}{\text{B's money}} \times \frac{\text{B's money}}{\text{C's money}} = \frac{3}{4} \times \frac{6}{5} = \frac{9}{10},$$

$$\therefore \frac{\text{A's money}}{\text{C's money}} = \frac{9}{10}. \quad \therefore \text{ A's money : C's money} = 9 : 10.$$

6. The ratio of A to B is 2 : 3, of B to C is 4 : 5 and of C to D is 6 : 7. Find the continued ratio of A, B, C and D.

[First Method] $\dfrac{A}{B} = \dfrac{2}{3}, \ \dfrac{B}{C} = \dfrac{4}{5}, \ \dfrac{C}{D} = \dfrac{6}{7}.$

In order to find the continued ratio we should change the terms of the ratios in such a way that each antecedent may be equal to the preceding consequent.

Now, $\dfrac{A}{B} = \dfrac{2}{3} = \dfrac{2 \times 4 \times 6}{3 \times 4 \times 6} = \dfrac{48}{72}, \ \dfrac{B}{C} = \dfrac{4}{5} = \dfrac{4 \times 3 \times 6}{5 \times 3 \times 6} = \dfrac{72}{90}$

$$\frac{C}{D} = \frac{6}{7} = \frac{6 \times 3 \times 5}{7 \times 3 \times 5} = \frac{90}{105}$$

$\therefore$ A : B = 48 : 72, B : C = 72 : 90, C : D = 90 : 105,

$\therefore$ A : B : C : D = 48 : 72 : 90 : 105 = 16 : 24 : 30 : 35.

[Second Method] A : B = 2 : 3,

B : C = 4 : 5 = 1 : $\frac{5}{4}$ = 1 × 3 : $\frac{5}{4}$ × 3 = 3 : $\frac{15}{4}$,

C : D = 6 : 7 = 1 : $\frac{7}{6}$ = 1 × $\frac{15}{4}$: $\frac{7}{6}$ × $\frac{15}{4}$ = $\frac{15}{4}$: $\frac{35}{8}$,

$\therefore$ A : B : C : D = 2 : 3 : $\frac{15}{4}$: $\frac{35}{8}$ = 16 : 24 : 30 : 35.

7. A mixture of 7 Dl. 2 litres contains milk and water in the ratio 5 to 3 ; find the quantities of milk and water in the mixture.

7 Dl. 2 litres = 72 litres

If the mixture be divided into (5+3) or 8 equal parts, 5 parts wiil be milk and 3 parts water.

$\therefore$ The quantity of milk = $\frac{72}{8}$ × 5 litres = 45 litres.

and the quantity of water = $\frac{72}{8}$ × 3 litres = 27 litres.

8. A mixture of 48 gallons contains wine and water in the ratio of 7 : 5, how much wine must be added to it that the ratio of wine to water may be 3 : 2 ?

$7+5=12$; $\therefore$ the quantity of wine in the mixture

$=\frac{7}{12}$ of 48 gallons=28 gallons.

The quantity of water in the mixture=(48 − 28) or 20 gallons.

In the second mixture the quantity of water remains the same 20 gallons and the ratio =3 : 2.

$$\therefore \quad \frac{\text{The total quantity of wine}}{\text{The quantity of water}}=\frac{3}{2},$$

$$\therefore \quad \frac{\text{The total quantity of wine}}{\text{20 gallons}}=\frac{3}{2},$$

$\therefore$ the total quantity of wine$=\frac{3}{2}\times 20$ gallons$=30$ gallons.

$\therefore$ (30 − 28) or 2 gallons of wine must be added.

Exercise 17

Find a fourth proportional to

1. 4, 6, 8.
2. 10, 12, 25.
3. 33, 22, 18 P.
4. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$.
5. 5, $7\frac{1}{2}$, 12.
6. ·12, ·21, 8.
7. Rs. 2. 3as., Rs. 2. 10 as., Re. 1. 9 as.
8. Rs. 2.50 P., Rs. 3.20 P., 25 metres.

Find a third proportional to

9. 12, 18.
10. $3\frac{1}{3}$, $3\frac{1}{3}$.
11. ·16, ·18
12. 1 hour 20 minutes, 1 hour 40 minutes.

The following quantities are in proportion ; find the missing one :

13. 12, 16, *, 20.
14. 4, *, 9, $13\frac{1}{2}$.
15. 6, 14, *, Rs. 35.

Find a mean proportional between

16. 25, 81.
17. $1\frac{4}{7}$, $6\frac{2}{7}$.
18. 1·4 grams, 5·6 grams.
19. ·9, ·036.

20. Find the number which has the same ratio to 39 that 6 has to 13.

21. What quantity bears the same ratio to 30 minutes as Rs. 3. 6 as. bears to Rs. 2. 4 as. ?

22. Are 14, 16, 35, 42 in proportion ? If not, what should be the fourth term for which they will be in proportion.

23. Are $3\frac{1}{3}$, $6\frac{1}{4}$, $6\frac{1}{8}$ and $12\frac{1}{2}$ in proportion ? If not, find the third term for which they will be in proportion.

24. If 49 is to a quantity in the same ratio as that quantity is to 81, find the quantity.

25. Seven numbers are in continued proportion. The first and the second are 2 and 4 respectively. Find the seventh number.

26. Of the four numbers the first : the second : : 2 : 3, the second : the third : : 4 : 5 and the third : the fourth $=6:7$. Find their continued ratio.

27. The ratio of the age of Ram to that of Hari$=3:4$, the ratio of the age of Hari to that of Jadu$=12:13$. Find the ratio of the ages of Ram and Jadu.

28. $A:B=2:3$, $B:C=4:7$, $C:D=5:6$. Find the ratio of A to D. Also compare A, B, C, D, *i.e.*, find the continued ratio of $A:B:C:D$.

29. A's age$=\frac{3}{4}$ of B's and C's age $=1\frac{1}{2}$ of B's. Find the ratio of A's age to C's age. If the age of C is 30 years, find the age of A.

30. Divide Rs. 1224 among A, B and C so that their shares are as $3:4:5$.

31. The ratio of two numbers is $3:4$ and their L. C. M. is 180. Find the numbers.

32. $\frac{2}{3}$ of Ram's money $=\frac{1}{2}$ of Hari's money. They have altogether Rs. 1400. Find the amount of money each has.

33. The ratio of two numbers is $5:8$ and their difference is 69. Find the numbers.

34. The ratio of the present age of A to that of B is $4:5$. After 5 years their ages will be as $5:6$. Find their present ages.

35. 4 years ago the ratio of A's age to B's was $11:14$ and 4 years hence their ages will be as $13:16$. First their present ages.

36. The sum of the ages of A and B is 60 years. 3 years ago their ages were as $4:5$. Find the ratio of their ages 3 years hence.

37. A mixture of 180 litres contains water and wine in the ratio of 2 : 3. Find the quantities of water and wine in the mixture.

38. A mixture of 35 kilograms contains milk and water in the ratio of 5 : 2, how much water must be added to it that the ratio of milk to water may be 2 : 1.

39. A mixture of equal quantities of milk and water weighs 4 Kg. 2 Hg. How much milk must be added to it that the ratio of milk and water may be 4 : 3 ?

40. A Chowkidar pursues a thief and takes 4 steps for every 5 steps of the thief, but 6 steps of the Chowkidar are equal to 8 steps of the thief ; compare the rates of speed of the Chowkidar and the thief.

DIVISION INTO PROPORTIONAL PARTS

To divide a given quantity into porportional parts is to divide it into parts which will be proportional to certain numbers.

Suppose that Rs. 240 is to be divided among Ram, Hari and Jadu so that their shares may be in the proportion of 3, 5 and 4. It is understood that if Ram gets 3 parts, Hari will get 5 and Jadu 4 parts, *i.e.*, if we divide Rs. 240 into (3+5+4) or 12 parts, then A will have 3, B will have 5 and C will have 4 of these parts.

Process in brief is this :—

$3+5+4=12$; Rs. $240 \div 12=$ Rs. 20.

$\therefore$ Ram's share = Rs. $20 \times 3=$ Rs. 60,

Hari's share = Rs. $20 \times 5=$ Rs. 100,

Jadu's share = Rs. $20 \times 4=$ Rs. 80.

Therefore, the rule may be stated thus :—

Divide the given quantity by the sum of the numbers representing the proportion and then multiply the quotient by each of the numbers.

Examples [18]

1. (*i*) Divide between Ram aud Hari Rs. 350 in the ratio 3 : 4 and (*ii*) 589 dollars in the ratio $\frac{3}{4} : \frac{5}{6}$.

(*i*) The given ration=3 : 4 ; 3+4=7.

Now, dividing Rs. 350 into 7 equal parts, 3 of these parts should be given to Ram and 4 to Hari.

$\therefore$ Ram's share$=\frac{\text{Rs. }350}{7}\times 3=$Rs. 150,

and Hari's share$=\frac{\text{Rs. }350}{7}\times 4=$Rs. 200.

[or, Hari's share=Rs. 350 − Rs. 150=Rs. 200.]

(2) [Here $\frac{3}{4}$ and $\frac{5}{6}$ should be expressed with the lowest common denominators.]

$$\frac{3}{4}=\frac{3\times 3}{4\times 3}=\frac{9}{12},\quad \frac{5}{6}=\frac{5\times 2}{6\times 2}=\frac{10}{12},$$

$\therefore$ the given ratio$=\frac{9}{12} : \frac{10}{12}=9 : 10$. Now, $9+10=19$.

$\therefore$ Ram's share$=\frac{589 \text{ dollars}}{19}\times 9=31$ dollars$\times 9=279$ dollars.

and Hari's share$=\frac{589 \text{ dollars}}{19}\times 10=31$ dollars$\times 10=310$ dollars.

[or, Hari's share=(589 − 279) or 310 dollars]

2. A sum of money was divided among three persons in proportion to 3 : 7 : 8. It was found that the second man received Rs. 72 more than the first. What was the sum ?

3+7+8=18 ; the first man gets Rs. 3 and the second man Rs. 7 out of Rs. 18.

$\therefore$ the second man gets Rs. (7 − 3) or Rs. 4 more when the the total sum is Rs. 18 ;

$\therefore$ he gets Re. 1 more when the total sum=Rs. $\frac{18}{4}$

$\therefore$,, Rs 72 ,, ,, ,, Rs. $\frac{18}{4}\times 72=$Rs. 324.

$\therefore$ the total sum=Rs. 324.

3. Divide Rs. 940 among A, B and C so that A's share : B's share=6 : 5 and B's share : C's share=15 : 14.

$\frac{\text{A's share}}{\text{B's share}}=6:5=\frac{6}{5}=\frac{6\times3}{5\times3}=18:15$ and $\frac{\text{B's share}}{\text{C's share}}=15:14$,

$\therefore$ A's share : B's share : C's share=18 : 15 : 14.

$\therefore$ their total shares=18+15+14=47

$\therefore$ 1 share=Rs. 940÷47=Rs. 20.

$\therefore$ A's share=Rs. 20×18=Rs. 360,

B's share=Rs. 20×15=Rs. 300,

C's share=Rs. 20×14=Rs. 280.

[N. B. It is given that if A get 6, B gets 5 and if B get 15, C gets 14. To get the continued proportion of the shares of the three persons, B's share in both the ratios must be made equal. $\therefore$ in the first ratio B's share has been made 5×3 or 15, which is also equal to B's share in the second ratio. Now see that in the first ratio B's share has been multiplied by 3 and $\therefore$ A's share has also been multiplied by 3. Now it is seen that if A get 18, B gets 15 and if B get 15, C gets 14 parts.

$\therefore$ A's share : B's share : C's share=18 : 15 : 14.]

4. Divide Rs. 750 among 4 men, 5 women and 6 boys in such a way that if each man get 8 annas, each woman gets 5 annas and each boy gets 3 annas.

[Previously you have worked out such a problem in a different way. Now see that it can also be worked out by the help of ratio and proportion.]

share of 1 man : share of 1 woman : share of 1 boy=8 : 5 : 3.

$\therefore$ share of 4 men : share of 5 women : share of 6 boys =32 : 25 : 18.

32+25+18=75 ; Rs. 750÷75=Rs. 10.

$\therefore$ Share of each man=Rs. 10×8=Rs. 80,

share of each woman=Rs. 10×5=Rs. 50,

and share of each boy=Rs. 10×3=Rs. 30.

5. Divide 800 mangoes among 4 men, 10 women and 16 boys so that $\frac{1}{3}$ of each man's share, $\frac{1}{2}$ of each woman's share and $\frac{3}{4}$ of each boy's share may be equal.

$\frac{1}{2}$ of each woman's share $=\frac{1}{3}$ of each man's share,

$\therefore$ each woman's share $=\frac{2}{3}$ of each man's share.

Again, $\frac{3}{4}$ of each boy's share $=\frac{1}{3}$ of each man's share,

$\therefore$ each boy's share $=\frac{4}{9}$ of each man's share.

$\therefore$ each man's share : each woman's share : each boy's share

$=1 : \frac{2}{3} : \frac{4}{9}=9 : 6 : 4$ [multiplying each ratio by 9]

$\therefore$ share of 4 men : share of 10 women : share of 16 boys

$=36 : 60 : 64$.

Now, $36+60+64=160$, 800 mangoes$\div 160=5$ mangoes.

$\therefore$ each man will get 5×9 or 45 mangoes,

each woman will get 5×6 or 30 mangoes,

and each boy will get 5×4 or 20 mangoes.

6. The three sides of a triangle are in the proportion of $7 : 9 : 12$, and the difference of its greatest and least sides is 15 cm. Find the length of the greatest side.

Let the three sides of the triangle be $7a$, $9a$. and $12a$ cm.

$\therefore$ the difference of the greatest and the least sides is $12a-7a=5a$.

$\therefore$ by the condition of the problem, $5a=15$ cm., $\therefore$ $a=3$ cm.

$\therefore$ the greatest side$=12a$ cm.$=12\times 3$ cm.$=36$ cm.

7. Some rupee-, 50 P.- and 25 P.- coins make up Rs. 93.75 P. and their numbers are proportional to 3, 4 and 5. Find the number of each kind of coin.

Number of rupees : number of 50 P. coins : number of 25. P. coins $=3 : 4 : 5$. $\therefore$ the values of the three groups of coins are as

3 rupees : 4×50 P. : 5×25 P.

or, as 12×25 P. : 8×25 P. : 5×25 P., or, as $12 : 8 : 5$

Now, $12+8+5=25$, and Rs. 93.75 P.$=$Rs. $\frac{375}{4}$

$\therefore$ the value of the rupee coins$=$Rs. $\frac{375}{4\times 25}\times 12=$Rs. 45

$\therefore$ number of rupee$=45$.

The value of 50 P.-coins$=$Rs. $\frac{375}{4\times 25}\times 8=$Rs. 30.

$\therefore$ number of 50 P.-coins$=30\times 2=60$.

Again, the value of 25 P.-coins$=$Rs. $\frac{375}{4\times 25}\times 5=$Rs. $\frac{75}{4}$

$\therefore$ number of 25 P.-coins$=\frac{75}{4}\times 4=75$.

8. There are 240 coins in half rupees, quarter rupees and two-anna pieces of which the values are proportional to 5 : 3 : 1. Find the number of each coin.

The values of the coins are as 5 : 3 : 1

or as Rs. 5 : Rs. 3 : Re 1

$\therefore$ numbers of the coins are as 10 : 12 : 8

[$\because$ Rs. 5=10 half rupees, Rs. 3=12 quarter rupees Re. 1=8 two-anna pieces.] $10+12+8=30$.

$\therefore$ the number of half rupees$=\frac{240\times10}{30}=80$,

the number of quarter rupees$=240\times\frac{12}{30}=96$ and the number of two-anna pieces$=240\times\frac{8}{30}=64$.

9. A man worked for 4 days, a woman for 6 days and a boy for 5 days to finish a certain work and received altogether Rs. 66 as their wages. If their daily work be in the proportion of $\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$, how much wages should each get ?

4 days' work of a man : 6 days' work of a women : 5 days' work of a boy$=4\times\frac{1}{3}:6\times\frac{1}{4}:5\times\frac{1}{5}=\frac{4}{3}:\frac{6}{4}:\frac{5}{5}=16:18:10$ [multiplying by 12. the L. C. M. of the denominators,]

Now, $16+18+10=44$; Rs. $66\div44=$Re 1. 50 P.

$\therefore$ The wages of a man=Re. 1. 50 P.$\times16=$Rs. 24,

the wages of a woman=Re. 1. 50 P.$\times18=$Rs. 27,

and the wages of a boy=Re. 1. 50 P.$\times10=$Rs. 15. } Ans.

10. A, B and C are owners of an estate and their shares are in the proportion of $4:2\frac{1}{4}:1\frac{3}{4}$; A gave $\frac{3}{4}$ of his own share to C and C sold 50 kilo ares of land to B, whereby the shares of A and C became equal. How much land had each originally ?

A's share : B's share : C's share$=4:\frac{9}{4}:\frac{7}{4}=16:9:7$.

Suppose they had at first $16a$, $9a$ and $7a$ ares of land. Now, A gave $16a\times\frac{3}{4}$ or $12a$ ares of land to C and $\therefore$ C's land was then $(7a+12a)$ or $19a$ ares.

Then C sold 50 ares of land to B and $\therefore$ C's land was $(19a-50)$ ares and B's land was $(9a+50)$ ares at the end.

$\therefore$ by the condition of the problem,

$19a-50=9a+50$, or $10a=100$, $\therefore a=10$.

$\therefore$ A had 16×10 or 160 ares, B had 9×10 or 90 ares and C had 7×10 or 70 ares of land originally.

11. 5 men, 6 women and 7 boys finished a work in 5 days and got wages of Rs. 51.75 P. If the work of 1 man, 1 woman and 1 boy be in the proportion of $\frac{1}{2} : \frac{1}{3} : \frac{1}{4}$, how much wages did each get ?

Here the work of 5 men : work of 6 women : work of 7 boys $= 5\times\frac{1}{2} : 6\times\frac{1}{3} : 7\times\frac{1}{4} = \frac{5}{2} : 2 : \frac{7}{4} = 10 : 8 : 7.$

Now, $10+8+7=25$; Rs. 51.75 P.$\div 25=$Rs. 2.07 P.,

$\therefore$ Wages of 1 man=Rs. 2.07 P.$\times\frac{10}{5}$=Rs. 4.14 P.,

wages of 1 woman=Rs. 2.07 P.$\times\frac{8}{6}$=Rs. 2.76 P.

and wages of 1 boy=Rs. 2.07 P.$\times\frac{7}{7}$=Rs. 2.07 P.

Exercise 18

1. (i) A milkman mixed water with milk in the ratio of 3 : 8 and sold 33 kilolitres of the mixture. How much water did the mixture contain ?

(ii) What sum of money is to be divided among 3 men in the ratio 3 : 4 : 5 so that the third man receives Rs. 10 only.

2. Divide Rs. 750 into 3 parts in the ratio of 4 : 5 : 6.

3. Divide Rs. 340 among A, B and C so that their shares are proportional to 2, 5 and $1\frac{1}{2}$.

4. Gun-powder has 15 parts of charcoal, 10 parts of sulphur and 75 parts of saltpetre as its ingredients. What quantity of eath ingredient is required to prepare 1 kilogram of gun-powder ?

5. Some mangoes are divided among Ram, Hari and Jadu in the ratio 4 : 3 : 5. If Jadu gets 60 mangoes more than Hari, find the total number of mangoes.

6. Divide 154 into four parts which are proportional to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$.

7. Divide Rs. 350 among A, B and C so that A's share : B's share=2 : 3 and B's share : C's share=4 : 5.

8. Gun-powder contains 75 parts of saltpetre, 10 parts of sulphur and 15 parts of charcoal. Find the quantity of each ingredient in 10 cwt. of gun-powder.

9. Divide Rs. 450 among A, B and C so that for every Rs. 7 given to A, B may get Rs. 5 and C may get Rs. 3.

10. The cost of a house and its furniture is Rs. 9030. If $\frac{1}{4}$ of the cost of the house is equal to $\frac{4}{5}$ of the cost of the furniture, find the cost of each.

11. Three persons start business and make a profit of Rs. 1180. If their capitals are as $\frac{1}{5} : \frac{1}{6} : \frac{1}{8}$, how should the profit be divided ?

12. In a cricket match Naidu, Amarnath and Hazari make altogether 342 runs. If the runs of Naidu and Amarnath and those of Amarnath and Hazari are as to $3 : 2$ each, how many runs does each make ?

13. Three persons got Rs. 13,400 in a lottery. If $\frac{1}{4}$ of the first man's, $\frac{2}{3}$ of the second man's and $\frac{5}{6}$ of the third man's money be equal, find how much each got.

14. How many rupees, half rupees and quarter-rupees proportional to 8, 5 and 3 in number together make up Rs. 112.50 P. ?

15. Divide Rs. 400 among A, B and C so that for every Rs. 7 given to A, B gets Rs. 8 and for every Rs. 4 given to B, C gets Rs. 5.

16. 280 coins consist of half-rupees, quarter-rupees and two-anna pieces, the values of all coins of each kind being equal. Find the number of each coin and the total amount of money.

17. The sides of a quadrilateral are as $3 : 4 : 5 : 6$ and its perimeter is 72 centimetres. Find the length of its greatest side.

18. How many Rupees, 50 P. and 25 P. coins, of which the values are as $3 : 4 : 5$, are together worth Rs. 240 ?

19. Three angles of a triangle are as $5 : 6 : 7$, find the magnitude of each angle.

20. The sum of the ages of 3 men is 150 years. 10 years ago their ages were as $7 : 8 : 9$. Find their present ages.

21. What sum of money should be divided into parts proportional to 3·4, 5·7 and 4·9 so that the value of the least part is Rs. 170 ?

22. The work of 4 men is equal to that of 6 women and the work of 5 women is equal to that of 8 boys. Divide Rs. 940 among 5 men, 15 women and 11 boys in proportion to their work.

23. Divide Rs. 402 among A, B and C so that $\frac{1}{3}$ of A's share $= \frac{4}{5}$ of B's share $= \frac{5}{4}$ of C's share.

24. Of three boys the first boy had 4 pieces and the second 3 pieces of loaves and they ate them equally. The third boy who had no loaf with him paid 21 P. as the price of his share. How should this amount be divided between the other two?

25. Divide Rs. 870 among A, B and C so that ·5 of A's share = ·6 of B's share = ·75 of C's share. [D. B. 1924]

26. Divide Rs. 500 among 3 men, 5 women and 8 boys so that for every 37·5 P. given to a man, a woman gets 25 P. and a boy gets 9·375 P.

27. 7 Dg. 5g. of tea are divided among A, B and C in a certain proportion so that A gets 1 Dg. 5g. of tea more than C and B gets 2 Dg. 2g. 5 dg. of tea less than C. Divide Rs. 480 in the above proportion.

28. One man, 1 woman and 1 boy work together and get Rs. 49.35 P. as wages. They work for 6, 5 and 4 days respectively and their works are as $\frac{1}{3} : \frac{1}{4} : \frac{1}{5}$. Find the amount of wages each gets.

29. The volumes of three kinds of materials are as 3 : 4 : 7 and the weights of equal volumes of the three materials are as 5 : 2 : 6. If they are mixed to form a material of 52 kilograms, find the weight of the third material in the mixture.

FELLOWSHIP OR PARTNERSHIP

The business which more than one person start jointly with their respective capitals is called a **joint business.** Each of those who contribute capitals to a joint business is called a **fellow or partner** of the business. The profit or loss in a business is divided into parts proportional to the respective capitals of the partners. This method is called **Partnership or Fellowship.** It is of two kinds, viz., the simple partnership and the compound partnership.

Simple Partnership : When the capitals of the several partners are invested in the business for the same period of time, the method of dividing the profit or loss is called simple partnership.

Compound fellowship : When the capitals of the several partners are invested in the business for different periods of time, the method of dividing gain or loss is called compound fellowship. If the capitals of the partners be invested for different periods of time, the periods of time must be taken into consideration in calculating their respective gain or loss.

In such cases the profits or losses of the partners will be proportional to the products of their money and corresponding measure of time. It is evident that it is similar to division into proportional parts.

Examples [19]

1. A and B enter into partnership with Rs. 1200 and Rs. 800 respectively. They gain Rs. 300 in 1 year. How should the profit be divided ?

[**First Method**] Rs. $1200+$Rs. $800=$Rs. 2000.

Profit on Rs. $2000=$Rs. 300,

$\therefore$ " " Re. 1 $=$Rs. $\frac{300}{2000}=$Rs. $\frac{3}{20}$.

$\therefore$ A will get Rs. $\frac{3}{20}\times1200$ or Rs. 180,

and B will get Rs. $\frac{3}{20}\times800$ or Rs. 120.

[**Second Method**] A's capital : B's capital

$=1200:800=3:2$; $3+2=5$

$\therefore$ A's profit$=$Rs. $300\times\frac{3}{5}=$Rs. 180,

and B's profit$=$Rs. $300\times\frac{2}{5}=$Rs. 120.

2. A, B and C enter into partnership. A puts in Rs. 800 on 1st January, 1950, B Rs. 600 on 1st May and C Rs. 500 on 1st July. They make a profit of Rs. 348 in all that year. How should the profit be divided ?

Here, capitals of A, B and C have been employed for 12, 8 and 6 months respectively.

$\therefore$ A's share : B's share : C's share

$=800\times12 : 600\times8 : 500\times6=9600 : 4800 : 3000$

$=16 : 8 : 5.$ $\quad 16+8+5=29.$

$\therefore$ A's profit $=$ Rs. $348\times\frac{16}{29}=$ Rs. 192.

B's profit $=$ Rs. $348\times\frac{8}{29}=$ Rs. 96.

and C's profit $=$ Rs. $348\times\frac{5}{29}=$ Rs. 60.

3. A, B and C start a business with capitals of Rs. 1000, Rs. 800 and Rs. 600 respectively. After a few months A invests Rs. 300 more in that business. The total profit amounts to Rs. 300 in one year and C gets Rs. 72 as his share of profit. When does A put in Rs. 300 in the business ?

C gets the same profit by investing Rs. 600 for 12 months as by investing Rs. 600×12 or Rs. 7200 for 1 month.

$\therefore$ profit on Rs. 7200 for 1 month $=$ Rs. 72

$\therefore$,, ,, Rs. 1 ,, ,, $=$ Rs. $\frac{72}{7200}=$ Rs. $\frac{1}{100}$

$\therefore$ profit on B's capital $=\frac{1}{100}\times$(Rs. 800×12) $=$ Rs. 96

and profit on A's capital of Rs. 1000 $=\frac{1}{100}\times$(Rs. 1000×12) $=$ Rs. 120.

Rs. 72 + Rs. 96 + Rs. 120 = Rs. 288

$\therefore$ the remaining profit $=$ Rs. 300 $-$ Rs. 288 $=$ Rs. 12.

$\therefore$ A gets a profit of Rs. 12 for his extra capital of Rs. 300.

Now, profit on Rs. 300 invested for 1 month

$=\frac{1}{100}\times$ Rs. 300 $=$ Rs. 3.

$\therefore$ Rs. 300 is employed for (Rs. 12 $\div$ Rs. 3) or 4 months.

$\therefore$ A puts in his extra capital of Rs. 300 after 8 months.

4. A starts a business with a capital of Rs 900 on the 1st of January and after 3 months admits a partner B. What capital should B invest then, if the profit of A and B be equal in the year ?

Here A's capital is invested for 12 months and B's capital for 9 months.

Now, if the profit of A and B be equal, then B's capital must be $\frac{12}{9}$ or $\frac{4}{3}$ times A's capital.

$\therefore$ B's capital $=$ Rs. $900\times\frac{4}{3}=$ Rs. 1200.

5. A and B start a business, A invests Rs. 500 for 9 months and B his capital for 6 months. The total profit amounts to Rs. 69 and B gets a profit of Rs. 46. What is B's capital?

[B. U. 1925]

$\because$ B's profit = Rs. 46, $\therefore$ A's profit = Rs. $(69-46)$ = Rs. 23.

Suppose, B's capital = x rupees.

Now, profit on B's capital of Rs. x for 6 months = profit on Rs. $6x$ for 1 month and profit on A's capital of Rs. 500 for 9 months = profit on Rs 4500 for 1 month,

$\therefore$ $\frac{6x}{4500}=\frac{46}{23}=2$, or, $6x=2\times4500$ $\therefore$ $x=\frac{2\times4500}{6}=1500$.

$\therefore$ B's capital = Rs. 1500.

6. A invested Rs. 1800 and B Rs. 1000 for 9 months in a business. If the profit of A and B be equal, find the period of time for which A's capital was invested.

Suppose, A's capital was invested for x months.

$\because$ profits of A and B are equal. $\therefore$ $1800\times x=1000\times9$,

$\therefore$ $x=\frac{1000\times9}{1800}=5$, $\therefore$ A invested his capital for 5 months.

7. A invested Rs. 600 and B Rs. 900 for 4 months in a business. If A's profit be $\frac{5}{11}$ of the total profit, find for what period A's capital was invested.

$\because$ A's profit = $\frac{5}{11}$ of the total profit.

$\therefore$ B's profit = $(1-\frac{5}{11})$ or $\frac{6}{11}$ of the total profit.

$\therefore$ A's profit : B's profit = $\frac{5}{11}:\frac{6}{11}=5:6$.

Suppose, A's capital was invested for x months.

$\therefore$ $\frac{600\times x}{900\times4}=\frac{5}{6}$ $\therefore$ $\frac{x}{6}=\frac{5}{6}$ $\therefore$ $x=\frac{6\times5}{6}=5$.

$\therefore$ A invested his capital for 5 months.

8. The capitals of A, B and C invested in a joint business are as $\frac{1}{6}:\frac{1}{5}:\frac{1}{4}$; after 4 months A withdraws half of his capital. If after 8 months more the total profit amounts to Rs. 1212, find what profit each will get.

The given capitals are as $\frac{1}{6}:\frac{1}{5}:\frac{1}{4}$ or as $10:12:15$, *i.e.*, if A invests Rs. 10, then B invests Rs. 12 and C Rs. 15.

$\therefore$ A invests Rs. 10 for 4 months and Rs. 5 for the remaining 8 months.

Profit on Rs. 10 for 4 months = profit on Rs. 40 for 1 month ; and profit on Rs. 5 for 8 months = profit on Rs. 40 for 1 month.

$\therefore$ A's profit = profit on Rs. 80 for 1 month.

Similarly, B's profit = profit on Rs. 12×12 or Rs. 144 for 1 month, and C's profit = profit on Rs. 15×12 or Rs. 180 for 1 month,

$\therefore$ the profits of A, B and C are as 80 : 144 : 180,

or, as 20 : 36 : 45. Now, $20+36+45=101$,

$\therefore$ A's profit = Rs. $1212\times\frac{20}{101}$ = Rs. 240,

B's profit = Rs. $1212\times\frac{36}{101}$ = Rs. 432,

and C's profit = $1212\times\frac{45}{101}$ = Rs. 540.

Exercise 19

1. Ram and his friend start a joint business with Rs. 400 and Rs. 500 respectively. If after one year the total profit amounts to Rs. 180, how will the profit be divided ?

2. A and B invest Rs. 600 and Rs. 750 respectively in a business. If after one year there is a loss of Rs. 72, how much loss will each bear ?

3. A, B and C start a business with a total capital of Rs. 1500 and after one year A gets Rs. 80, B Rs. 100 and C Rs. 120 as profit. Find the respective capitals invested.

4. A, B and C put in 100 guineas, 100 sovereigns and 100 crowns respectively in a joint business. If they incur a loss of £23 after 1 year 6 months, what is C's loss ?

5. Three milkmen hired a pasture at an annual rent of Rs. 45. The first man tends in it 25 cows, the second man 30 cows and the third 35 cows. How much of the rent should each pay ?

6. A and B make a profit of Rs. 150 in a joint business. If A's capital is Rs. 600 and his profit is Rs. 90, find B's capital.

7. Three persons start a business and their capitals are as 3 : 8 : 5. If the first man gets Rs. 60 less than the third as his profit, find the total profit.

8. Three persons enter into partnership. The first man gets $\frac{1}{2}$ of the total profit, the second $\frac{1}{3}$ and the third the remaining part of the profit. If the capital of the first man is Rs. 700 more than that of the second man, find the capital of the third man.

9. A, B and C make a profit of Rs. 1000 in a joint business. If A's capital : B's capital : : 2 : 3 and B's capital : C's capital : : 2 : 5, how should the profit be divided among A, B and C ? [C. U. 1932]

10. A starts a business with a capital of Rs 500. B and C join that business after 3 and 5 months respectively. If B invests Rs. 600 and C Rs. 800 in that business and if the profit amounts to Rs. 340 in one year, how much profit should each get ?

11. A, B, C and D start a business jointly. A puts in Rs. 1200 on the first January, B Rs. 1500 on the first April, C Rs. 1800 on the 1st July and D Rs. 2100 on the 1st October. If the profit amounts to Rs. 900 in 1 year, how shouid the profit be divided ? [D. B. 1932]

12. The capitals of three partners in a business are Rs. 713.15 P., Rs. 964.85P. and Rs. 2391.15P. respectively. If the profit is Rs. 2231, how should it be divided among the partners ?

13. A, B and C start a business. A invests Rs. 9100 for 3 months, B Rs. 6825 for 2 months and C Rs. 8190 for 5 months. If the total profit amounts to Rs. 4158, how much profit should each get ? [P. U. 1930]

14. A and B hire a meadow for £54. A puts in 23 horses for 27 days and B puts in 21 horses for 39 days. How much of the rent should each pay ? [C.S.]

15. A and B start a business with capitals of Rs. 3000 and Rs. 4500 respectively. After 8 months A puts in Rs. 2500 more and after 7 months more the total profit amounts to Rs. 520. How much profit will each get ? [P. U. 1926]

16. A and B enter into partnership with Rs. 300 and Rs. 500 respectively. After 6 months A puts in Rs. 400 more, but B withdraws Rs. 100. If the profit in the business in one year is Rs. 61.75P., find the profit received by A and B.

17. In a joint-stock business the capitals of A, B and C are as $\frac{1}{2} : \frac{1}{3} : \frac{1}{4}$. After 4 months A withdraws half of his capital and after 8 months more the profit is Rs. 2024. Find what profit A will have. [P. U. 1910]

18. In a joint-stock business B's capital was half as much again as A's. After 8 months B withdraws half of his capital and after 2 months more A withdraws $\frac{1}{4}$ of his capital. If the profit for the year be Rs. 530, how much profit will each receive? [C. S.]

19. A, B and C enter into partnership with capitals of Rs. 600, Rs. 800 and Rs. 900 respectively. After a few months A puts in Rs. 300 more. After a year the profit amounts to Rs. 300 and C receives Rs. 108 as his share of profit. When does A put in Rs. 300?

20. In a joint concern A's capital : B's capital $= 5 : 4$. After 3 months A withdraws $\frac{3}{4}$ of his capital and B $\frac{2}{3}$ of his capital. If after a year the profit is Rs. 670, how should it be divided?

UNITARY METHOD

[*Income-tax*]

You have already learnt Unitary Method. A further discussion is made here.

Income-tax: Every man, merchant or joint family has to pay a tax on his income. This is called income-tax. The government of a country levies and realises this income-tax. It is fixed at a certain rate per rupee. As 4 P. (say) in the rupee, or as 3d. in the pound. The Government exempts income to a certain extent from income-tax and levies income tax on the rest of a man's income, if it exceeds that limit. Suppose income up to Rs. 3000 is exempted from tax. If a man has an annual income of Rs. 3000 or less, he is exempted from paying any income-tax; but a man having an annual income of Rs. 5000 will have to pay an income-tax on Rs. $(5000-3000)$ or Rs. 2000 at a certain rate. This rate increases with higher and higher levels of income. Government prepares a schedule of these rates of income-tax.

If a man pays **premium** for life insurance, then the income-tax on the amount of premium is exempted from tax. Generally the amount paid as premium to the extent of $\frac{1}{4}$ of the total income is free from income-tax. But if $\frac{1}{4}$ of the total income exceeds Rs. 8000, then only Rs. 8000 of the premium is exempted from income-tax. To ascertain income-tax in such cases first find the income-tax on the whole taxable income and then determine the income-tax on the exempted amount of premium at the same rate. The difference of the two taxes is the required income-tax.

The solution of problems on income-tax by the Unitary method is shown below :

Examples [20]

1. Find the income-tax on Rs. 13420 at 5 P. in the rupee.

Income-tax on Re. 1 = 5 P.

$\therefore$ " " on Rs. 13420 = 5×13420 P. = Rs. 671.

2. After paying an income-tax at 4 P. in the rupee a man has Rs. 5760 left ; find his gross income.

He has (Re. 1 − 4 P.) or $\frac{96}{100}$ rupee left out of Re. 1.

$\therefore$ Re. 1 is left when the income is Rs. $\frac{100}{96}$.

$\therefore$ Rs. 5760 is left when the income is

Rs. $\frac{100 \times 5760}{96}$ or Rs. 6000.

$\therefore$ the gross income = Rs. 6000.

3. After paying an income-tax at the rate of 2% a man finds that his income decreases by Rs. 200. What is his whole income ?

Income decreases by Rs. 2 on the total income of Rs. 100,

$\therefore$ " " " Re. 1 " " " " " Rs. $\frac{100}{2}$

$\therefore$ " " " Rs. 200 " " " Rs. $\frac{100 \times 200}{2}$

or Rs. 10000

$\therefore$ the reqd. income = Rs. 10,000.

4. When income-tax is 7 cents in the dollar, a person has to pay 52 dollars 60 cents more as income-tax than when the tax is 5 cents in the dollar. Find his income.

7 cents − 5 cents = 2 cents ; 52 dollars 60 cents = 5260 cents.

Income-tax increases by 2 cents on the income of 1 dollar.

$\therefore$ " " " " 5260 cents " " $\frac{1}{2} \times 5260$ dollars

= 2630 dollars.

$\therefore$ the reqd. income = 2630 dollars.

5. The income of a man decreased by Rs. 360; but the income-tax having been increased from 4 P. to 5 P. in the rupee he had to pay the same amount of income-tax as before. What was his income at first?

Since the income decreases by Rs. 360, the income-tax should have decreased by 360×4 P., or 1440 P., But now the rate of income-tax increases by $(5-4)$ or 1 P. in the rupee and $\therefore$ he has to pay income-tax of 1440 P. more at 1 p. per rupee on the present income, $\therefore$ his present income is (1440 P.$\div$1 P.) rupees or Rs. 1440.

$\therefore$ the required first income$=$Rs. 1440$+$Rs. 360$=$Rs. 1800.

6. The income-tax on the income above Rs. 3600 at 6 P. in the rupee is Rs. 126. What is the gross income?

6 P. or Re. $\frac{6}{100}$ is the income-tax on the income of Re. 1.

$\therefore$ Re. 1 " " " Rs. $\frac{100}{6}$

$\therefore$ Rs. 126 " " " Rs. $\frac{100\times126}{6}$ or Rs. 2100.

$\therefore$ the reqd. gross income$=$Rs. 3600$+$Rs. 2100$=$Rs. 5700.

7. The first Rs. 3000 of the income is not subject to an income-tax. A man pays Rs. 120 as income-tax at 9 pies in the rupee on the rest of his income. What is his gross income and what rate per rupee does he pay on the average on his whole income? [W. B. S. F. '58]

9p.$=$Re. $\frac{9}{192}$$=$Re. $\frac{3}{64}$.

$\therefore$ Re. $\frac{3}{64}$ is the income-tax on the income of Re. 1,

$\therefore$ Re. 1 " " " Rs. $\frac{64}{3}$

$\therefore$ Rs. 120 " " " Rs. $\frac{64\times120}{3}$ or Rs. 2560

$\therefore$ the gross income$=$Rs. 3000$+$Rs. 2560$=$Rs. 5560.

Again, income-tax on Rs. 5560$=$Rs. 120

$\therefore$ income-tax in the rupee on the average

$=$Re. $\frac{120}{5560}$$=$Re. $\frac{3}{139}$$=4\frac{20}{139}$ p.

8. The first Rs. 3000 of a man's income is exempted from income-tax and tax is paid at 3 paise in the rupee on the next Rs. 2500 of his income and at the rate of 5% on a further income up to Rs. 2000. If his monthly income be Rs. 600, find the amount of income-tax he has to pay in a year.

The annual income of the man$=$Rs. $600\times12$$=$Rs. 7200.

Of this income the first Rs. 3000 is not subject to income-tax.

∴ Of (Rs. 7200 − Rs. 3000) or Rs. 4200 he has to pay an income-tax at 3 P. in the rupee on Rs. 2500 and at the rate of 5% on his remaining income of (Rs. 4200 − 2500) or Rs. 1700.

Now, income-tax on Re. 1 = 3 P.

∴ income-tax on Rs. 2500 = 3 P. × 2500 = Rs. 75.

Again, income-tax on Rs. 100 = Rs. 5

∴ income-tax on Rs. 1700 = Rs. 5 × 17 = Rs. 85.

∴ the man has to pay Rs. (75 + 85) or Rs. 160 as the income-tax in a year.

9. An income up to the first Rs. 2500 is free from income-tax, but a further income of Rs. 2320 is subject to an income-tax at 4 P. in the rupee and the income above this is subject to an income-tax at 6 P. in the rupee. If a man pay altogether Rs. 169.60 P. in a year as income-tax, find his annual income.

Income-tax on Rs. 2320 = 4 P. × 2320 = 9280 P.
= Rs. 92.80 P.

∴ income-tax on the income above Rs. 2320
= Rs. 169.60 P. − Rs. 92.80 P. = Rs. 76.80 P.

An income-tax is levied on this income at 6 P. in the rupee.

∴ his income above Rs. 2320
= (Rs. 76.80 P ÷ 6 P.) rupees = Rs. 1280

∴ his total annual income
= Rs. 2500 + Rs. 2320 + Rs. 1280 = Rs. 6100.

10. The annual income of a person is Rs. 7225. The first Rs. 3000 of his income is free from income-tax and the income above this is subject to an income-tax at 3 P. in the rupee. But he is exempted from paying an income-tax at the rate of 3% on his annual provident fund deposit of Rs. 150. Find the total amount of the income-tax he has to pay in a year.

Rs. 7225 − Rs. 3000 = Rs. 4225.

Rs. 3000 is free from income-tax.

∴ income-tax on Rs. 4225 = 3 P. × 4225 = Rs. 126.75 P.

An income-tax on Rs. 150 at 3% is exempted from this income-tax due.

Income-tax on Rs. 150 at 3% = Rs. 4.50 P.

∴ he has to pay an income-tax of
(Rs. 126.75 P. − Rs. 4.50 P.) or Rs. 122.25 P. annually.

11. The annual income of a man is Rs. 5000 and he pays Rs. 1360 as premium of life insurance in a year. The first Rs. 3000 of his income is free from income-tax and $\frac{1}{4}$ of the total income or Rs. 8000 (whichever is less) is exempted from an income-tax for payment of premium. Find the amount of income-tax he has to pay per year at the rate of 5 P. in the rupee.

The man has to pay an income-tax on (Rs. 5000 − Rs. 3000) or Rs. 2000.

Income-tax on Re. 1 = 5 P.

$\therefore$ income-tax on Rs. 2000 = 5 P. × 2000 = Rs. 100.

The man pays a premium of Rs. 1360, but $\frac{1}{4}$ of his total income of Rs. 5000 = Rs. 1250, $\therefore$ the income-tax on Rs. 1250 in proportion to the income-tax on his whole income will be exempted.

$\therefore$ income-tax on Rs. 5000 = Rs. 100,

$\therefore$,, Re. 1 = Rs. $\frac{100}{5000}$ or Rs. $\frac{1}{50}$

$\therefore$,, Rs. 1250 = Rs. $\frac{1}{50}$ × 1250 = Rs. 25

$\therefore$ he has to pay an income-tax of (Rs. 100 − Rs. 25) or Rs. 75 annually.

Exercise 20

1. Find the income-tax on Rs. 2200 at 3 P. in the rupee.

2. A man pays Rs. 472.50 P. as income-tax at the rate of 5 P. in the rupee ; find his income.

3. What net income will remain after paying an income-tax on Rs. 450 at the rate of 2·5% ?

4. After paying an income-tax at 4 P. in the rupee a man has Rs. 480.24 P. left, find his gross income.

5. If the rate of income-tax increases by 1 P. in the rupee, the net income of a man after paying income-tax decreases by Rs. 12.10 P. What is his gross income ?

6. When the income-tax is 7 P. in the rupee a person has to pay Rs. 170.80 P. more than when the tax is 5 P. in the rupee. Find his income.

7. The income of a man decreases by Rs. 650, but the rate of income tax being raised from 4 P. to 6 P. in the rupee he has to pay the same amount of income-tax as before. What was his income at first ?

8. The total income of a person is Rs. 4650. He has to pay an income-tax at 3 P. in the rupee on his income above Rs. 2500. What is his net income after payment of income tax ?

9. A person pays an income-tax of £68 at the rate of 8*d.* in the pound on his income above £1500 up to which he is exempted from paying an income-tax. What is his net income after payment of income-tax ?

10. After paying an income-tax at 8 P. in the rupee a man has Rs. 552 left. If he pays the income-tax at 7 P. in the rupee, what income will be left after payment of income-tax ?

11. A person pays on his salary an income-tax of 9 pies in the rupee and contributes one-anna in the rupee to Provident Fund. If he has Rs. 445. 5a. left after this payment, what is his salary ? [C. U. '32]

12. The annual income of a person is Rs. 9875 and the first Rs. 2000 of his income is not subject to income-tax. After paying the income-tax he has Rs. 9481.25 P. left. Find the rate of income per rupee.

13. The monthly income of a man is Rs. 625. He has to pay no income-tax on the first Rs. 3000 of his income. He pays an icome-tax at 4 P. in the rupee on his further income of Rs. 3500 and then at the rate of 5% on an income above this. Find the amount of income-tax he has to pay in a year.

14. The annual income of a man is Rs. 5600 and he deposits a premium of Rs. 1520 for life insurance in a year. The first Rs. 3000 of his income is free from income-tax and $\frac{1}{4}$ of the total income or Rs. 8000 (whichever is less) is not subject to an income-tax for payment of premium. Find the amount of income-tax he has to pay in a year at the rate of 5 P. in the rupee.

FOREIGN EXCHANGE AND CHAIN RULE

The table of principal foreign coins.

Country	*Principal coins*	*Country*	*Principal coins*
England, Australia, New Zealand	...1 pound (£)	America, Canada	...1 Dollar ($) (=100 cents)
Italy	...1 lira	Japan	...1 yen (=100 sen).
France, Switzerland, Belgium	...1 franc	China	...1 tael
Russia	...1 rouble =(100 copecks)	Ceylon	...1 Rupee (=100 cents)
Germany	...1 marc.	Pakistan	...1 Rupee (=100 pices)
		India	...1 Rupee (=100 P.)

There are different kinds of coins made of different metals in different countries. The coin of one country is not current or accepted in another country. *Exchange* means the giving or receiving of a sum of money of one country equal in intrinsic value to a given sum of money of another country. Money transactions are thus made in commercial field.

Gold is the medium of this exchange.

The relation or ratio of the actual value of gold contained in a coin of one country to the intrisic or real value of gold contained in a coin of another country is called *Par of Exchange* : Pound is an English coin and Franc is a French coin. If one pound contains 25'2 times the quantity of gold contained in one franc, the *par of exchange* of a pound is 25'2 francs.

The rate of Exchange is the actual or marketable value at any time of a coin of one country, as estimated in terms of a coin of the other country. It may be equal to, greater than or less than the par of exchange. When the rate of exchange is greater than the par of exchange, it is said to be at a *permium.*

When the rate of exchange is less than the par of exchange, it is said to be at a *discount.*

This exchange between two countries may be made direct or through the medium of one or more countries.

Money transactions between one country and another are usually carried on by means of *Foreign Bill of Exchange* or briefly *Foreign Bills* or by Draft, Hundi etc.

The following is the usual process :—Suppose an Indian merchant wants to transmit a sum of money to another merchant in London. He goes to a banker and pays in local money a sum equivalent to the amount payable in London at the current rate of exchange. The Bank sends the *bill* or *hundi* or *draft* to the merchant in London, who presents it to the local branch of the bank and receives the amount. Discussion about '*draft*' has been made hereafter.

Problems on Exchange can be easily worked out by the help of the *Chain Rule.*

The brief method of repeated applications of unitary method is called the *Chain Rule.* In the chain rule the different quantities should be so placed that no two quantities of the same kind are in the same column. Note the following examples :

Example [21]

1. If 8 sheep cost as much as 12 goats, 6 goats as much as 40 cocks and 20 cocks as much as 32 ducks, find the value of one sheep, one duck being worth Re. 1.20 P.

[Solution by the Chain Rule]

Cost of 8 sheep = cost of 12 goats,

Cost of 6 goats = cost of 40 cocks,

Cost of 20 cocks = cost of 32 ducks,

Cost of 1 duck = Re 1. 20 P. = Rs. $\frac{6}{5}$.

$\therefore$ the reqd. cost of 1 sheep = Rs. $\frac{12\times40\times32\times6}{8\times6\times20\times5}$ = Rs. 19.20 P.

2. If 5 sheep cost as much as 8 goats, and 30 goats cost as much as 3 cows, how many cows will cost as much as 50 sheep ?

[G. U.'62]

Suppose the required number of cows $= x$.

Now, Cost of x cows = cost of 50 sheep,

Cost of 5 sheep = cost of 8 goats,

Cost of 30 goats = cost of 3 cows.

$\therefore$ x or the required number of cows $=\frac{50\times8\times3}{5\times30}=8$.

3. If 2 sheep cost 185 francs, cost of 2 calves be equal to $\frac{1}{3}$ of the cost of 1 bullock, 15 sheep cost as much as 2 bullocks and if 55'50 francs are worth £2., how many calves will you get at the cost of £25 ? [C. S.]

Suppose, the required number of calves $= x$.

Now, cost of x calves $=$ £25

£2 $=$ 55'50 francs

185 francs $=$ cost of 2 sheep

Cost of 15 sheep $=$ cost of 2 bullocks

Cost of $\frac{1}{3}$ bullock $=$ cost of 2 calves

$\therefore$ x or the reqd. number of calves $= \dfrac{25 \times 55'50 \times 2 \times 2 \times 2}{2 \times 185 \times 15 \times \frac{1}{3}} = 6.$

4. If 9 Kg. of rice be worth 4 Kg. of sugar, 14 Kg. of sugar be worth 1'5 Kg. of tea, 2 Kg. of tea be worth 5 Kg. of coffee, and if 2 Kg. 5 Hg. of rice cost Re. 1.25 P., then what is the cost of 11 Kg. of coffee ?

Suppose, the reqd. cost of 11 Kg. of coffee $= x$ rupees.

$\therefore$ Rs. $x =$ cost of 11 Kg. of coffee,

Cost of 5 Kg. of coffee $=$ cost of 2 Kg. of tea

Cost of 1'5 Kg. of tea $=$ cost of 14 Kg. of sugar

Cost of 4 Kg. of sugar $=$ cost of 9 Kg. of rice

Cost of 2'5 Kg. of rice $=$ Rs. 1.25 P.

$\therefore$ $x = \dfrac{11 \times 2 \times 14 \times 9 \times 1'25}{5 \times 1'5 \times 4 \times 2'5}$ rupees $=$ Rs. 46.20 P.

$\therefore$ the reqd. price of 11 Kg. of coffee $=$ Rs. 46.20 P.

5. 19 dollars $=$ 80 marcs, 16'1 marcs $=$ 100 francs, 25 francs $=$ 1 pound and 1*s*. 4*d*. $=$ Re. 1. How many rupees are equal to 3059 dollars ? [P. U. 1916]

Suppose, the reqd. number of rupees $= x$.

$\therefore$ Here the chain should be as follows :

Rs. $x =$ 3059 dollars

19 dollars $=$ 80 marcs

16'1 marcs $=$ 100 francs

25 francs $=$ £ 1 $=$ 240 *d*.

1*s*. 4*d*. or 16*d*. $=$ Re. 1.

$\therefore$ $x = \dfrac{3059 \times 80 \times 100 \times 240 \times 1}{19 \times 16'1 \times 25 \times 16} = 48000$

$\therefore$ the reqd. rupees $=$ Rs. 48000.

6. A merchant of New York purchased goods worth 4004 francs at Geneva. If the rate of exchange between London and New York be 4·865 dollars=£1, and that between London and Geneva be 25·48 francs=£1; what should be the cost of the goods in dollars? [C. U. '38, '42]

Suppose, cost of the goods=x dollars.

$\therefore$ x dollars=4004 francs

25·48 francs=£1

£1=4·865 dollars

$$\therefore \quad x=\frac{4004\times 4{\cdot}865}{25{\cdot}48}\$=764{\cdot}5\ \$ \text{ (nearly)}$$

$\therefore$ the cost of the goods=764·5 dollars (nearly).

7. The rate of exchange between London and Calcutta is 16 and that between London and Paris is 26·4. What is the rate of exchange between Calcutta and Paris through the medium of London?

From the given condition we see that the rates of exchange of Calcutta and Paris with the same country are 16 and 26·4 respectively. $\therefore$ Rs. 16=26·4 francs

$\therefore$ Re. 1=26·4 francs÷16=1·65 francs.

$\therefore$ the reqd. rate of exchange is Re. 1=1·65 francs.

8. A merchant of Bombay owes Rs. 1410 to a marchant of Berlin. He repays his debt through a London Bank. If the rates of exchange be as follows: Re. 1=1*s.* 4*d.* and 1 marc=$11\frac{3}{4}$ *d.*, how many marcs will the merchant of Berlin get? [I. I. B.]

Suppose, the merchant of Berlin gets x marcs. $\therefore$ by the chain rule, x marcs=Rs. 1410

Re. 1=1*s.* 4*d.*=16*d.*

$11\frac{3}{4}$ or $\frac{47}{4}$ *d.*=1 marc

$$\therefore \quad x=\frac{1410\times 16\times 1}{1\times \frac{47}{4}} \text{ marcs}=1920 \text{ marcs.}$$

$\therefore$ the merchant of Berlin will get 1920 marcs.

9. A Bombay merchant has to repay a debt of 10000 francs to a Paris merchant. Which is more advantageous for him to remit, directly to Paris or through the medium of London? The rate of exchange in the first case is 1 franc=10 as. 6 p. and in the second case it is 1*s.* 3*d.*=Re. 1. and 25 francs=1 sovereign. [C. U. '34]

In the first case, when the bill is drawn directly on Paris the expenses=10 as. 6p.×10000=Rs. 6562. 8 as.

In the second case, suppose x rupees$=$10000 francs

25 francs$=$£1$=$240 $d.$

1$s.$ 3$d.$ or 15 $d.$$=$Re. 1.

$$\therefore \quad x=\frac{10000\times 240}{25\times 15} \text{ rupees}=\text{Rs. }6400.$$

$\therefore$ if money be remitted through London it will be less costly and therefore more advantageous.

10. A Madras merchant owes £ 395. 4$s.$ to a London merchant. He finds that if instead of sending the money direct to London, he sends it through Paris, he may save Rs. 104. The rates of exchange are as follows: Re. 1$=$1·71 francs., £1$=$25·2 francs. Find the rate of exchange between London and Madras.

Here, we have first to find how many rupees are worth £395. 4$s.$, suppose it is equal to x rupees.

$\therefore$ by the chain rule, £1$=$25·2 francs,

1·71 francs$=$Re. 1

x rupees$=$£ 395. 4$s.$$=$£$\frac{1976}{5}$

$$\therefore \quad x=\frac{25{\cdot}2\times 1\times 1976}{1\times 1{\cdot}71\times 5} \text{ rupees}=\text{Rs. }5824.$$

$\because$ the merchant has to spend Rs. 104 more, if he sends money direct from Madras to London,

$\therefore$ the rate of exchange between Madras and London is

Rs. (5824$+$104) or Rs. 5928$=$£$\frac{1976}{5}$.

$\therefore$ Re. 1$=$£$\frac{1976}{5\times 5928}$$=$£$\frac{1}{15}$$=$$\frac{4}{3}$ $s.$$=$1 $s.$ 4$d.$

$\therefore$ the reqd. rate of exchange is Re. 1$=$1$s.$ 4$d.$

11. The par of exchange is 3 francs$=$1 rupee. If the Paris money be at a premium of 10%, how many francs can be had in exchange for Rs. 2310 ?

At the par of exchange 3 francs$=$Re. 1.

$\therefore$ at the premium of 10% 3 francs$=$Rs. $\frac{110}{100}$$=$Rs. $\frac{11}{10}$

$\therefore$ at the premium Rs. $\frac{11}{10}$ is exchanged for 3 francs,

$\therefore$,, ,, Re. 1 is exchanged for $\frac{30}{11}$ francs.

$\therefore$ Rs. 2310 is exchanged for $\frac{30}{11}\times 2310$ or 6300 francs.

12. The par of exchange is 1*s*. 6*d*. per rupee, At what discount should the rate of exchange be so that Rs. 662. 50 P. may be exchanged for £47. 14*s*. ?

Rs. 662.50 P. is exchanged for £47.14*s*.

$\therefore$ Rs. $662\frac{1}{2} = £47\frac{7}{10}$

$\therefore$ Re. $1 = £\left(\frac{477}{10} \times \frac{2}{1325}\right) = £\frac{9}{125}$

Again, at par of exchange, Re. $1 = \frac{3}{2}s. = £\frac{3}{40}$

$\therefore$ the discount on $£\frac{3}{40} = £\left(\frac{3}{40} - \frac{9}{125}\right) = £\frac{3}{1000}$

$\therefore$ " " $£\ 1 = £\frac{3 \times 40}{1000 \times 3} = £\frac{1}{25}$

$\therefore$ " " $£\ 100 = £\frac{100}{25} = £4.$

$\therefore$ the rate of exchange is at a discount of 4%.

Exercises 21

1. If the rate of exchange is 1 *s*. $4\frac{8}{33}$ *d*. = 1 rupee, how many pounds etc. can be had in exchange for Rs. 100000.

2. If the cost of one metre of cloth is 16·5 francs and Rs. 100 = 780 francs and 1 yard = 91·44 cm., find the cost of 1 yd. of cloth in rupees and annas.

3. If the rate of exchange be 1*s*. 6*d*. = 1 rupee, how many rupees can be had in exchange for £2. 5*s* ?

4. The rate of exchange between Calcutta and Yokohama is Re. 1. $9\frac{1}{4}$ as. per Yen, betweeu London and Calcutta 1*s*. $5\frac{1}{2}$*d*. per rupee and between Yokohama and New York ·52 dollar per yen. What should be the rate of exchange between London and New York through the medium of Yokohama ?

5. A bill of exchange is purchased at Calcutta at the rate of 1*s*. 3*d*. per rupee and sold at Paris at the rate of 24 francs per pound. Find the rate of exchange between Calcutta and Paris.

6. One ounce of gold is worth £7. 6*d*. in London. If 1 oz. = 28·35 grams and £1 = 4·86 dollars, find the cost of 21 grams of gold in dollars. [P. U.]

7. A Calcutta merchant wants to remit £300 to a London merchant. If the rate of exchange be 1*s*. 4*d*. = 1 rupee, how many rupees should be deposited in a Calcutta Bank to have a bill of exchange for £300 ?

8. If the rates of exchange of Calcutta and Paris with London are 15 and 25·23 respectively, find the rate of exchange between Calcutta and Paris through the medium of London.

9. It costs Rs. 12. 1a. including postage of Re. 1. 2as. to bring a book from London. The book-seller allows a commission of 2*d*. per shilling on the face value of the book and the rate of exchange is 1*s*. 4*d*.=1 rupee. What is the face value of the book in English money ? [C. U. 1906]

10. If 4 goats cost as much as 3 sheep, 7 sheep cost as much as 2 cows and 9 cows cost as much as 7 horses, what is the price of a goat when a horse costs Rs. 90 ?

11. 24 cows can be had in exchange of 6 horses, 8 buffaloes can be had for 10 cows, 15 asses for 4 buffaloes and 32 sheep can be had for 8 asses. What is the price of one horse when 9 sheep cost Rs. 25 ? [D. B. 1926]

12. What is the value of a draft of £1030. 7*s*. 6*d*. on a London Bank at the rate of exchange of 1*s*. $3\frac{15}{16}$*d*. per rupee ?

13. A Bombay merchant is to remit £ 1000 to a London merchant. If instead of sending the money direct to London he remits the amount through Paris, he has to spend Rs. 200 less. The rate of exchange between Bombay and Paris is Rs. 617=2016 francs and the rate of exchange between Paris and London is 50·40 francs=£1. What is the rate of exchange between London and Bombay ? [B. U. 1922]

14. A merchant in New York has to pay the price of goods purchased at 5000 dollars in London, 1 dollar being equal to 4*s*. 6*d*. Bill for what sum in English money must be drawn when bills are at a premium of $9\frac{1}{2}$% in London ? [C. U. 1945]

15. Exchange Rs. 2910 for English money when it is at a discount of 3%, given that at par Re. 1=1*s*. 6*d*.

METRIC SYSTEM

Metric Linear Measure

Metre. In this system the unit of length or linear unit is named metre. Adjective from 'metre' is 'metric'; so this system goes by the name of **Metric System.**

Metre is taken to be the ten-millionth part of quarter of the circumference of the earth (i.e. of the distance from the pole to the equator).

$$\therefore \quad 1 \text{ metre} = \frac{\text{Circumference of the Earth}}{40000000}$$

Thus, 1 metre = 39·370113...inches. In brief, 1 metre = 39·37 inches.

Units of length

10 millimetres (mm.) = 1 centimetre (cm.)
10 centimetres = 1 decimetre (dm.)
10 decimetres = 1 metre (m.)
10 metres = 1 decametre (Dm.)
10 decametres = 1 hectometre (Hm.)
10 hectometres = 1 kilometre (Km.)
10 kilometres = 1 myriametre (Mm.)

Conversion table

1 inch = ·025399...metre = 25·4 mm.
1 foot = ·3048...metre = 30·48 cm.
1 yard = ·91438...metre = ·91 m. (App.).
1 mile = 1609·3149...metres = 1·61 Km. (App.).
1 Km. = $\frac{5}{8}$ mile = ·62 mile (App.)
1 metre = $1\frac{3}{32}$ yards = 1·09 yards (approximately).

Metric Square Measure

The area of a square whose each side is 1 metre is 1 square metre (1m. by 1m.). In this way, the area of a square whose each side is 1 decametre is 1 square decametre. One square decametre is the unit of square measure in metric system. It is called an are.

$\therefore$ 1 are = 1 square decametre = 100 square metres.

Table

	1 are = 120 sq. yards (App,)
10 centiares = 1 deciare,	1 hectare = $2\frac{1}{2}$ acres (App.)
10 deciares = 1 are,	1 acre = ·40 hectare
10 ares = 1 decare, etc,	1 sq. yard = ·84 sq. m.
100 hectares = 1 sq. Km.	1 sq. inch = 6·5 sq. cm.
	1 square mile = 2·59 sq. Km.

Metric Cubic Measure

The unit of volume is the **cubic metre** which is the volume of a cube whose length, breadth and height or thickness or depth are each a metre. This cubic unit is called a *Stere* and is used in measuring huge objects, wood, etc. 10 steres make a decastere, etc.

The unit of capacity, for measuring liquid is called the **litre** and is equal to one cubic decimetre, i.e., the capacity of a cube whose length, breadth and depth are each one decimetre.

1 Litre = ·22 gallon, 1 gallon = 4·55 litres.

1 cubic metre = 1·31 cubic yards, 1 cubic yard = ·76 cubic metre.

Table

1000 cu. millimetres = 1 cu. centimetre, etc.

10 millilitres (ml.) = 1 centilitre (cl.), etc.

Metric Weight Measure

The unit of weight is the **gram** (or gramme) which is the weight of one cubic centimetre of distilled water at its maximum density (at 4° centigrade).

1 kilogram = 1000 gram = weight of 1000 cu. centimetre of water = weight of 1 cu. decimetre of water = weight of 1 litre of water.

100 kilograms = 1 quintal.

1000 kilograms = 1 Tonne or Metric ton,

1 tonne = 0·98 ton ; 1 ton = 1·02 tonne.

1 quintal = 1·97 cwt. (hundred weights).

1 gram = ·09 tola ; 1 tola = 11·66 gram ;

1 kilogarm = 1·07 seers ; 1 seer = 0·93 kg.

1 kg. = $2\frac{1}{5}$ lbs. (Avoir.) = 86 tola (App.).

1 lb. = 0·45 kg. ; 1 maund = 0·37 quintal.

1 quintal = 2·68 maund ; 1 chhatak = 58 grams (App,).

French Money

10 centimes = 1 decime

10 decimes = 1 franc (fr.) = $\frac{4}{5}$ shilling (nearly).

20 francs = 1 Napoleon.

Examples [22]

Reduction in Metric System

1. Reduce 217032005 millimetres to metres, kilometres etc.

[As in writing the digits of a number, first place the digit in the units' place of the given number of millimetres just below the millimetre's place and then put each digit to the left one by one just below each respective higher units. Now read off the number of units from the left to the right]

Myria	Kilo	Hecto	Deca	Metre	deci	centi	milli
21	7	0	3	2	0	0	5

∴ 217032005 millimetres = 21 Mm. 7 Km. 3 Dm. 2m. 5mm.

2. Express 7 Mg. 2 Kg. 4 Dg. 8g. 5 cg. in grams.

[Here also place the given quantity as in the previous method. Then put the decimal point just to the right of the unit in which the given quantity is to be expressed.]

Myria	Kilo	Hecto	Deca	Gram	deci	centi	milli
7	2	0	4	8.	0	5	0

∴ the given quantity = 72048·05 grams.

3. Express 4·203608 kilolitres as a compound quantity.

[Here 4 is a whole number, ∴ it should be taken as 4 kilolitres and each figure to the right should be taken one by one as the successive sub-unit.] Thus.

4·203608 kilolitres = 4 kilolitres 2 hectolitres 3 litres 6 decilitres 8 millilitres.

Addition and subtraction. In metric system the processes of addition and subtraction are just like those of addition and subtraction of compound quantities.

4. Multiply 8 Hm. 6 Dm. 7 dm. by 28.

[Multiplication may be effected by the ordinary method.]

[Alternative method] The given quantity = 8607 dm.

$$\begin{array}{r} 8607 \text{ dm.} \\ \times 28 \\ \hline 68856 \\ 17214 \\ \hline 240996 \text{ dm.} \end{array}$$

∴ the required product = 2 Mm. 4 Km. 9 Dm. 9 m. 6 dm.

5. Divide 5 Mg. 3 Kg. 8 Dg. 7 g. by 125.

The given quantity = 53087 g. ; 53087 g. ÷ 125 = 424·696 g.

∴ the reqd. quotient = 4 Hg. 2 Dg. 4 g. 6 dg. 9 cg. 6 mg.

6. Express 3 miles in kilometres.

∵ 1 mile = 1609·31 metres

= 1·61 Km. (correct to 2 decimal places)

∴ 3 miles = 1·61 Km. × 3 = 4·83 Km.

7. How many gallons are equal to one kilolitre ?

∵ 4·55 litres = 1 gallon, ∴ 1 litre = $\frac{1}{4\cdot55}$ gallon.

∴ 1 kilolitre = $\frac{1000}{4\cdot55}$ gallons = 219·78 gallons.

8. If one chhatak of sugar cost 5 P., what does one kilogram of sugar cost ?

1 Kg. = 1·07 seers = 1·07 × 16 chhataks

∴ the required cost = 1·07 × 16 × 5 P. = 85·6 P.

Solution of Miscellaneous Problems

1. Metre is equal to the ten-millionth part of a quarter of the circumference of the earth. If 1 metre = 39·37079 inches, what is the circumference of the earth in miles ?

$$1 \text{ metre} = \frac{\text{circumference of the earth}}{40000000}$$

∴ the required circumference = 40000000 metres

= 40000000 × 39·37079 inches = 400 × 3937079 inches

= $\frac{400 \times 3937079}{12 \times 3 \times 1760}$ miles

= 24855·2...miles = 24855 miles (approximately).

2. The velocity of light is 3×10^8 metres per second and light takes 8 minutes to reach the earth from the sun. What is the distance of the earth from the sun in miles ? [1 m. = 39·37 in.]

[C.U. '43 ; D.B. '34]

$\because$ In 1 second light travels 3×10^8 metres,

$\therefore$ in 8 minutes ,, ,, $3\times10^8\times60\times8$ metres

$=3\times10^8\times60\times8\times39{\cdot}37$ inches

$=\dfrac{3\times10^6\times60\times8\times3937}{12\times3\times1760}$ miles $=89477272\frac{8}{11}$ miles.

$\therefore$ the reqd. distance $=89477272{\cdot}7\dot{2}$ miles.

3. What does it cost to fence a square plot of land 2·5 acres in area at the rate of Rs. 39.37 P. per kilometre ?

[1 metre = 39·37 inches]

Area of the square plot = 2·5 acres

$=2{\cdot}5\times4840$ square yards $=25\times484$ sq. yds.

$\therefore$ each side of the square plot

$=\sqrt{25\times484}$ yards $=5\times22$ yds. $=110$ yds.

$\therefore$ its perimeter $=110$ yds. $\times4=110\times4\times3\times12$ inches

$=\dfrac{110\times4\times3\times12}{39{\cdot}37}$ metres $=\dfrac{110\times4\times3\times12}{39{\cdot}37\times1000}$ kilometres

$\therefore$ the required expenses

$=\frac{110\times4\times3\times12}{39370}\times$ Rs. 39.37 P. $=\frac{110\times4\times3\times12}{39370}\times3937$ P.

$=11\times4\times3\times12$ P. = Rs. 15.84 P.

4. If 1 metre $=39\frac{3}{8}$ inches, find the nearest whole number of litres in a cubic foot.

[C.U. '11 ; D.B. '38]

$39\frac{3}{8}$ inches or 39·375 inches = 1 metre = 100 centimetres.

1 inch $=\dfrac{100}{39{\cdot}375}$ cm. ; $\therefore$ 1 foot $=\dfrac{100\times12}{39{\cdot}375}$ cm.

$\therefore$ 1 cubic foot $=\left(\dfrac{1200}{39{\cdot}375}\right)^3$ cubic centimetres $=(30{\cdot}4\ldots)^3$ cu.cm.

$=\dfrac{(30{\cdot}4)^3}{1000}$ cu. dm. or litres [$\because$ 1 litre = 1 cu. dm.]

$=28$ litres (approximately).

5. How many litres of water weigh 1000lbs., given that 1 cubic foot of water weighs 1000 oz. and 1 metre=39'37 inches ? [C. U. '37 ; P. U. '18 ; D. B. '47]

[$\because$ 1 cu. dm.=1 litre, $\therefore$ first find how many cu. dm. are equal to 1 cu. foot.]

1 cu. foot=12×12×12 cu. inches$=\dfrac{12^3}{(39{\cdot}37)^3}$ cu. metres

$$=\frac{12^3}{(39{\cdot}37)^3}\times10^3 \text{ cu. dm}=\frac{12^3\times10^3}{(39{\cdot}37)^3}\text{ litres}$$

Now, 1000 lbs.=16000 oz.=weight of 16 cubic feet of water

=weight of $\dfrac{16\times12^3\times10^3}{(39{\cdot}37)^3}$ litres of water

=weight of 453'0696 litres of water.

6. Given that 1 pound=7000 grains and 1 gram=15'432 grains, find the number of grains in an ounce (avoir) correct to 3 decimal places. [C. U. '49]

1 ounce$=\frac{1}{16}$ pound$=\frac{7000}{16}$ grains

$=\frac{7000}{16\times15{\cdot}432}$ grams=28'350 grams (approximately).

7. Find in kilograms the weight of 16 hectoliters of mercury, given that mercury is 13'6 times as heavy as water. [G. U. '49]

$\because$ 16 Hl.=1600 litres=1600 cu. dm.=1600000 cu. cm.

$\therefore$ weight of 16 Hl. of water

=weight of 1600000 cu. cm. of water

=1600000 grams=1600 kilograms.

$\therefore$ weight of 16 Hl. of mercury=1600×13'6Kg.=21760 kilograms.

8. If the circumference of the earth be 40000 kilometres and the circumference is $\frac{22}{7}$ times the diameter, find the radius of the earth in miles. (1 metre=39'3709 inches.) [C. U. (High) '50]

$\frac{22}{7}\times$diameter=40000 km., $\therefore$ $\frac{44}{7}\times$radius=40000 km.

$\therefore$ Radius$=\frac{40000\times7}{44}$ km.$=\frac{10000\times7\times1000}{11}$ metres

$$=\frac{10000\times7\times1000\times39{\cdot}3709}{11}\text{ inches}$$

$=\frac{7\times1000\times393709}{11\times12\times3\times1760}$ miles=3954'2 miles.

9. Find the length in feet of a tank which is 2·56 metres deep and holds 300000 litres of water, the length being 3 times the breadth. (1 metre = 39·37 inches). [C. U. '50 sp.]

The capacity of the cistern (i.e. length × breadth × depth) = 300000 litres = 300000 cu.dm. ; its depth = 2·56m. = 25·6 dm.

$\therefore$ length × breadth $= \frac{300000}{25 \cdot 6}$ sq.dm. $= \frac{3000000}{256}$ sq. dm.

or, 3 breadth × breadth $= \frac{3000000}{256}$ sq. dm.

or, $(\text{breadth})^2 = \frac{1000000}{256}$ sq. dm., $\therefore$ breadth $= \frac{1000}{16}$ dm.

$\therefore$ the reqd. length $= \frac{3000}{16}$ dm. $= \frac{300}{16}$ m. $= \frac{300}{16} \times 39 \cdot 37$ in.

$= \frac{300 \times 39 \cdot 37}{16 \times 12}$ ft. $= \frac{3937}{64}$ feet = 61·5 feet.

10. An open rectangular tank with a square base contains 60750 litres of water. Find the cost of lining its inner surface with lead at Rs. 5 per square metre, the height of the tank being 3 metres. [C.U. (High) '50]

The volume of the tank = 60750 litres = 60750 cu. dm.

or, the area of the base × height = 60750 cu. dm.

But the height = 3 m. = 30 dm.,

$\therefore$ the area of the square base $= \frac{60750}{30}$ sq. dm. = 2025 sq. dm.

$\therefore$ length or breadth of the tank $= \sqrt{2025}$ dm. = 45 dm.

Now, the area of the 4 perpendicular interior surfaces

= 45 dm. × height × 4 = 45 dm. × 30 dm. × 4 = 5400 sq. dm.

$\therefore$ the area of all internal surfaces including the square base = (2025 + 5400) sq. dm. = 7425 sq. dm. = 74·25 sq. m.

($\because$ 1 sq. m. = 100 sq. dm.)

$\therefore$ the reqd. cost = Rs. 5 × 74·25 = Rs. 371·25 = Rs. 371.25 P.

11. A gallon of water weighs 10 lbs. ; find its volume in cubic centimetres. (1 Kg. = $2\frac{1}{5}$ lbs.) [C. U. '48]

$\because$ $2\frac{1}{5}$ lbs. = 1 Kg. = 1000 g., $\therefore$ 1 lb. $= \frac{1000 \times 5}{11}$ g.

$\therefore$ Weight of 1 gallon of water = 10 lbs.

$= \frac{1000 \times 5 \times 10}{11}$ g. $= \frac{50000}{11}$ g.

$\therefore$ Volume of 1 gallon of water $= \frac{50000}{11}$ c.c. = $4545 \cdot \dot{4}\dot{5}$ cu.cm.

12. A decilitre of air weighs ·1293 gram. Find the weight of a cubic inch of air in grains correct to 4 decimal places ; given 1 foot=30·4 cms., 1 gram=15·435 grains. [C. U. '50]

$\because$ 1 foot=30·4 cm., $\therefore$ 1 inch$=\dfrac{30·4}{12}$ cm.$=2·533$ cm.

$\therefore$ 1 cubic inch$=(2·533)^3$ cubic cm.

$=\dfrac{(2·533)^3}{1000}$ cu. dm.$=\dfrac{(2·533)^3}{1000}$ litres$=\dfrac{(2·533)^3}{100}$ decilitres.

$\therefore$ Weight of 1 cubic inch of air=weight of $\dfrac{(2·533)^3}{100}$ decilitres of air$=\dfrac{(2·533)^3}{100}\times·1293$ gram$=\dfrac{(2·533)^3}{100}\times·1293\times15·435$ grains

$=·3245$ grains (App).

13. Given that 1 yard$=0·9144$ metre and £ 1$=25$ francs ; find, correct to a farthing, the difference of the fares for the railway journey of 250 miles in England and in France, if the railway fare in France be 6 centimes per kilometre and $1\frac{1}{2}$ d. per mile in England. [G. U. '50]

250 miles$=250\times1760$ yards$=250\times1760\times·9144$ metres$=$ 402336 m.$=402·336$ Km. $\therefore$ the fare of 250 miles in France $=402·336\times6$ centimes$=\dfrac{402·336\times6}{100}$ francs$=\dfrac{2414·016}{100}$ francs

$=24·14016$ francs$=\dfrac{24·14016}{25}$ pound$=$£ ·9656064.

Again, the fare for 250 miles in England$=1\frac{1}{2}$ d.$\times250=$ $\dfrac{3\times250}{2\times12\times20}$ pounds$=$£ $\dfrac{25}{16}=$£ 1·5625.

$\therefore$ the difference of the fares in the two countries $=$£ 1·5625 $-$ £ ·9656064$=$£ ·5968936$=$11s. 11d. 1q. (App).

14. A gallon contains 277·274 cubic inches, a cubic decimetre is 61 cubic inches and a kilogram is $2\frac{1}{5}$ lbs. Calculate the weight in pounds of 1 gallon of water. [D. B. '42]

Volume of 1 gallon$=277·274$ cubic inches. $\therefore$ the weight of 1 gallon of water$=$weight of 277·274 cubic inches of water

$=\dfrac{277·274}{61}$ cubic decimetres of water$=\dfrac{277·274\times1000}{61}$ cu. cm. of water$=\dfrac{277274}{61}$ grams [$\because$ weight of 1 cu. cm. of water$=1$ g.]

$=\dfrac{277274}{61\times1000}$ Kg. $=\dfrac{277274}{61000}\times\dfrac{11}{5}$ pounds$=10$ lbs. (approximately).

15. A litre of good milk weighs 1·032 Kg. After purchasing 6 litres one morning, I found that the weight was 6·128Kgs. How many c.c. of water had been added ? [C. S. '31 ; D. B. '64]

The weight of 6 litres of pure milk $=1·032$ Kg. $\times 6=6·192$ Kg.

Again, the weight of 6 litres of milk mixed with water $=$ 6·128 Kg.

$\therefore$ the weight of 6 litres of milk mixed with water is $(6·192-6·128)$ or ·064Kg. less than the wt. of 6 litres of pure milk.

$\because$ 1 litre $=1000$ cu. cm., $\therefore$ the weight of 1 litre of water $=$ the weight of 1000 cu. cm. of water $=1000$ grams $=1$ Kg.

$\therefore$ if 1 litre of water be mixed instead of 1 litre of pure milk the weight decreases by $(1·032-1)$ or ·032 Kg. per litre.

Now, the weight decreases by ·032 Kg. in 1 litre,

$\therefore$ " " ·064 Kg. in $\frac{·064}{·032}$ or 2 litres

$\therefore$ 2 litres or 2000 c.c. of water were mixed.

16. The earth's circumference measured along a meridian is 40 million metres or 21600 Nautical miles. Assuming that a metre $=39·3708$ inches, find the nearest whole number of feet in a Nautical mile. [C. U. '45 ; G. U. '51]

$\because$ 21600 Nautical miles $=40000000$ metres,

$\therefore$ 1 Nautical mile $=\frac{40000000}{21600}$ metres

$=\frac{400000\times 39·3708}{216}$ inches $=\frac{40\times 393708}{216\times 12}$ feet $=6076$ feet (App.).

17. If 1 inch $=2·54$ cm., 1 kg. $=2·2$ lbs. and the pressure of the atmosphere to each square inch be 15 lbs. (Avoir), find in grams the pressure per square centimetre. [D. B. '28]

2·54 cm. $=1$ inch, $\therefore$ 1 cm. $=\frac{1}{2·54}$ inch

$\therefore$ the atmospheric pressure on 1 sq. cm.

$=$ the atmospheric pressure on $\frac{1}{(2·54)^2}$ sq. inch $=\frac{15}{(2·54)^2}$ pounds

$=\frac{15}{(2·54)^2\times 2·2}$ Kg. $=\frac{15\times 1000}{(2·54)^2\times 2·2}$ grams $=1056·8=$ grams.

18. A shop-keeper used a kilogram weight in place of a seer weight by mistake in selling sugar. If 1 gram = 15'432 grains and 1 tola = 180 grains, find the percentage of his gain or loss.
[C. U. '45 ; G. U. '52]

1 seer = 80 tolas = 80×180 grains

$$=\frac{80\times180}{15\text{'}432}\text{ grams}=\frac{80\times180}{15\text{'}432\times1000}\text{ kg.}=\frac{14400}{15432}\text{ kg.}$$

$\therefore$ $1\text{ kg.}=\frac{15432}{14400}\text{ seers}=1\frac{129}{1800}\text{ seers}$

$\therefore$ the grocer gives $1\frac{129}{1800}$ seers in place of 1 seer, and so he loses $(1\frac{129}{1800}-1)$ or $\frac{129}{1800}$ seer per $1\frac{129}{1800}$ seers,

$\therefore$ On 100 seers he loses $\frac{129\times1800\times100}{1800\times1929}$ seers or 6'7 seers (App.)

$\therefore$ his loss = 6'7%(App.).

19. If the great wall of China is said to be 2400 Km. long and 7625 mm. thick at the bottom, find to the nearest square feet, the area of the ground it stands upon. [1 m = 39'37 inches]
[P. U. '20 ; D. B. '43]

Here the breadth of the required area is equal to the thickness of the wall.

$\therefore$ length = 2400 Km. = 2400000 m $=\frac{2400000\times39\text{'}37}{12}$ feet

and breadth = 7625 mm. $=\frac{7625}{1000}$ m. $=\frac{7625\times39\text{'}37}{1000\times12}$ feet.

$\therefore$ the reqd. area $=\frac{2400000\times39\text{'}37}{12}\times\frac{7625\times39\text{'}37}{1000\times12}$ sq. ft.

$$=\frac{50\times39\text{'}37\times7625\times39\text{'}37}{3}\text{ sq. ft.}=196\text{'}978773\text{ sq. ft. (App.).}$$

20. The length of a rectangular cistern is 3 times its breadth, and its depth is 3 metres. If it can contain 81000 litres of water, find its length in decimetres.
[C. U. '42 ; E. B. S. B. '51]

The depth of the cistern = 3m. = 300 cm.

The cistern can hold 81000 litres of water, i.e., its volume is 81000 litres or 81000000 cu. cm.

$\therefore$ Length $\times$ breadth $\times$ depth of the cistern $= 81000000$ cu. cm.

$\therefore$ length $\times$ breadth $= \dfrac{81000000 \text{ cu. cm.}}{300 \text{ cm.}} = 270000$ sq. cm.

$\therefore$ 3 breadth $\times$ breadth $= 270000$ sq. cm.

[$\because$ length $= 3 \times$ breadth]

$\therefore$ $(\text{breadth})^2 = 90000$ sq. cm.

$\therefore$ breadth $= \sqrt{90000}$ cm. $= 300$ cm.

$\therefore$ the reqd. length $= 900$ cm. $= 90$ dm.

21. Given that a cubic foot of water weighs 1000 oz. and an inch $= 2{\cdot}54$ cm., find the nearest whole number of grams in one pound. [wt. of 1c.c. of distilled water is 1 gram.]

[C. U. '49]

Weight of 1 cu. cm. of water $= 1$ gram. Here, we have to find how many cubic cms. of water weigh 1 pound.

1 foot $= 12$ inches $= 2{\cdot}54 \times 12$ cm. $= 30{\cdot}48$ cm.

$\therefore$ 1 cu. ft. $= (30{\cdot}48)^3$ cu. cm.

$\because$ Weight of 1 cu. ft. of water $= 1000$ oz. $= \frac{1000}{16}$ lbs. $= \frac{125}{2}$ lbs.

$\therefore$ $\frac{125}{2}$ lbs. $=$ weight of $(30{\cdot}48)^3$ cu.cm. of water $= (30{\cdot}48)^3$ gms.

$\therefore$ 1 lb. $= \dfrac{(30{\cdot}48)^3 \times 2}{125}$ grams $= 453$ grams (App.).

22. The length of a rectangular plot of land is to its breadth as 7 : 3, and its area is $4{\cdot}116$ hectares. Find its length. [D.B.'45]

Here, $\dfrac{\text{length}}{\text{breadth}} = \dfrac{7}{3}$, $\therefore$ breadth $= \dfrac{3}{7} \times$ length.

Now, area $= 4{\cdot}116$ hectares $= 411{\cdot}6$ ares $= 411{\cdot}6$ sq. decametres

i.e., length $\times$ breadth $= 411{\cdot}6$ square decametres,

or, length $\times \frac{3}{7}$ length $= 41160$ square metres

or, $(\text{length})^2 = \frac{41160 \times 7}{3}$ sq. m. $= 96040$ sq. m.

$\therefore$ the reqd. length $= \sqrt{96040}$ m. $= 310$ m. (App.).

23. If a litre is equal to a cubic decimetre and contains 1·76 pints. How many pints will be contained in a tank whose capacity is 22 cubic hectometres ? [G. U. '52]

$$22 \text{ cubic hectometres} = 22\times(100)^3 \text{ cubic metres}$$
$$= 22\times(100)^3\times(10)^3 \text{ cubic decimetres}$$
$$= 22\times(10)^6\times(10)^3 \text{ cu. dm.} = 22\times(10)^9 \text{ cu. dm.}$$

$\because$ 1 cu. dm. contains 1·76 pints,

$\therefore$ $22\times(10)^9$ cu. dm. contain $1{\cdot}76\times(10)^9\times22$

or $176\times22\times(10)^7$ or $3872\times(10)^7$ pints.

24. The weight of a cubic inch of air is 31 grains. Find the weight of 10 litres of air in grams, taking a cubic metre = 35·3 cubic feet and 1 gram = 15·43 grains. [W. B. S. F. '53]

10 litres = 10000 cubic centimetres.

Again, 1 cu. m. = 35·3 cu. ft. = $35{\cdot}3\times(12)^3$ cubic inches.

$\therefore$ $(100)^3$ cu. cm. = $35{\cdot}3\times(12)^3$ cu. inches,

$\therefore$ 1 cu. cm. = $\dfrac{35{\cdot}3\times(12)^3}{1000000}$ cu. inches.

$\therefore$ 10 litres or 10000 cu. cm. = $\dfrac{35{\cdot}3\times(12)^3}{100}$ cu. inches.

$\therefore$ weight of 10 litres of air = wt. of $\dfrac{35{\cdot}3\times(12)^3}{100}$ cu. in. of air

$= \dfrac{35{\cdot}3\times(12)^3}{100}\times31$ grains [$\because$ wt. of 1 cu. in. of air = 31 grains.]

$= \dfrac{353\times1728}{1000}\times\dfrac{31}{15{\cdot}43}$ grams = 1225·5 grams (App.).

Exercise 22

1. The circumference of a wheel is 1 m. 2 dm. 5 cm. How many times will it revolve to go a distance of ·4875 Km. ?

2. If a camp require 2 Kg. 5 Hg. 9 g. 2 dg. of salt at 2 g. 4 cg. per soldier, find the number of soldiers.

3. Find the length of each side of a square whose area is 2 square metres. [C. U. '17]

4. Express 1 g. 6 dg. as the decimal of 2 Kg., correct to 3 places of decimals. [C. U. '29]

5. If 1 metre = 39·37 inches, express 10 feet in centimetres. [C. U. '48]

6. If a metre is 3·2809 feet and the length of a line drawn on the earth from the North pole to the Equator be 10000000 metres, find the circumference of the earth correct to the nearest mile. [C. U. '12]

7. The circumference of the earth is 40000 kilometres, find it in miles. (1 metre = 39·3709 inches). [C. U. '38]

8. Express 5 miles in kilometres and metres correct to the nearest metre. (1 metre = 39·37 inches.) [Pat. U. '19]

9. Given 1 metre = 39·3701 inches, show that 981 cm. = 32 ft. approximately. [E. B. S. B. '50]

10. Assuming that 1 Kg. = 2·2 lbs. and 1 metre = 1·09 yards, find to 3 places of decimals the weight in pound of 100 yards of wire, 1 metre of which weighs 55 grams. [D. B. '46]

11. A room is 20 metres long and 10 metres broad. Find the number of square yards in the area of the floor. (1 metre = 39·37 inches). [C. U. '13]

12. If a metre be equal to 39·37 inches, find in square metres the area of a floor 15 ft. 6 in. long and 14 ft. 2 in. broad, correct to 2 places of decimals. [C. U. '46]

13. Find the cost of surrounding a square field, having an area of 40804 square kilometres, with a fence at 2·5 francs per metre. [G. U. '49]

14. The palace of the king of Babylon contained a thousand rectangular court-yards each 60 metres long and 54 metres broad. The court-yards were all paved with marble slabs, 18 inches long by 18 inches broad. Required the total number of slabs. (1 metre = 39·37 in.) [C. U. '15, '51]

15. Express one litre in cubic inches. (1 m. = 39·3701 inches.)

16. Find the dimensions of a tank which is 2·56 metres deep and which holds 3000 litres, the length of the tank being 3 times the width. [C. U. '18, ; D. B. '14]

17. A tank is 20·5 m. long, 10·2 m. broad and $\frac{1}{2}$ m. deep. How many litres of water does it hold ?

18. Find in grams the weight of the water contained in a cistern measuring 7·5 m. by 3·2 dm. by 10 cm.

19. If glass is 2·5 times as heavy as water, what is the weight in kilograms of a cubic metre of glass ? (1 c. c. of water weighs 1 gram.) [C. U. '34]

20. Mercury is 13·6 times as heavy as water. Find the weight of 525 c.c. of mercury in kilograms. [C. U. '35]

21. Express 0·04375 Kg.+0·3775 g.+0·72 mg. as the decimal of a pound (avoir). [Given 1 gram = 15·432 grains, 1 lb. (avoir.) = 7000 grains.] [C. U. '16]

22. The length of a rectangular plot of land is to its breadth as 5 : 3 and its area is 6·534 hectares. Find its length in metres.

23. The third class railway fare in France is 5 centimes per kilometre and in England 1 d. per mile. Given that 1 yard = ·9144 m. and £ 1 = 25·17 francs, find in English money the difference of the fares for a journey of 100 miles in the two countries, correct to within a farthing. [C. U. '51, D. B. '44]

24. An open tank with a square base contains 28900 litres of water. Find the cost of lining its inner surface with lead at Rs. 5 per square metre, the height of the tank being 2·5 metres. [D. B. '24]

25. The ratio of the length of a rectangular field to its breadth is 3 : 2, and its area is 1109400 square metres. Find the cost of fencing it at 2·5 francs per metre of the boundary. [C. U. '41]

26. The driving wheel of a locomotive is 12·5 metres in circumference, and it makes 2·5 revolutions in a second. How long will it take to travel 100 miles, if 1 mile = 1·6 kilometres ?

27. Given that 1 yard = 0·9144 metres, £ 1 = 25 francs ; find correct to a farthing the difference of the fares for a railway journey of 250 miles in England and in France, if the railway fare be 5 centimes per kilometre in France and 1 pence per mile in England. [C. U. '44]

28. The length of a rectangular cistern is 3 times its breadth, the height being 3 metres. If its capacity be 81000 litres, find its

length and breadth. Find the cost of painting the inside of the four walls at Rs. 200 per are. [G. U. '54]

29. A grocer gave a kilogram of sugar in place of a seer by mistake. If 1 seer $= \cdot 9\dot{3}$ kilogram, find his gain or loss per cent.

30. Find the cost of surrounding a square field having an area of $432\frac{18}{25}$ square kilometres with a fence at the rate of Rs. 18. 75 P. per 100 feet ? [1 metre $= 39 \cdot 37$ inches]

APPROXIMATION

Approximate Value. In our daily transactions it is often inconvenient and not always possible to find the exact value of a quantity or to ascertain accurate values, lengths, weights etc. of a thing. So in calculations involving values, lengths, weights etc. and to carry on our work in such cases, we have to take the approximate values just a little above or a little below the accurate values. This is called the approximate value of a quantity or the nearest value or measurement of a thing.

Suppose the sum of Rs. 27 is to be equally divided among 11 boys. Each boy will then get (Rs. $27 \div 11$) or Rs. $2.45\frac{5}{11}$ P. Now, how is this amount to be given to each boy ? As the current coin of minimum value is a Paisa coin, it is not possible to actually pay $\frac{5}{11}$ P. Now, if every boy be given Rs. 2.45 P., he will get $\frac{5}{11}$ P. less than his actual dues. Again, if each boy be given Rs. 2.46 P., he will get (6 P. $-$ $5\frac{5}{11}$ P.) or $\frac{6}{11}$ P. more. Neither Rs. 2.45 P. nor Rs. 2.46 P. are the real dues of each boy. Hence there will be an error in either payment. The error in the first case is less than that in the second case. So if each boy be given Rs. 2.45 P., he will be given the nearest or approximate value of his dues, correct to a paisa. Thus, Rs. 2 will be the approximate value of each boy's dues, correct to a rupee. The less is the amount of error, the greater is the correctness of the result.

Rule : If the approximate value of a quantity to a specific unit is to be found instead of its true value, the fraction of its last unit should be rejected, if it be less than $\frac{1}{2}$, and 1 should be added to its last unit, if the fraction be equal to or more than $\frac{1}{2}$.

Approximate value of an integral number.

If 5000 be written for the value of the number 5832, it is 832 less than the true value. If 6000 be written for 5832, it is (6000 − 5832) or 168 more than the true value. So the difference of 6000 from 5832 is less than the difference of 5000 from 5832.

∴ the approximate value of 5832 correct to the nearest thousand is 6000. Thus, the approximate value of 5832 correct to the nearest hundred is 5800 and correct to the nearest ten is 5830.

Approximate value of a decimal fraction.

If we are to find the approximate value of a decimal fraction to a certain place of decimals we should write all the digits up to that place and reject the following digits, but we should increase the last digit retained by 1 if the first digit rejected be 5 or greater than 5.

Thus, 3·65038=4, correct to the integer (here the first digit rejected after the units' place is 6 which is greater than 5 ; so 1 is added to the whole number 3 to make the approximate value 4.).

Similarly, 3·65038=3·7, correct to one place of decimals
=3·65 correct to two places of decimals
=3·650 „ „ three „ „
=3·6504 „ „ four „ „

[N.B. (1) The approximate value and the value correct to a few specific places of decimals are the same.

'To a certain decimal place' and 'Correct to a certain decimal place' are not the same.

Thus, 1·4068=1·406 to 3 places of decimals,
=1·407 correct to 3 places of decimals.

Here, the first case does not give the approximate value, whereas the second does.

(2) The value of 1·345 correct to 2 places of decimals may be both 1·35 and 1·34, the first being (1·35 − 1·345) or ·005 more than the true value and the second being (1·345 − 1·34) or ·005 less than the true value. But as per convention its value correct to 2 places of decimals should be taken as 1·35.

Significant figures. A number is formed with the digits from 1 to 9. These digits are called significant figures. One or more ciphers (zeroes) in between two significant figures are also significant. If a decimal fraction begins with a decimal point with one or more zeroes after it, then the zeroes are not significant figures, but only the figures following them are significant. The zeroes at the end of a whole number or a decimal fraction may or may not be significant according to the degree of accuracy implied.

Examples : (1) 30·23046=30·23, correct to 4 significant figures. But it is equal to 30·2305 correct to 4 places of decimals.

(2) ·12065=·1207, correct to 4 places of decimals or correct to 4 significant figures.

(3) ·00147=·001, correct to one significant figure
=·0015, correct to 2 significant figures.

(4) ·000240079=·00024, correct to 5 places of decimals
=·00024008, correct to 5 significant figures.

(5) ·44598=·4460, correct to 4 decimal places as well as correct to 4 significant figures.

(Here 0 is a significant figure ; the digit rejected being 8, 1 is added to 9, the last digit of ·4459 so as to obtain ·4460 as the approximate result.)

(6 163289=163300, correct to 4 significant figures (here the two zeroes are not significant figures.)

(7) 705769=706000, correct to the nearest thousand
=705800, correct to the nearest hundred.

(In both cases the last zeroes are not significant figures.)

Error : It is an error if we take any value other than the true value. In practical field it is desirable that the magnitude of this error should be as little as possible. Suppose, I roughly weigh a maund of coal : If I weigh it more minutely, perhaps its weight may be $39\frac{3}{4}$ seers. The loss in this case may be negligible as the cost of $\frac{1}{4}$ seer of coal is negligible ; but if it be gold instead of coal the loss would be too much to be negligible.

∴ the goldsmith is bound to weigh his gold more minutely.

Absolute Error, Relative Error, Percentage Error.

The **absolute error** in an approximation is the actual difference between the approximate value and the true value of a quantity, i.e., Absolute error = True value $\sim$ approximate value,

The **relative error** in an approximation is the ratio of the absolute error and the true value of a quantity, i.e.,

$$\text{Relative error} = \frac{\text{absolute error}}{\text{true value}}.$$

The **percentage error** is the percentage of the absolute error with respect to the true value. So the percentage error

$$= \frac{\text{absolute error} \times 100}{\text{true value}} = \text{relative error} \times 100.$$

Examples [23]

1. Express Rs. 4325, correct to the nearest hundred rupees. Also find its absolute error, relative error and percentage error.

Rs. 4325 = Rs. 4300, correct to the nearest hundred rupees.

$\therefore$ the absolute error = Rs. 4325 − Rs. 4300 = Rs. 25.

$$\text{The relative error} = \frac{\text{Rs. } 25}{\text{Rs. } 4325} = \frac{1}{173},$$

and the percentage error = relative error $\times$ 100

$= \frac{1}{173} \times 100 = {}^{\cdot}58$ (correct to 2 places of decimals).

2. Find the approximate value of 3·068 correct to first two significant figures and hence find the absolute error.

The first two significant figures in the given number are 3 and 0.

$\therefore$ the approximate value, correct to first two significant figures = 3·1. $\therefore$ the absolute error $= 3{\cdot}068 \sim 3{\cdot}1 = {}^{\cdot}032$.

3. A person measures a path $12\frac{7}{200}$ metres long and takes it for 12·01 metres. What are the absolute error, relative error and percentage error ?

$12\frac{7}{200} = 12{\cdot}035$. $\therefore$ Absolute error = 12·035 m. − 12·01 m. = ·025 m.

$$\text{Relative error} = \frac{{}^{\cdot}025 \text{ metre}}{12{\cdot}035 \text{ metres}} = \frac{5}{2407}.$$

and percentage error $= \frac{5}{2407} \times 100$

$= \frac{500}{2407} = {}^{\cdot}21$ (correct to 2 places of decimals.)

Approximate Summation and Subtraction of decimals.

Rule. To find the approximate value of the sum or difference of decimals to a certain place of decimals we should correctly know the first digit to be rejected in the result.

Thus to find an approximate answer, correct to a certain decimal place, the results to two more places of decimals should first be found.

4. Find the sum of 4·3074, ·0028391 and $9{\cdot}36\dot{4}$ correct to 5 decimal places.

4·30740	
·00283	91
9·36464	64
13·67488	55

∴ the required sum =13·67489.

Here, to find the sum correct to 5 decimal places, we should know correctly the digit in the 6th decimal place of the sum. ∴ the numbers are added taking one digit more *i.e.* taking digits up to 7 places of decimals.

5. Find the difference between $30{\cdot}4\dot{0}\dot{6}$ and $23{\cdot}1\dot{3}\dot{9}$ correct to 4 places of decimals.

30·4064	06
23·1393	93
7·2670	13

∴ the required difference = 7·2670.

6. Find the sum of $7{\cdot}\dot{3}0\dot{2}$, ·0865, $32{\cdot}\dot{8}\dot{7}$ and 3·02 to 3 significant figures.

32·8	78
7·3	02
·0	86
3·0	2
43·2	86

∴ the required sum = 43·3.

[N. B. Approximation concerning multiplication, division and summation of series being out of syllabus, is not dealt with here.]

Exercise 20

Express the following numbers correct to the nearest thousand and hundred :

1. 7432 2. 9672 3. 3726

4. Find the value of 3·74036 correct to 2, 3, 4 decimal places.

5. Express 7·562 correct to the nearest whole number.

Find the approximate value of the following, correct to 3 significant figures :

6. 3·265 7. 5·072 8. 72·083 9. ·007876.

10. Express the following numbers correct to 2 decimal places and find the absolute error, relative error and percentage error in each case :

(1) ·3216 (2) 2·447 (3) ·0269

11. Express the following numbers correct to 3 significant figures and find the absolute error, relative error and percentage error in each case :

(i) ·3276 (ii) ·02145 (iii) 2·034

12. One boy gives 17·9 and another boy 17·8 as the product of 3·8 and 4·7. What is the percentage error in each case ?

13. One person makes an error of Rs. 2 in counting Rs. 370 and another makes an error of Rs. 5 in counting Rs. 1110. Find the ratio of their relative errors.

Find the sum correct to 4 decimal places and 4 significant figures :

14. $32{\cdot}036+{\cdot}7\dot{2}\dot{8}+5{\cdot}035213$ 15. ${\cdot}004872+13{\cdot}72\dot{5}+{\cdot}3\dot{6}$

16. $5{\cdot}2167+1{\cdot}02\dot{3}\dot{5}+12{\cdot}\dot{5}0\dot{7}$

Find the difference correct to 3 decimal places and correct to 3 significant figures :

17. $95{\cdot}3064,\ 90{\cdot}7\dot{6}$ 18. ${\cdot}78205,\ {\cdot}83\dot{1}\dot{2}$ 19. $7{\cdot}5\dot{3}1\dot{2}-2{\cdot}04\dot{6}$

20. What is the percentage error, if Rs. 5, 5 as. be paid in place of Rs. $5{\cdot}\dot{3}$

COMPOUND INTEREST

The interest is called **Compound Interest**, when at the end of an assigned period (say a year, or 6 months or 3 months) the interest on the sum lent is added to the principal and the whole (*i.e.*, the amount) bears interest at the same rate for another equal period and so on.

In such cases the principal increases from period to period and the interest for each period is the interest on the amount at the end of each preceding period.

For example, suppose Rs. 100 is lent at 4% compound interest, the interest being payable yearly. If the interest of Rs. 4 for the first year be not paid, the interest for the second year will be charged on Rs. (100+4) or Rs. 104 at 4% and not on the original principal of Rs. 100. The interest thus calculated is called the compound interest.

In simple interest the capital remains always the same.

Amount at Compound Interest

The final amount is obtained by adding the compound interest at the end of the given period with the original principal.

The method of finding compound Interest

(1) If Rs. 350 is lent at 3%, the interest at the end of 1 year $=\text{Rs. }\frac{350\times3}{100}=\text{Rs. }\frac{1050}{100}=\text{Rs. }10{\cdot}5$.

So it is seen that the interest is obtained by multiplying the principal by the rate of interest and then dividing the product by 100, *i.e.*, by simply shifting the decimal point two places to the left in the product.

(2) If the rate of interest be a mixed fraction (such as $3\frac{1}{2}\%$, $4\frac{3}{4}\%$ etc.) the compound interest is to be obtained by means of aliquot parts.

(3) If the time be a mixed fraction (such as, $2\frac{1}{2}$ or $3\frac{1}{3}$ years or 4 years 3 months etc.), first find the compound interest for the whole number of years and then the simple interest on the new principal for the fractional year. The sum of this simple interest and the previous compound interest will be the required compound interest.

(4) If the principal be a compound quantity, it should be reduced to rupees or pounds and expressed in decimals.

(5) If the interest be payable half-yearly, quarterly or after every 4 months, it should be so mentioned. The interest should be taken to be payable yearly, if there be no mention of the period. If the interest be payable half-yearly, find the interest for every 6 months according to the previous method and add it to the preceding capital.

Or. Suppose the interest is due after every 6 months and we are to find the compound interest for 2 years at the rate of 4%. If the interest is payable half-yearly the result may be obtained by finding the interest for double the number of years at half the rate per cent. Here, the required compound interest will be equal to that for 4 years at the rate of 2% per year.

Thus, if the interest be payable after every 4 months the required compound interest at the end of a period will be equal to that for thrice the number of years at $\frac{1}{3}$ of the rate per cent. ($\because$ 4 months $=\frac{1}{3}$ year).

Note the following examples very carefully.

Example [24]

1. Find the compound interest on Rs. 1000 for 3 years at 5% per annum.

Rs. 1000 = Principal for the 1st year.
×5
Rs. 50·$\not 0\not 0$ = Interest for the 1st year.
Rs. 1000
Rs. 1050 = Principal for the 2nd year.
×5
Rs. 52·5$\not 0$ = Interest for the 2nd year.
Rs. 1050
Rs. 1102·5 = Principal for the 3rd year.
×5
Rs. 55·125 = Interest for the 3rd year.
Rs. 1102·5
Rs. 1157·625 = Amount at the end of the 3rd year.

$\therefore$ the required compound interest
= Rs. 1157·625 − Rs. 1000 = Rs. 157·625 = Rs. 157. 62·5 P.

[N. B. Multiplying the principal Rs. 1000 by the rate of interest 5 we get Rs. 5000. Placing the decimal point after two figures from the right to the left (*i.e.*, dividing by 100) we get Rs. 50·00 or Rs. 50, which is the interest for the 1st year. Adding this interest to the first principal Rs. 1000 we get Rs. 1050 as the principal for the 2nd year. Again multiplying Rs. 1050 by 5 and dividing the product as before by 100 we get Rs. 52·50 or Rs. 52·5 as the interest for the second year. It is added to Rs. 1050, the principal for the 2nd year and we get Rs. 1102·5 as the principal for the 3rd year. Thus, we get Rs. 1157·625 for the

final amount at the end of the third year. Now we subtract the original principal from this amount and get Rs. 157·625 as the compound interest for 3 years.]

2. Find the amount at compound interest on £550 for 2 years at $3\frac{3}{4}$ per cent per annum (correct to the nearest pence.).

[Here $3\frac{3}{4}\% = 3 + \frac{3}{4} = 3 + \frac{1}{4}$ of 3]

£550	= Principal for the 1st year,
£16·5	= Interest at 3% $= \left(\frac{550 \times 3}{100}\right)$
£4·125	= Interest at $\frac{3}{4}\%$ (= 16·5 ÷ 4)
£570·625	= Principal for the 2nd year.
17·11875	= Interest at 3%
4·2796875	= Interest at $\frac{3}{4}\%$
£592·0234375	= Amount at the end of 2 years
20	
·46875*s*.	[= ·0234375 × 20*s*.]
12	
5·625*d*.	

∴ the amount at compound interest = £592.6*d*. (App.).

3. Find the compound interest on Rs. 350. 50 P. for 2 years 3 months at $3\frac{1}{2}\%$ per annum, the interest being payable yearly.

Rs. 350. 50P. = Rs. 350·5, $3\frac{1}{2} = 3 + 3 \div 6$; 2 yrs. 3 months $= 2\frac{1}{4}$yrs.

Rs. 350·5	= Principal for the 1st year.
10·515	= Interest at 3% $\left[= \frac{350{\cdot}5 \times 3}{100}\right]$
1·7525	= Interest at $\frac{1}{2}\%$ [= 10·515 ÷ 6]
362·7675	= Principal for the second year
10·8830...	= Interest at 3% $\left[= \frac{\text{2nd principal} \times 3}{100}\right.$ to 4 decimal paces]
1·8138...	= Interest at $\frac{1}{2}\%$ [= 10·8830 ÷ 6]
375·4643	= Principal for the third year
2·8159	= Interest at 3% for $\frac{1}{4}$ year [Interest at 3% for 1 year ÷ 4]
·4693	= Interest at $\frac{1}{2}\%$ for $\frac{1}{4}$ year [= 2·8159 ÷ 6]
378·7495	= Amount at compound interest for $2\frac{1}{4}$ years.
350·5	= 1st principal
(sub.)Rs. 28·2495	= the compound interest.

∴ the required compound interest = Rs. 28.25 P. (approx.).

4. Find the compound interest on Rs. 500 for $1\frac{1}{2}$ years at 2% per annum, the interest being payable half-yearly.

Here the interest is payable half-yearly.

$\therefore$ the required compound interest will be equal to that for twice the time (*i. e.* 3 years) at half the rate *i. e.* 1% (the interest being payable yearly).

Rs. 500 = Principal for the 1st year.
5 = Interest " " " " (at 1%)

505 = Principal for the 2nd year.
5'05 = Interest for the 2nd year $(\frac{505\times1}{100})$

510'05 = Principal for the 3rd year.
5'1005 = Interest for the 3rd year

515'1505 = Amount at the end of the 3rd year.
500 = The first principal

Rs. 15'1505 = Compound interest.

$\therefore$ the required compound interest=Rs. 15.15 P. (approx.).

[N.B. $1\frac{1}{2}$ years contain 3 six-months and the interest for 6 months at 2% per annum is Re. 1. $\therefore$ the interest for 3 years at 1% payable yearly is the same as the interest for $1\frac{1}{2}$ years at 2%, the interest being payable half-yearly].

5. The difference between the simple interest and the compound interest on a sum of money for 2 years at 4% per annum is 80 P. Find the sum.

The simple interest on Rs. 100 for 2 years at 4%=Rs. 4×2=Rs. 8.

Again, Rs. 100 = First principal
Rs. 4 = Interest for the 1st year

Rs. 104 = The 2nd principal
Rs. 4'16 = Interest for the 2nd year

Rs. 108'16 = Amount at the end of two years.
Rs. 100 = 1st principal

(Subtracting) Rs. 8'16 = Compound interest

$\therefore$ Compound interest − simple interest
= Rs. 8'16 − Rs. 8 = Re. '16 = 16 P.

The difference is 16 P. when the principal is Rs. 100

$\therefore$ „ „ 1 P. „ „ „ Rs. $\frac{100}{16}$

„ „ 80 P. „ „ „ Rs. $\frac{100}{16}\times 80$

or Rs. 500

$\therefore$ the required principal = Rs. 500.

[N.B. In such cases first find out the simple interest and the compound interest on Rs. 100.]

General formula for finding the compound interest.

Suppose, Re. 1 is lent for 2 years at 4% per annum compound interest. Interest on Re. 1 for 1 year at 4% = Rs. $\frac{4}{100}$.

$\therefore$ after 1 year Re. 1 will amount to Rs. $(1+\frac{4}{100})$

Now, let us see what the amount will be at the end of the 2nd year.

The principal for the 2nd year is Rs. $(1+\frac{4}{100})$

Accordiug to the previous calculation if the principal be Re. 1, the amount is Rs. $(1+\frac{4}{100})$ for the 2nd year.

$\therefore$ if the principal be Rs. $(1+\frac{4}{100})$, the amount is Rs. $(1+\frac{4}{100})\times(1+\frac{4}{100})$ or Rs. $(1+\frac{4}{100})^2$ at the end of the 2nd year.

$\therefore$ briefly the general formula is :

$$\text{Amount} = \text{Principal}\times\left(1+\frac{\text{rate of interest}}{100}\right)^{\text{number of years}}.$$

or, $A=P\left(1+\dfrac{r}{100}\right)^y$. $\left[\begin{array}{l}\text{Here, } A=\text{amount, } P=\text{principal,}\\ r=\text{rate p.c., } y=\text{number of yrs.}\end{array}\right]$

Miscellaneous problems on compound interest.

6. Prove that the amount is 1·0609 times the principal in 2 years at 3% per annum compound interest.

Amount at compound interest

$$= \text{Principal}\times\left(1+\frac{\text{rate of interest}}{100}\right)^{\text{number of years}}$$

$= \text{Principal}\times(1+\frac{3}{100})^2 = \text{Principal}\times(1+{\cdot}03)^2$

$= \text{Principal}\times(1{\cdot}03)^2 = \text{Principal}\times 1{\cdot}0609.$

[N. B. Taking Re. 1 as the principal you may work out this sum according to the Example 2.]

7. The population of a town increases 10% at the end of each year. If the population be 399300 at the end of 3 years, what was the original population ?

Here, rate of increase=10%,

the amount=total number with the increase=399300,

number of years=3, Principal=Original population.

$\therefore$ by the formula, Principal $\times(1+\frac{10}{100})^3$
=Amount at compound interest,

we have original population $\times(1\cdot1)^3=399300$,

$\therefore$ the required original population $=\dfrac{399300}{(1\cdot1)^3}$

$=\frac{399300\times10\times10\times10}{11\times11\times11}=300000.$

8. What sum will amount to Rs. 49691. 87·5 P. in 3 years at $7\frac{1}{2}\%$ per annum compound interest ?

Amount at compound interest=Rs. 49691. 87·5 P.

=Rs. 49691·875, rate of interest $=7\frac{1}{2}\%=7\cdot5\%$.

By the formula,

$$\text{Amount at compound interest}=\text{Principal}\left(1+\frac{\text{rate of interest}}{100}\right)^{\text{yrs.}}$$

$$\therefore\quad 49691\cdot875=\text{Principal}\times\left(1+\frac{7\cdot5}{100}\right)^3=\text{Principal}\times(1\cdot075)^3$$

$$\therefore\quad \text{the required principal}=\text{Rs. }\frac{49691\cdot875}{(1\cdot075)^3}=\text{Rs. }40000.$$

[Here the value of $(1\cdot075)^3$ and then the division of the numerator by the denominator should be worked out by the student.]

9. A sum of money is lent at 8% per annum compound interest. If the interest for the second year exceeds that for the first year by Rs. 32, find the original principal. [C. U. '45]

[**First method**], Suppose, the principal=Rs. 100.

Rs.	100	=First principal,
Rs.	8	=Interest for the 1st year,
Rs.	108	=The second principal
Rs.	8·64	=Interest for the 2nd year.

$\therefore$ the difference of interests for the 2nd and 1st year
=Rs. $(8\cdot64-8)$=Rs. ·64=Rs. $\frac{64}{100}$.

Rs. $\frac{64}{100}$ is the difference of interests on the principal Rs. 100,
Rs. 32 " " " Rs. $\frac{100\times100\times32}{64}$ or Rs. 5000

$\therefore$ the original principal=Rs. 5000.

[Second method] In the first year the interest is calculated only on the original principal ; but in the second year the interest =the interest on the original principal+the interest on the interest of the first year.

$\therefore$ Rs. 32 is equal to the interest on the interest of the first year. Rs. 32 is the interest for 1 year at 8% per annum on the principal Rs. 400.

$\therefore$ the interest on the reqd. principal for the first year at 8% =Rs. 400.

$\therefore$ the required principal=Rs. $(\frac{100}{8}\times 400)$=Rs. 5000.

10. A sum of money lent at compound interest amounts to Rs. 525 in 1 year and to Rs. 551. 25 P. in 2 years. Find the sum.

Here it is evident that Rs. 525 amounts to Rs. 551. 25 P. in 1 year.

$\therefore$ when the amount is Rs. $551\frac{1}{4}$, the principal =Rs. 525

$\therefore$ " " Re. 1 " =Rs. $\frac{525\times 4}{2205}$

$\therefore$ " " Rs. 525 " =Rs. $\frac{525\times 4\times 525}{2205}$

=Rs. 500.

11. The simple interest on a certain sum for one year is £80 and the compound interest on the said sum for two years is £164. Find the sum and the rate of interest. [U. P. '22]

The interest on the sum for 1 year=£80,

$\therefore$ the interest for the second year=£(164 − 80)=£84

$\therefore$ £80 (the interest on the sum for 1 year)
+interest on £80 for 1 year=£84

$\therefore$ interest on £80 for 1 year=£84 − £80=£4.

$\therefore$ " " £100 " " =£$(\frac{4}{80}\times 100)$=£5.

Again, if £5 be the interest for 1 year, the principal=£100,

$\therefore$ " £80 " " " " " " =£$(\frac{100}{5}\times 80)$=£1600.

$\therefore$ the required principal=£1600, and rate of interest=5%.

12. A money lender borrows a certain sum of money at 3% simple interest, and invests the same at 5% compound interest (compounded annually). After 3 years he makes a profit of Rs. 541. Find the amount borrowed by him. [C. U. '50]

Suppose, the man borrowed Rs. 100.

Simple interest on Rs 100 for 3 years at 3%=Rs. 3×3=Rs. 9.

Now, the compound interest on Rs. 100 for 3 years at 3% is to be found.

Rs. 100 = 1st principal
Rs. 5 = Interest for the 1st year

Rs. 105 = 2nd principal
Rs. 5·25 = Interest for the 2nd year

Rs. 110·25 = Principal for the 3rd year
Rs. 5·5125 = Interest for the 3rd year

Rs. 115·7625 = Amount at compound interest at the end of the 3rd year.

Rs. 100 = 1st principal

Rs. 15·7625 = compound interest for 3 years.

∴ profit = Rs. (15·7625 − 9) = Rs. 6·7625

If the profit be Rs. 6·7625, the principal = Rs. 100,

∴ „ „ Rs. 541 „ „ = Rs. $\frac{100 \times 541}{6 \cdot 7625}$ = Rs. 8000.

∴ the man borrowed Rs 8000.

13. A man saves Rs. 200 at the end of each year and lends the money at 5% compound interest. How much will he be worth at the end of 3 years?

Rs. 200 = Savings at the end of the 1st year, *i.e.*
= Principal for the 2nd year.
Rs. 10 = Interest for the 2nd year.

Rs. 210 = Amount at the end of the 2nd year.
Rs. 200 = Savings at the end of the 2nd year.

Rs. 410 = Principal of the 3rd year.
Rs. 20·5 = Interest for the 3rd year.

Rs. 430·5 = Amount at the end of 3rd year.
Rs. 200 = Savings at the end of the 3rd year.

Rs. 630·5 = total amount at the end of the 3rd year.

∴ the man will be worth Rs. 630. 50 P. in 3 yrs.

14. A person borrowed Rs. 3320 for 2 years at $7\frac{1}{2}$ % compound interest. 'If he wants to pay up the debt by two equal payments at the end of the first and the second year respectively, what should be his annual payment ? [C. U. '40]

[First method] Suppose, Re. 1 is payable at the end of each year.

Amount at compound interest on Re. 1 at the end of the first year=Rs. $\frac{107\frac{1}{2}}{100}$=Rs. $\frac{43}{40}$.

$\therefore$ If Rs. $\frac{43}{40}$ be payable at the end of 1 year, the principal=Re. 1.

$\therefore$ If Re. 1 be payable at the end of the first year, the principal=$\frac{40}{43}$.

Again, the amount on Re. 1 at the end of the 2nd year=Rs. $(\frac{43}{40})^2$.

$\therefore$ If Rs $(\frac{43}{40})^2$ be payable at the end of 2nd year, the principal=Re. 1

$\therefore$,, Re 1 ,, ,, ,, ,, ,, ,, =Rs.$(\frac{40}{43})^2$

$\therefore$ if the total principal be $\{\frac{40}{43}+(\frac{40}{43})^2\}$ or Rs. $\frac{83\times40}{43\times43}$, the money payable at the end of each year=Re. 1,

$\therefore$ if the principal be Rs. 3320, the money payable at the end of each year=Rs. $\frac{1\times43\times3320\times43}{83\times40}$=Rs. (43×43)=Rs. 1849

$\therefore$ the required annual payment=Rs. 1849.

[Alternative method] Suppose, the person has paid x rupees at the end of each year. The interest on Rs. 3320 in the first year=Re. $\frac{15\times3320}{2\times100}$=Rs. 249

$\therefore$ Amount at the end of the first year=Rs. (3320+249) =Rs. 3569.

Of this amount, the sum of x rupees is paid and there remain Rs. $(3569-x)$ as debt unpaid. At the end of the second year if the principal be Rs. $(3569-x)$, its amount at compound interest will be $\frac{107\frac{1}{2}}{100}(3569-x)$ or $\frac{43}{40}(3569-x)$ rupees, on payment of which at the end of the second year the loan will be repaid ;

$\therefore$ $x=\frac{43}{40}(3569-x)$, or, $\frac{40}{43}x=3569-x$, or, $\frac{40}{43}x+x=3569$, or, $\frac{83}{43}x=3569$, or, $x=\frac{3569\times43}{83}=1849$

$\therefore$ Rs. 1849 will be payable at the end of each year.

15. A sum of money is borrowed and paid back in two equal annual instalments of Rs. 882 each, allowing 5% compound interest (interest being added yearly). What was the sum borrowed? [W. B. S. F. '52]

$\because$ the loan is repaid on payment of Rs. 882 at the end of the second year, $\therefore$ the debt at the beginning of the second year amounts to Rs. 882 in 1 year at 5%

$\therefore$ the debt at the beginning of the second year

$=\text{Rs.}\ (\frac{100}{105}\times 882)=\text{Rs.}\ 840.$

$\therefore$ This loan remained unpaid after paying Rs. 882 at the end of the first year.

$\therefore$ the loan at the end of the first year amounts to Rs. (840+882) or Rs. 1722.

$\therefore$ the loan at the beginning of the first year $=\text{Rs.}\ (\frac{100}{105}\times 1722)$ or Rs. 1640. $\therefore$ the reqd. debt=Rs. 1640.

16. If the difference between the simple interest and the compound interest on a certain sum of money for 2 years at 5% per annum is Rs. 12.50 P., find the sum.

Simple interest on Rs. 100 for 2 years at 5%=Rs. 10.

Again, Rs.	100	=Principal of the first year.
Rs.	5	=Interest for the first year,
Rs.	105	=Principal of the second year.
Rs.	5·25	=Interest for the second year.
Rs.	110·25	=Amount at compound interest for 2 years.

$\therefore$ Compound interest=Rs. (110·25 − 100)=Rs. 10·25=Rs. $10\frac{1}{4}$

$\therefore$ the difference of the two interests=Rs. $10\frac{1}{4}$ − Rs. 10=Re. $\frac{1}{4}$.

If Re. $\frac{1}{4}$ is the difference of two interests, the principal=Rs. 100,

$\therefore$ if Re 1 " " " =Rs. 400,

$\therefore$ if Rs. $12\frac{1}{2}$ " " " Rs. $\frac{400\times 25}{2}$=Rs. 5000.

$\therefore$ The reqd. sum=Rs. 5000.

17. A sum of money lent at compound interest amounts to Rs. 1102. 8 as. in 2 years and to Rs. 1157. 10 as. in 3 years. Find the sum and the rate of interest.

Here it is evident that in the third year interest has been calculated on Rs. 1102. 8 as.

$\therefore$ interest on Rs. $1102\frac{1}{2}$ for 1 year

$=$(Rs. 1157. 10as. $-$ Rs. 1102. 8as.)$=$Rs. 55. 2as.$=$Rs. $\frac{441}{8}$.

$\therefore$ interest on Re. 1 for 1 year$=$Rs. $\frac{441\times 2}{8\times 2205}$$=$Re. $\frac{1}{20}$

$\therefore$ interest on Rs. 100 for 1 year$=$Rs. $\frac{1}{20}\times 100$$=$Rs. 5

$\therefore$ the required rate of interest$=5\%$.

Now, by the formula, Rs. $1102\frac{1}{2}$$=$Principal$\times(1+\frac{5}{100})^2$,

or, Rs. $1102{\cdot}5$$=$Principal $\times(1{\cdot}05)^2$

$\therefore$ the principal$\times 1{\cdot}1025$$=$Rs. $1102{\cdot}5$

$\therefore$ the required principal$=$Rs. $\frac{1102{\cdot}5}{1{\cdot}1025}$$=$Rs. 1000.

Exercise 24

Find the compound interest on—

1. Rs. 10000 at 5% for 3 years. [G.U. '49 ; C.U. '40]
2. Rs. 625 at 4% for 2 years. [B. U.]
3. £500 at 5% for 3 years. [C. U. '49]
4. Rs. 5000 for 2 years at $4\frac{1}{2}\%$
5. Rs. 526. 60 P. at 3% for 3 years.
6. Rs. 450 at 4% for $2\frac{1}{2}$ years.
7. £375. 10s. at $3\frac{3}{4}\%$ for 2 years 4 months.

Find the amount of (the interest being compounded annually)

8. £1000 at 5% in 6 years. [B.U.]
9. Rs. 500000 at 6% in 3 years [C. U. '43]
10. Rs. 740 at 5% in 2 years.
11. Rs. 1750 in 3 years at $3\frac{1}{2}\%$
12. Rs. $450{\cdot}5$ at 4% in 2 yrs. 6 months.
13. Find the difference between the compound interest and the simple interest on Rs. 2000 for 3 years at 5%. [C. U. '46]
14. What sum will amount to Rs. 3528 in 2 years at 5% compound interest ?
15. *A* invested Rs. 5000 at 5% compound interest (compounded annually) and *B* invested an equal sum at $5\frac{1}{4}\%$ per annum simple interest. Who was the gainer at the end of 3 years and by how much ? [G. U. '53]
16. Find the compound interest on Rs. 500 for $1\frac{1}{2}$ years at 4%, payable half-yearly.

17. What will be the amount from Rs. 400 in 1 year 4 months at 12% payable at the end of every 4 months ?

18. Find the compound interest on Rs. 450 for 9 months at 8% per annum payable quarterly.

19. Find the difference between the simple and the compound interest on Rs. 400 for 3 years at 5%. [P. U. '20]

20. The difference between the simple and the compound interest on a certain sum of money for 2 years at 4% is Rs. 20. Find the sum. [P. U. '12]

21. The difference between the simple interst and the compound interest on a certain sum of money for 3 years at 5% is Rs. 76. 25P. Find the sum. [G. U. '50]

22. The difference of compound interest and the simple interest on a sum of money for 2 years at 6% per annum is Rs. 13. 50 P. ; find the sum.

23. Prove that the amount at compound interest for 2 years at 4% is 1·0816 times the principal.

24. The simple interest on Rs. 10000 for 3 years amounts to Rs. 1500, find the compound interest on the sum for 3 years.

25. What will be the compound interest on Rs. 1000 at 5% for the time in which it produces Rs. 150 as simple interest at the same rate ?

26. A sum was lent at 4% compound interest. If the interest of the second year exceeds that of the first year by Re. 1, find the sum.

27. What sum at compound interest will amount to Rs. 650 at the end of the first year and to Rs, 676 at the end of the second year ? [A. U.]

28. The population of a town is 24000, and its annual increase is 5% ; what will be the population at the end of 3 years ? [C. U. (High) '50]

29. The population of a town increases 4% in every 10 years. If its present population be 100000, what will it be 20 years hence ?

30. If the simple and the compound interest on a sum of money for 2 years be Rs. 40 and Rs. 41 respectively, find the sum and the rate per cent.

31. The simple interest on a sum of money for 1 year is Rs. 50 and the compound interest on it for 2 years is Rs. 102. Find the sum and the rate per cent.

32. A man borrows Rs. 4100 for 2 years at 5% compound interest. If he wants to clear the debt by two equal annual payments, how much has he to pay at the end of each year ?

33. A man deposits Rs. 100 at the end of each year in a bank, paying 10% interesi compounded annually. How much will he get from the bank at the end of 4 years ?

34. Rs. 10300 is lent for 2 years at 6% compound interest per annum. If the borrower prefers to pay money back by two equal annual payments, how much should he pay at the end of each year ? [C. U. '43]

35. A money lender borrows Rs. 5000 at 4% per annum and pays the interest at the end of the year. He lends it at 6% per annum payable half-yearly and realises the interest at the end of the year. How much does he gain a year by this means ?

BILL OF EXCHANGE

In wholesale trade when a merchant sells his goods to another merchant, the latter instead of paying the price in cash, generally gives the former a promise in writing to pay the price after some stipulated time, *i.e.*, on some future date. The man who sells goods on credit is called a **Creditor** and the man who purchases goods on credit is called a **Debtor.**

23. Bill of Exchange.

The creditor sells goods to the debtor on credit and usually draws up a *bill of exchange* demanding the debtor to pay unconditionally a certain sum of money at the end of a fixed period to himself or order (*i.e.*, to the bearer of the bill or to another person as directed by him) for the value of the goods. The Bill is then forwarded to the debtor for his acceptance. The debtor writes the word '*accepted*' on it and puts in his signature under the word. He then returns it to the creditor.

The debtor becomes legally liable for payment of the money only after this *acceptance*.

This bill is drawn up on a stamped paper, the value of the stamp depending on the amount of the Bill.

To write a bill in the above way is called '*drawing of the bill*'. He who writes the bill (*i.e.*, the creditor or the seller of goods), is called the '*Drawer of the Bill*'. He may also be called 'Sender of the bill or Hundi'.

Again, the person who is ordered to make payment by the drawer of the Bill (*i.e.*, the debtor or buyer of goods) is called the '*Drawee*'.

Now you should notice the following :—

(1) It is advantageous to make transactions of money in a business by means of the Bill or Hundi or Draft.

(2) The drawer of the Bill puts his own signature in the Bill.

(3) The drawee sends back the bill to its drawer after writing the word '*accepted*' on the bill over his own signature.

(4) The preson who is ordered to receive payment of the money written in the bill from the drawee is called a '*Payee*'.

(5) The time at the end of which the money of the bill is to be paid is called the *term* of the bill.

(6) The amount of money is written in words in the bill or the draft and also in figures generally on its left hand corner.

(7) The value of the stamp varies as the amount of money written in the bill.

24. Discounting a Bill.

The creditor keeps the bill legally accepted by the debtor with him and at the end of the fixed period as specified in the bill presents it to the debtor to receive payment of his money. The whole transaction then closes.

Again, if the creditor is in want of money before the specified time, he may take the bill to a Banker and ask him to give him cash for it.

Usually the Banker deducts from the face value of the bill *the simple interest* on it at the rate which then prevails for the number of days, remaining till the due date agreed upon, together with the **3 days of Grace**. This simple interest deducted by the

Banker is called the **Commercial or Banker's Discount**. The Bill is **nominally** due on the date agreed upon. But there is a custom, which has the force of law, by which a bill is not **legally** due till 3 days after the date agreed upon. These are known as *three days of grace.*

Thus to receive payment of money for a bill from a Banker is called *discounting of bill.* The banker receives payment of the face value of the bill from the debtor at the end of the specified period.

Example. A person draws a bill of Rs. 500 on the 5th April payable after 6 months. If the bill is discounted on the 27th July at 5%, how much does the holder of the bill receive and what is the banker's discount ?

The bill drawn on the 5th April is payable after 6 months, $\therefore$ it is *nominally* due on the 5th October, but is *legally* due on the 8th October, taking the 3 days of grace into account. The bill is discounted on the 27th July.

$\therefore$ the total time from the 28th July to the 8th October $=(4+31+30+8)$ or 73 days $=\frac{1}{5}$ year.

The banker will deduct his banker's discount which is equal to the interest on Rs. 500 for $\frac{1}{5}$ year at 5%.

Now, interest on Rs. 100 for 1 year = Rs. 5.

$\therefore$,, Rs. 100 for $\frac{1}{5}$ year = Rs. $5\times\frac{1}{5}$ = Re. 1

$\therefore$,, Re. 1 ,, ,, = Re. $\frac{1}{100}$

$\therefore$,, Rs. 500 ,, ,, $=(\frac{1}{100}\times 500)$ = Rs. 5.

$\therefore$ the required banker's discount = Rs. 5 ; and the holder of the bill will receive Rs. $(500-5)$ or Rs 495.

24 (*a*). The bill may be used in another way :

The holder of the bill may endorse (*i.e.*, transfer) the bill in the name of his creditor to repay his debt to him. To endorse a bill in this way the holder of the bill should put down the name of his creditor (in whose favour the bill is endorsed) over his own signature on the reversed side of the bill. It is called the *endorsing of a bill.* The person who endorses the bill in favour of another person is called an *Endorser* and the person in whose

favour the bill is endorsed is called an *Endorsee*. This endorsee may again endorse the bill in favour of some other person. Thus the bill may be transferred from hand to hand. The endorsee becomes the owner of the money of the bill.

Specimen : Suppose Sri Rasik Dhar has purchased books worth Rs. 300 from Sri Paresh Bhowal on credit on condition that he will repay the price after two months. Here see below how Paresh Bhowal will draw up the bill or the draft.

Stamp

1/1, College Square
Calcutta, 5th April, '63

Rs. 300/-

Two months after date, pay to me or my order, the sum of Rupees Three Hundred for value received.

To

Sri Rasik Dhar
10, Beadon Street
Calcutta-6

Paresh Bhowal

Explain the following bill :—

Stamp

32, Wellington Street
Calcutta, 15th June, 1962

Rs. 1000/-

On demand pay to Mr. Haridas Sen or order, the sum of Rupees One Thousand only for value received.

Rabindranath Palit

M. C. Laha, Esq.
Calcutta

Explanation : This is an **Inland bill**. Rabindranath Palit is the drawer of the bill. M. C. Laha is the drawee and Haridas Sen is the payee of the bill. Rabindranath Palit hereby issues an order to the debtor M. C. Laha that the latter should pay,

on demand, one thousand rupees to Haridas Sen or to any endorsee of Haridas Sen.

25. Bank draft

The bill drawn up by a bank on one of its branches is called a **Bank draft.** If a certain sum of money with corresponding commission be deposited in a local bank to pay off the dues to a person in his own or a foreign country, the bank orders its branch nearer to the payee to pay the money. The bill ordering such payment is the **bank draft.**

Promissory notes

The **Promissory note** is a document in which a debtor promises to pay the creditor a sum of money on demand or at the end of a certain time. See a specimen of the Promissory note below :—

> Stamp
>
> 17, Russa Road
> Madras, 10th March, 1963
>
> Rs. 500/-
>
> Six months after date, I promise to pay Yakub Ahmed the sum of Rupees Five Hundred only, for value received.
>
> Sukur Hossain.

Explanation : It is understood from the above specimen of the promissory note that the debtor Sukur Hossain promises to pay rupees five hundred to his creditor Yakub Ahmed six months after 10th March.

N. B. If money is to be paid **on demand**, write 'on demand' in place of 'six months after date' in the above specimen.

Difference between a bill of exchange and a promissory note :—

(1) Bill is an order written by the creditor on the debtor ; but a promissory note is a promise in writing made by the debtor to the creditor and it is not an order.

(2) There are three parties, viz., Drawer, Drawee and Payee, concerned in a bill. There are only two parties, viz , the creditor and the debtor in a promissory note.

(3) 'Acceptance' of the drawee is required in a bill, but it is not required in a promissory note.

Cheque

27. Bank : If a person or a marchant do not want to keep his surplus money over and above his daily necessity in his own custody, he may deposit it in his own name in a bank. Thus an account is opened in his name in the bank. He may, if he likes, often deposit more money in the bank or withdraw money from it whenever necessary. The account opened in a bank in such a way may be a **current account** or **Savings Bank account.** There are various other kinds of account. Specimen signature of the depositor is kept in the bank for this purpose.

The following are the advantages of opening an account in the bank :

(1) It ensures the safety of money.

(2) The bank pays interest on the deposited money at a fixed rate.

(3) If the depositor wants to make payment to his creditor, he may issue a cheque in his name on the bank.

28. Cheque : Whenever a person opens an account in a bank, the bank supplies him with a **Paying-in-Book** and a **Cheque Book.**

If that person wants to deposit in his Bank any money in cash or a cheque in his name received from elsewhere, he should do so on entering the same in the Paying-in-Book. Every leaf of that book is divided into two parts. There are spaces for writing the account number, name, amount of money (cash, note or cheque) etc. in every part. These should be first filled in in writing and then the money or cheque should be deposited in the bank. Then the bank tears off one part of the leaf for its use, puts the seal of the bank and signature of the receiving officer on the other part and returns the book to the depositor.

Paying-in-slip-এর নমুনা

Date................

Notes......			
Silver.......			
Gold.......			
Cheques...			
Rs.			

Cashier.........

Particulars of Payment

Notes.........			
Silver.........			
Gold...........			
Cheques			
,,			
,,			
Rs.			

Entd......... *Cashier*.........

SOUTHERN BANK OF INDIA

Calcutta........*196*

Paid to the credit of................

....

the sum of Rupees....................

....

in Current Deposit Account.........

By.......................

Folio......... *Ledger-keeper*.........

The money thus deposited is, as usual, credited to his account. This is the method of depositing money in a bank.

Now note the process of withdrawing money from the bank :

If a person wants to pay some money to his creditor, he may issue a cheque in his name on the bank where he has an account. He issues the cheque to his payee after writing his name, amount of money etc. over his own signature.

Every leaf of the cheque book has its two parts. One part, namely counter foil, remains with the owner of the cheque-book and the other part, namely the cheque, is given to the creditor or payee.

The cheque is, therefore, a written order of the depositor on the bank to pay on demand a certain amount of money to a certain payee or to another person as endorsed by the payee or to the bearer of the cheque.

There are three parties concerned in the cheque, viz., the bank (the Drawee) on which the cheque is issued, the giver of the cheque (the Drawer) and the Payee.

When the payee presents a cheque to the bank for payment, the bank first scrutinises if the cheque is correctly written, if the signature of the drawer tallies with his specimen signature kept in the bank and if the amount of money mentioned in the cheque does not exceed the money deposited in his bank account. When all these points are satisfied, the bank pays the money to the payee and receives his signature on the back of the cheque.

See the specimen of a cheque given here. The money mentioned in the cheque is payable either to the **Bearer** or payable to **Order**. If the writer of the cheque himself wants to withdraw money, he should write the word 'Self' after the word 'Pay' instead of writing the name of any other payee.

29. Crossed Cheque : There are two kinds of cheques, viz. crossed cheque and uncrossed or open cheque.

The specimen of an open cheque has already been shown. If two parallel straight lines be drawn on or at the corner of an open cheque it is called a crossed cheque.

The object of crossing a cheque is that no person other than the real payee can encash it even if he gets possession of it any

how. A crossed cheque cannot be encashed simply by presenting it to a bank. It has to be encashed through some bank. Suppose you have received a crossed cheque in your name on the Punjab Bank. You will not receive money on presenting the cheque direct to the Punjab Bank. In order to encash the cheque you have to submit it to the Bank (say State Bank of India) in which you have opened an account. Your Bank will then have the cheque cashed from the Punjab Bank and credit the money to your account.

Crossed cheques may be of different kinds. See the specimens below :—

(1)

(2)
& Co.

(3)
Not Negotiable

(4)
Not Negotiable
& Co.

The above crossed cheques are called **'generally Crossed Cheques'.**

The object of writing the words 'Not Negotiable' within the two parallel straight lines on a crossed cheque is that if anybody having stolen or got it accidentally endorses the cheque in the name of another person, the Bank will not accept that endorsement.

The following crossed cheques are called **'specially crossed cheques'.**

(5)
The State Bank of India

(6)
The Bank of India, Ltd.
A/c. Payee only

(7)
The Southern Bank, Ltd.
Not Negotiable

(8)
The Punjab Bank
Under Rupees sixty

A specially crossed cheque means that money is to be received only from the bank, the name of which has been entered within the two parallel straight lines.

A/c means '**Account**'. The purpose of writing 'A/c Payee only' is that the Bank through which the cheque is cashed is instructed to credit the money cashed to the account of the payee only. Suppose, a cheque of Rs. 58 is crossed with the words 'Under Rupees Sixty' written within the parallel straight lines. [Vide the specimen (8)] The object of writing these words is that the amount of money written in the cheque cannot be fraudulently changed or tampered with.

30. **Cheque Dishonoured.** If the Drawer of the cheque issues a cheque for a certain amount of money on a bank and the money deposited in his account in that bank is found short of that amount, the bank will refuse to pay the money to the payee. The cheque is thus said to be **dishonoured.**

A cheque may also be dishonoured if the signature of the drawer of the cheque do not tally with his specimen signature kept in the bank or if there be other defects in writing the cheque.

Exercise 25

1. What is called **the Bill of Exchange** ?
2. Illustrate by a specimen how the bill of Exchange is to be written.
3. What is a **Promissory Note** ? Give a specimen.
4. What is called a **Cheque** ? How is it to be written ?
5. If you receive a cheque, what will you do to encash it ?
6. What is the difference among a Bill, a Promissory Note and a cheque.
7. What is the object of crossing a cheque ?
8. Show how a cheque is to be crossed.
9. What is the object of endorsing a bill ?
10. Explain the specimens given in the next page :—

(a)

(1)

No. 30251

Rs. 1500/-

Name............Ramhari Roy

Last Balance...	3208/-
Deposit...	502/-
Total...	3710/-
Withdrawal...	1500/-
Balance...	2210/-

Date 5. 6. 56.

Fol............

No. $\frac{\text{B. O.}}{\text{D.}}$ 13026

& Co.

Entd............

Calcutta, 5th June, 1956

PUNJAB BANK Ltd.

51, Russa Road, Calcutta

Pay Ramhari Roy............*Or Bearer*

Rupees One Thousand Five Hundred Only.

Rs. 1500/-

Mitra & Co.

(b)

Stamp

8, Linton Street
Bombay, 8th May, 1950

Two months after date, pay to Ramdas Chetty or order the sum of Rupees Three Thousand for value received.

Rs. 3000/-

To
S. Chetty
20, Mudaliar Rd.,
Bombay

R. Naidu

(c)

Stamp

11, Clive Street
Calcutta, 10th July, 1963

Rs. 3000/-

On demand I promise to pay Sri Jatin Das the sum of Rupees Three Hundred only for value received.

Ram Charan Koley

Miscellaneous Examples worked out

1. Rs. 49. 75 P. was divided amongst 150 children so that each boy received 50 P. and each girl 25 P. How many boys were there ?

If 150 children be given 25 P. each we have to spend 25 P.$\times$150 or 3750 P. Here the total amount spent =Rs. 49. 75 P. or 4975 P.

$\therefore$ (4975 − 3750) P. or 1225 P. still remains to be paid.

This amount is to be divided among the boys at (50 − 25) or 25 P. each so that each boy may get 50 P. each.

$\therefore$ the number of boys=(1225$\div$25)=49.

2. In dividing a certain sum of money among some boys, I find that if I give them Rs. 6 each, I have Rs. 18 left over, but if I give them Rs. 10 each, I want Rs. 22 more. Find the number of boys as well as the sum of money I have.

If the boys are given Rs. 6 each, I have Rs. 18 left and if they are given Rs. 10 each, I want Rs. 22 more, so in the latter case Rs. 22 over and above Rs. 18 left in the former case is spent in excess over the sum spent in the former case. $\therefore$ if the boys be given (10 − 6) or 4 rupees more each, we require (18+22) or 40 rupees more in the second case than in the first.

$\therefore$ the reqd. number of boys=(Rs. 40$\div$Rs. 4)=10.

$\therefore$ The total sum of money I have=Rs. 6$\times$10+Rs. 18=Rs. 78.

3. The cost of 4 sheep and 5 goats is Rs. 155 and that of 3 sheep and 7 goats is Rs. 165. Find the cost of each sheep and each goat.

Suppose the cost of 1 sheep is x rupees and that of 1 goat is y rupees. $\because$ by the condition of the problem,

$4x+5y=155\ldots(1)$ and $3x+7y=165\ldots(2)$

Multiplying (1) by 3 and (2) by 4 we get

$$12x+15y=465$$
$$12x+28y=660$$

(subtracting) $-13y=-195$, $\therefore$ $y=\dfrac{-195}{-13}=15$

Now, from (1) $4x+75=155$, or, $4x=80$. or, $x=20$.

$\therefore$ the cost of 1 sheep=Rs. 20 and the cost of 1 goat=Rs. 15.

4. A box contains Rs. 42.25 P. in rupees, 50 P., 25 P. and 5 P. coins. If there are three times as many 50 P., twice as many 25 P. and five times as many 5 P. bits as there are rupees, find the number of each coin.

If the number of rupees be 1, the number of 50 P. coins is 3, that of 25 P. coins is 2 and that of 5 P. bits is 5.

The value of 1 rupee = Re. 1
The value of three 50 P. coins = Re. 1.50 P.
" " " two 25 P. coins = 50 P.
" " " five 5 P. bits = 25 P.

∴ their total value = 325 P.

There are Rs. 42.25 P. or 4225 P. in the box. If a packet is made with one rupee, three 50 P. coins, two 25 P. coins and five 5 P. bits; the value of each packet is 325 P.

The total value being 4225 P., the number of packets $= 4225 \div 325 = 13$.

∴ the number of rupees in the box = 13, number of 50 P. coins $= 3 \times 13 = 39$, the number of 25 P. coins $= 2 \times 13 = 26$ and the number of 5 P. bits $= 5 \times 13 = 65$.

5. Divide Rs. 248 between A and B so that $\frac{3}{4}$ of A's money may be equal to $\frac{4}{5}$ of B's money.

Let A's money be x rupees. Then B's money $= (248 - x)$ rupees.

Now, $\frac{3}{4}x = \frac{4}{5}(248 - x)$, or, $\frac{3}{4}x = \frac{992}{5} - \frac{4x}{5}$, or, $15x = 3968 - 16x$,

or, $31x = 3968$, ∴ $x = \frac{3968}{31} = 128$

∴ A will get Rs. 128 and B will get Rs. (248 − 128) or Rs. 120.

6. A labourer was engaged for $\cdot 41\dot{6} \times 84$ days on the condition that for every day he worked he would get 2 as. 6 p. and for every day he was absent he would be fined 1 a. 6 p. At the end of the time he received Rs. 3. 7 as. 6 p. only. How maney days was he absent? [C. U. 1938]

$\cdot 41\dot{6} \times 84$ days $= \frac{416 - 41}{900} \times 84$ days $= \frac{375}{900} \times 84$ days $= 35$ days.

If he worked for 35 days, he would have got 2 as. 6 p. × 35 or Rs. 5. 7 as. 6 p.; but being absent for a few days he got only Rs. 3. 7 as. 6 p.

∴ he lost (Rs. 5. 7 as. 6 p. − Rs. 3. 7 as. 6 p.) or Rs. 2 in all

For one day's absence he loses (2 as. 6 p. + fine of 1 a. 6 p.) or 4 as.

$\therefore$ he was absent for (Rs. 2 $\div$ 4 as.) days or 8 days.

7. A merchant mixed a quantity of sugar worth Rs. 8. 12 as. per maund with twice as much sugar worth Rs. 7. 10 as. per maund and by selling the mixture at Rs. 8. 11 as. per maund gained Rs. 33. How much did he mix of each kind ? [C.U. 1935]

Price of 1 maund of sugar of the first kind = Rs. 8. 12 as.

Price of 2 maunds of sugar of the second kind

= Rs. 7. 10 as. $\times$ 2 = Rs. 15. 4 as.

$\therefore$ the price of 3 maunds of the mixed suggar

= Rs. 8. 12 as. + Rs. 15. 4 as. = Rs. 24.

$\therefore$ the average price of 1 maund of sugar = Rs. 24 $\div$ 3 = Rs. 8.

$\therefore$ the gain on selling 1 maund of the mixed sugar

= (Rs. 8. 11as. − Rs. 8) = 11 as. = Re. $\frac{11}{16}$

Here, the total gain is Rs. 33. $\therefore$ the total quantity of sugar

= $(33 \div \frac{11}{16})$ maunds or 48 maunds.

$\because$ One part of the first kind of sugar is mixed with two parts of the second kind of sugar, $\therefore$ the quantity of the first kind of sugar is $\frac{1}{3}$ of the total quantity of sugar. $\therefore$ the quantity of the first kind of sugar = 48 maunds $\times \frac{1}{3}$ = 16 maunds, and the quantity of the second kind of sugar = 16 maunds $\times$ 2 = 32 maunds.

8. A pail, when $\frac{7}{8}$ full of water, weighs 19 seers 6 chhataks, $\frac{4}{5}$ full of water weighs 18 seers 7 chataks. Find the weight of the pail when empty. [C. U. 1935]

$\frac{7}{8} - \frac{4}{5} = \frac{35-32}{40} = \frac{3}{40}$. The weight decreases by (19 seers 6 chh. − 18 seers 7 chh.) or 15 chhataks, if $\frac{3}{40}$ of the pail is not filled up. $\therefore$ the weight of water contained in $\frac{3}{40}$ of the pail = 15 chh.

$\therefore$ the weight of water contained in the whole pail $= \frac{15 \times 40}{3}$ chh. = 200 chh.

$\therefore$ the weight of water contained in $\frac{7}{8}$ of the pail

= 200 $\times \frac{7}{8}$ chh. = 175 chh. = 10 seers 15 chh.

But the weight of the pail $\frac{7}{8}$ full of water = 19 seers 6 chh.

$\therefore$ the weight of the empty pail

= 19 seers 6 chh. − 10 seers 15 chh. = 8 seers 7 chhataks.

9. A contractor makes an agreement on Monday evening. 10th March, to build a reservoir by the evening of 31st March. He employs 9 men, who begin on Tuesday, 11th March, in the morning, and the job is two-thirds finished by the evening of 25th march, How many more men must be employ to finish the job just in time? The men do not work on Sundays and work half time on Saturdays. [C. U. 1935 Addl.]

It is 15 days from 11th March to 25th March and this period contains 2 Sundays and 2 Saturdays. The men do not work on 2 Sundays and work half time on 2 Saturdays.

$\therefore$ Out of 15 days, work was done only for 12 full days.

$\therefore$ 9 men do $\frac{2}{3}$ of the work in 12 days.

Again, there are left only 6 days from 26th March to 31st March ; but this period contains 1 Sunday and 1 Saturday.

$\therefore$ there is to be no work for $1\frac{1}{2}$ days.

$\therefore$ the remaining $(1-\frac{2}{3})$ or $\frac{1}{3}$ of the work is to be finished in $(6-1\frac{1}{2})$ or $4\frac{1}{2}$ days more.

Now, in 12 days $\frac{2}{3}$ of the work is done by 9 men

$\therefore$,, 1	,, ,,	,, ,,	9×12 men
$\therefore$,, 1	,,	the whole work ,,	$\frac{9\times12\times3}{2}$ men
$\therefore$,, $4\frac{1}{2}$	,, ,,	,, ,, ,,	$\frac{9\times12\times3\times2}{2\times9}$ men
$\therefore$,, $4\frac{1}{2}$	,,	$\frac{1}{3}$ of the work ,, ,,	$\frac{9\times12\times3\times2}{2\times9\times3}$ men
			or 12 men.

$\therefore$ $(12-9)$ or 3 more men must be employed to finish the work in time.

10. Two-thirds of a certain number of poor persons received 1 s. 6 d. each, and the rest 2 s. 6 d. each, the whole sum spent on them being £ 2. 15 s. How many poor persons were there? [C. U. 1939 Sup. Addl.]

$\frac{2}{3}$ of the persons received 1s. 6d. each and $\frac{1}{3}$ of them received 2s. 6d. each.

The number of men in the first batch was twice the number of men in the second batch. 2 men of the first group received 1s. 6d. $\times$ 2 or 3s. and 1 man of the second group received 2s. 6d.

$\therefore$ 3 men of the two groups received (3s. + 2s. 6d.) or $5\frac{1}{2}$s.

But they together received £2. 15s. or 55s.

$\because$ $\frac{11}{2}$s. were spent on 3 men, $\therefore$ 55s. were spent on $\frac{3\times2}{11}\times55$ or 30 men. $\therefore$ There were 30 persons.

11. Two passengers have together 21 mds. of luggage and are charged for the excess above the weight allowed free Rs. 5. 14 as. 8 p. and Rs. 8. 11 as. 4 p. respectively. If the luggage belonged to one of them, he would have been charged Rs. 15. 3 as. Find the weight allowed free and also the charge per maund. [C. U. 1940]

As the luggages belong to two passengers separately, they have to pay a total charge of (Rs. 5. 14as, 8p + Rs. 8. 11as. 4p.) or Rs. 14. 10as. ; but if they belong to one of them, he has to pay Rs. 15. 3 as., i.e., (Rs. 15. 3as. − Rs. 14. 10as) or 9as. more.

$\therefore$ 9 annas is the charge for the weight which one passenger is allowed free.

$\therefore$ the charge on 21 mds. of the whole luggage
= Rs. 15. 3as. + 9as = Rs. 15. 12as. (when no weight is allowed free).

$\therefore$ the required charge for 1md. = Rs. 15. 12as $\div$ 21 = 12as.
Again, 12 as. = charge for 1md.,

$\therefore$ 9as. = charge for $\frac{1}{12} \times 9$ or $\frac{3}{4}$md. or 30 seers.

$\therefore$ the weight of 30 seers is allowed free.

12. Two persons going to the same place had 8 mds. of luggage between them and were charged excess for the luggage at Rs. 8 and Rs. 4 respectively. Had all the luggage belonged to one person, he would have been charged Rs. 14 for excess. Find how much luggage is allowed free and how much each had. [B. C. S. 1939]

[Vide Example 11] Rs. 8 + Rs. 4 = Rs. 12.

Rs. 14 − Rs. 12 = Rs. 2, which was the charge for the weight which one man could carry free of cost.

$\therefore$ the charge for 8 mds. of the whole luggage = Rs. 14 + Rs. 2 = Rs. 16.

$\therefore$ the charge for 1 md. = Rs. $\frac{16}{8}$ = Rs. 2 ;

$\therefore$ 1 md. of luggage was allowed free to one person.

Again, if the first person were allowed to carry no luggage free of cost, he had to pay a charge of (Rs. 8 + Rs. 2) or Rs. 10 and this was the charge for (10 $\div$ 2) or 5 maunds of luggage.

$\therefore$ the first man had 5 mds. of luggage and the second man had (8 − 5) or 3 mds. of luggage with them.

13. Divide Rs. 5501. 50 P. among 4 men, 6 women and 8 boys, giving to each man double that of a woman and to each woman triple that of a boy.

Money of 1 woman = money of 3 boys.

∴ money of 6 women = money of 18 boys.

Again, money of 1 man = money of 2 women = money of 6 boys.

∴ money of 4 men = money of 24 boys.

∴ money of 4 men, 6 women and 8 boys = money of (24+18+8) or 50 boys. ∴ 50 boys get Rs. 5501.50 P.

∴ 1 boy gets (Rs. 5501.50 P. ÷ 50) or Rs. 110. 3P.

∴ 1 woman gets (Rs. 110. 3P. × 3) or Rs. 330. 9P. and each man gets (Rs. 110. 3P. × 6) or Rs. 660. 18P.

14. Three tramps meet together for a meal., the first has 5 loaves, the second 3, and the third, who has his share of the bread, pays 48 P. How should they divide the money ?

The first two tramps have altogether 8 loaves, which have been eaten equally by the three tramps. ∴ each has taken $\frac{8}{3}$ loaves. The third man has no loaves with him. ∴ he has paid 48 P. for $\frac{8}{3}$ loaves eaten by him. Now, the first man has eaten $\frac{8}{3}$ loaves out of his 5 loaves. ∴ he gets the price of $(5-\frac{8}{3})$ ar $\frac{7}{3}$ loaves.

The price of $\frac{8}{3}$ loaves = 48 P.

∴ the price of 1 loaf = $\frac{48\times3}{8}$ P. = 18 P.

∴ the first man gets the price of his $\frac{7}{3}$ loaves.

i.e., 18 P. × $\frac{7}{3}$ or 42 P.

∴ the second man gets (48 P. − 42 P.) or 6 P.

15. Divide Rs. 870 among A, B and C so that ·75 of C's share shall be equal to ·5 of A's or ·6 of B's share. [D. B. 1924]

$·5=\frac{1}{2}$, $·6=\frac{6}{10}=\frac{3}{5}$, $·75=\frac{75}{100}=\frac{3}{4}$.

Now, $\frac{1}{2}$ of A's share = $\frac{3}{4}$ of C's share

∴ A's share = $\frac{3}{4}\times2$ or $\frac{3}{2}$ of C's share.

Again, $\frac{3}{5}$ of B's share = $\frac{3}{4}$ of C's share,

∴ B's share = $\frac{3}{4}\times\frac{5}{3}$ or $\frac{5}{4}$ of C's share.

∴ A's, B's and C's shares together = $(\frac{3}{2}+\frac{5}{4}+1)$ or $\frac{15}{4}$ of C'sshare.

∴ $\frac{15}{4}$ of C's share = Rs. 870, ∴ C's share = Rs. 870 × $\frac{4}{15}$ = Rs. 232

∴ A's share = Rs. 232 × $\frac{3}{2}$ = Rs. 348

and B's share = 232 × $\frac{5}{4}$ = Rs. 290.

16. At a circket match a contractor provided luncheon for 24 and fixed the price to gain $12\frac{1}{2}$ per cent on his outlay. Three persons were absent. The remaining 21 paid the fixed price and the contractor lost 2 rupees. What was the charge ?

[D. B. 1937]

Suppose, the contractor fixed charge at Rs. x per person and he spent y rupees in all. $\because$ He could gain Rs. $12\frac{1}{2}$ on Rs. 100,

$\therefore$ on y rupees he could gain $y\times\frac{12\frac{1}{2}}{100}$ or $\frac{y}{8}$ rupees.

Now, fram the given condition of the problem we have

$$24x=y+\frac{y}{8}\ldots(1) \text{ and } 21x=y-2\ldots(2).$$

From (1) we have $24x=\frac{9y}{8}$, or, $64x-3y=0\ldots(3)$

and from (2) we have $21x-y=-2\ldots(4)$

$$\left.\begin{array}{ll} \therefore & 64x-3y=0 \quad \ldots(3) \\ \text{and from } (4)\times 3, & 63x-3y=-6 \end{array}\right\}$$

$\therefore$ (subtracting) $x=6$. $\therefore$ He charged Rs. 6 per man.

17. A, B and C go into business and collect a profit of Rs. 1000. If A's capital : B's capital=2 : 3 and B's capital : C's capital=2 : 5, find the shares of the profit which go to each.

[B. U. '32]

The shares of the profit will be in proportion to the respective capitals of A, B and C.

Here, $\frac{\text{B's share}}{\text{C's share}}=\frac{2}{5}$, $\therefore$ B's share$=\frac{2}{5}$ of C's share ;

Again, $\frac{\text{A's share}}{\text{B's share}}=\frac{2}{3}$,

$\therefore$ A's share$=\frac{2}{3}$ of B's share$=\frac{2}{3}\times\frac{2}{5}$ of C's share$=\frac{4}{15}$ of C's share.

$\therefore$ shares of A, B and C are together $(\frac{4}{15}+\frac{2}{5}+1)$ of C's share $=\frac{5}{3}$ of C's share.

$\therefore$ $\frac{5}{3}$ of C's share=the total profit. $\therefore$ $\frac{5}{3}$ of C's share=Rs.1000

$\therefore$ C's share=Rs. $\frac{1000\times 3}{5}$=Rs. 600,

$\therefore$ B's share=Rs. $600\times\frac{2}{5}$=Rs. 240.

and A's share=Rs. $600\times\frac{4}{15}$=Rs. 160.

18. The expenses of a family when rice is at 12 seers for a rupee are Rs. 80 a month, when rice is at 15 seers for a rupee, the expenses are Rs. 77 a month. What will they be when rice is at 18 seers for a rupee [D. B. 1933]

When rice is at 12 seers for a rupee, the price of 1 seer=Re. $\frac{1}{12}$.

When rice is at 15 seers for a rupee, the price of 1 seer=Re. $\frac{1}{15}$,

and when rice sells at 18 seers a rupee, the price of 1 seer=Re. $\frac{1}{18}$,

$$\tfrac{1}{12}-\tfrac{1}{15}=\tfrac{1}{60} \text{ and } \tfrac{1}{15}-\tfrac{1}{18}=\tfrac{1}{90}.$$

Now, if the price per seer decreases by Re. $\frac{1}{60}$,

the total expenses decrease by Rs. (80 − 77) or Rs. 3.

∴ if the price per seer decreases by Re. $\frac{1}{90}$, the total expenses decrease by Rs. $\frac{3\times60}{90}$ or Rs. 5.

∴ the required montly expenses=Rs. (80 − 5) or Rs. 75.

19. A servant was engaged on an annual salary of Rs. 192 and a coat. He went away after 8 months and received Rs. 125 and the coat. What was the price of the coat ?

The total dues for 12 months=price of the coat+Rs. 192

and the total dues for 8 months=price of the coat+Rs. 125

(Subtracting) ∴ dues for 4 months=Rs. 67

∴ dues for 12 months=Rs. 67×3=Rs. 201

∴ the price of the coat+Rs. 192=Rs. 201

∴ the price of the coat=Rs. 201 − Rs. 192=Rs. 9.

20. A motor car drips one drop of oil every half minute and the drops are found to be 352 yards apart on the road. Find the speed of the car. [B. C. S. 1939]

[The motor car runs 352 yards in ½ minute. Now find how far it runs in an hour. Answer is 24 miles per hour.]

***21.** The grass growing uniformly in a meadow can be consumed by 30 oxen in 80 days ; if the same can be consumed by 36 oxen in 60 days, how many will consume it in 45 days ? [C.U. 1945]

Original grass+80 days' growth is consumed by 30 oxen in 80 days

∴ „ + „ „ by 1 ox in 80×30 or 2400 days...(1)

Again, original grass+60days' growth is eaten by 36 oxen in 60days,

∴ „ „ + „ „ „ by 1 ox in 60×36 or 2160 days...(2)

$\therefore$ Subtracting (2) from (1) it is found that

20 day's growth is eaten by 1 ox in 240 days.

$\therefore$ 1 day's growth is eaten by 1 ox in $\frac{240}{20}$ or 12 days.

$\therefore$ 200 days' growth maintains 1 ox for 12×200 or 2400 days...(3)

Now, from (1) and (3) it is found that

Original grass $=(200-80)$ or 120 days' growth.

Now, $\because$ 1 day's growth is eaten in 12 days by 1 ox

$\therefore$ " " " in 1 day by 12 oxen.

$\therefore$ " " " in 45 days by $\frac{12}{45}$ ox.

$\therefore$ Original grass $+45$ days' growth, *i.e.*, $(120+45)$ or 165 days' growth is eaten in 45 days by $\frac{12}{45} \times 165$ or 44 oxen.

$\therefore$ the required number of oxen $=44$.

22. The sum of Rs. 500 is to be divided amongst 3 men, 5 women and 8 boys so that for every 6 as. a man gets, a woman gets 4 as. and a boy 1a. 6p. Find the share of each man, woman and boy.

[C. U. 1945]

[Answer: Each man Rs. 60, woman Rs. 40 and boy Rs. 15.]

23. A has three times as much money as B, but only Rs. 25 more than C; the total sum of their money is Rs. 675. Find A's money.

[C. U. 1944]

Suppose, A has x rupees; $\therefore$ B has $\frac{x}{3}$ rupees and C has $(x-25)$ rupees. $\therefore$ $x+\frac{x}{3}+x-25=675$, or, $\frac{7x}{3}=675+25=700$,

$\therefore$ $x=\frac{700\times 3}{7}=300$. A has Rs. 300.

24. A sum of Rs. 63. 4 as. was paid in rupees and two-anna pieces. The total number of coins being 100, how many of each kind were used?

[Ans. 58 rupees, 42 two-anna pieces]

25. A man buys 100 kilograms of tea. He loses as much by selling 60 kilograms at Rs. 3 per kilogram as he gains by selling the rest at Rs. 4. 25 P. per kilogram. Find the cost price per kilogram.

[Answer=Rs. 3. 50 P.]

26. The distance between two places A and B is 15 miles. Coal sell at 9 as. 6 p. per md. at A and at 8 as. 6 p. per md. at B and the cost of carrying coal is 6 p. per md. per mile. Find a place between A and B where the total cost per md. will be the same whether coal is brought from A or B.

The cost of carrying coal per maund is equal in both cases. The price of 1 md. of coal at A is 1 anna more than that at B. So it is evident that the cost of carrying coal per md. from B to the required place is 1 anna more than the cost of carrying coal from A to the required place. But the cost of carrying 1 md. of coal for 2 miles is 1 anna. $\therefore$ the distance of the place from B is 2 miles more than its distance from A.

$15-2=13$, $\therefore$ the required place is $\frac{13}{2}$ or $6\frac{1}{2}$ miles away from A and $(15-6\frac{1}{2})$ or $8\frac{1}{2}$ miles away from B.

*27. A man paid one rupee more than half of his money to the first boy and then one rupee more than half of the remainder to the second boy. After giving his money to 4 boys in this way he had Rs. 2 left. How much had he at first?

[Proceed working out from the end] [Answer=Rs. 62]

28. I have to travel 132 miles. The steamer fare is 1s. for 20 miles and the train fare is 1d. per mile. If I have only 8s. with me, what least distance must I have to travel by steamer?

Suppose, I have to travel the least distance of x miles by steamer. $\therefore$ I have to travel $(132-x)$ miles by train. The train fare is $\frac{1}{12}$s for 1 mile and the steamer fare is $\frac{1}{20}$s. for a mile.

$$\therefore \quad \frac{x}{20}+\frac{132-x}{12}=\text{whole expenses}=8,$$

or, $3x+660-5x=480$, or, $-2x=-180$, $\therefore$ $x=90$.

$\therefore$ at least 90 miles are to be travelled by steamer.

*29. A man lent out $\frac{1}{3}$ of his capital at 6%, $\frac{2}{5}$ of the capital at $4\frac{1}{2}$%, and the remainder at 5%. It amounted to Rs. 3490. 8 as, in 8 months. Find the capital.

Let x rupees be the capital, $\therefore$ at the first rate of interest Rs. $\frac{x}{3}$, at the second rate Rs. $\frac{2x}{5}$ and at the third rate Rs. $(1-\frac{1}{3}-\frac{2}{5})x$ or Rs. $\frac{4}{15}x$ were lent out.

In the first case the interest on Rs. 100 for 1 year=Rs. 6.

$\therefore$ interest on Re. 1 for 8 months or $\frac{2}{3}$ year $=\left(\frac{6}{100}\times\frac{2}{3}\right)$ rupees.

$\therefore$ interest on $\frac{x}{3}$ rupees for 8 months$=\frac{6\times 2}{100\times 3}\times\frac{x}{3}=\frac{x}{75}$ rupees.

Thus, interest on $\frac{2x}{5}$ rupees for $\frac{2}{3}$ year at $4\frac{1}{2}\%=\frac{9}{100\times 2}\times\frac{2}{3}\times\frac{2x}{5}$ $=\frac{3x}{250}$ rupees. ; and interest on $\frac{4x}{15}$ rupees for $\frac{2}{3}$ year at 5% $=\frac{5}{100}\times\frac{2}{3}\times\frac{4x}{15}=\frac{2x}{225}$ rupees. $\therefore$ the total interest$=\left(\frac{x}{75}+\frac{3x}{250}+\frac{2x}{225}\right)$

$\therefore$ the amount$=x+\frac{x}{75}+\frac{3x}{250}+\frac{2x}{225}=3490\frac{1}{2}$

or, $\frac{2250x+30x+27x+20x}{2250}=\frac{6981}{2}$, or, $\frac{2327x}{2250}=\frac{6981}{2}$

$\therefore$ $x=\frac{6981}{2}\times\frac{2250}{2327}=3375$

$\therefore$ the required capital=Rs. 3375.

30. Owing to a defect in the engine a train had to reduce its speed by $\frac{1}{7}$ and reached its destination at 3-10 P.M. instead of at 2-30 P.M. When did it start ?

[Answer : at 10-30 A. M.]

31. A, B and C begin to fill a cistern, A brings a gallon every 5 minutes, B a pint every 3 minutes and C a quart every 4 minutes. If the capacity of the cistern be 58 gallons, in what time will it be filled ?

[Answer=192 minutes ; Vide Ex. 31 of time and work]

32. A sum of Rs. 192 consists of rupees, eight-anna pieces and four-anna pieces, of which the numbers are proportional to 4 : 5 : 6 ; find the number of each coin.

[Ans. 96 rupees, 120 eight-anna pieces, 144 four-anna pieces]

33. The first day of January, 1932 was Friday. What day of the week was 1st January 1933 ? [U. P. 1931]

The number of days from the 1st January, 1932 to the 1st January, 1933 (both the days inculsive)=1 year+1 day=366 days. But the year 1932 being a leap year the total number of days=367. 7 days make a week ; $367\div 7=52$, remainder 3.

$\therefore$ 367 days contain 52 complete weeks and 3 days more.

If the 1st January, 1932 be Friday, each week thereafter ends on Thursday. ∴ the last day of the 52 weeks is also Thursday.

Now counting onwards 3 days more after Thursday we get Sunday, which is therefore the required day of the week.

[N. B. Find the total number of days including the two given dates. Divide the total number of days by 7 and if the remainder be 1, the given day of the week will be the required day. If the remainders be 2, 3, 4, 5, 6 or 0, then the required day of the week will be the 2nd, 3rd, 4th, 5th, 6th or 7th day of the week on counting onwards taking the given day as the first day of the week. If you are to find a day previous to a fixed date you should count backwards in the similar way.]

34. 1st April, 1933 was Wednesday. What day of the week was 2nd April, 1931 ? [Ans.—Monday]

35. The first day of January, 1 A. D. was Monday. What day of week was 10th March, 1931 A.D. ? [C. U. 1943]

The number of days from the 1st January, 1 A. D. to the last day of 1930 A. D. (excepting the extra days for the leap years) $=365 \times 1930 = 704450$. Now it is required to find the number of leap years in 1930 years. If we divide 1930 by 4, the quotient is 482 ; ∴ there may be 482 leap years, but if the centuries be divisible by 400, they are leap years. So there are only 4 leap-years in 19 centuries in 1930 years and $(19-4)$ or 15 of them are not leap-years. The total number of leap years is, therefore, $(482-15)$ or 467. ∴ the total number of days in 1930 years (taking leap years into account) $=704450+467$. Now, the number of days from 1st January, 1931 to 10th March, 1931 $=69$. The total number of days $=704450+467+69=704986$.

If we divide 704986 by 7, the remainder is 2.

∴ the required day of the week is Tuesday.

36. Divide 250 into two parts such that 3 times the first part and five times the second part may be together equal to 950. [C. U. 1941]

[Hints : Let the first part be x, ∴ the second part $=250-x$. From the given condition, $3x+5(250-x)=950$.

Answer=150, 100].

37. Rs. 98 was divided among 300 children, each girl had 8 as. and each boy 4 as. How many boys were there ? [C.U. '41]

[Answer : 208 boys]

38. A supply of water suffices a besieged garrison for 80 days if 8 kilolitres leak off every day, but only for 75 days if 10 kilolitres leak off daily. Find the total quantity of water in the supply.

[cf. C. U. '46]

In the first case 80×8 or 640 kl. of water leak off in 80 days and in the second case 75×10 or 750 kl. of water leak off in 75 days. $\therefore$ in the second case $(750-640)$ or 110 kl. of water leak off more than in the first case, and so the supply lasts for $(80-75)$ or 5 days less.

$\therefore$ for 5 days is required 110 kilolitres of water

$\therefore$,, 1 day ,, ,, $\frac{110}{5}$ or 22 kl. ,,

$\therefore$,, 80 days ,, 22×80 or 1760 kl. ,,

and in 80 days there is a wastage of 640 kilolitres.

$\therefore$ the quantity of water in the supply $= (1760 + 640)$ kl. $= 2400$ kilolitres.

39. A, B, C, D enter into partnership ; on January 1st. A puts in Rs. 1200, on April 1st B puts in Rs. 1500, on July 1st C puts in Rs. 1800, and on October 1st. D puts in Rs. 2100. How should a profit of Rs. 900 be divided among them at the end of the year ?

[D.B. 1932 Addl.]

[Hints : A's share of profit $=$ Rs $\frac{1}{50} \times 14400 =$ Rs. 288.
B's ,, ,, ,, $=$ Rs. $\frac{1}{50} \times 13500 =$ Rs. 270
C's ,, ,, ,, $=$ Rs. $\frac{1}{50} \times 10800 =$ Rs. 216
D's ,, ,, ,, $=$ Rs. $\frac{1}{50} \times 6300 =$ Rs. 126]

40. The massacre of Cawnpore took place on 28th June, 1857. What day of the week was it ? [Ans. Sunday] [P.U. 1905]

[Take the first day of January, 1 A.D. to be Monday.]

41. A man travelled 60 miles partly by rail and partly by car in 3 hours. Had he travelled all the way by rail, he would have arrived at the destination one hour soon and saved $\frac{2}{3}$ of the time he was in car. How far did he travel by car ? [C.U.]

[Answer = 45 miles]

42. A man pays income-tax at 9 pies in the rupee, and contributes also to the Provident Fund at 1 a. in the rupee of his salary. If he draws a balance of Rs. 445. 5 as., find his salary. [C. U. '32] [Answer=Rs. 500]

43. An equal number of men, women and boys earned Rs. 2226 in 12 weeks, each man earned Rs. 2. 5 as. 4 p. a day each woman Re. 1. 4 as. and each boy 13 as. 4 p., how many were there of each ? [C. U. 48] [Answer=6 each]

44. A tailor hires a workshop for a year at an annual rent of Rs. 20. After $5\frac{1}{2}$ months he admits another to an equal share of it. How much rent should each pay ? [D. B. '35]

The second man should pay half rent for $6\frac{1}{2}$ months i.e., full rent for $3\frac{1}{4}$ months.

The rent for 12 months=Rs. 20, ∴ the rent for $3\frac{1}{4}$ months $=(\frac{20}{12}\times\frac{13}{4})$ rupees=Rs. $\frac{65}{12}$=Rs. 5. 6 as. 8 p.

∴ the second man should pay a rent of Rs. 5. 6 as. 8 p. and the first man should pay (Rs. 20 − Rs. 5. 6 as. 8 p.) or Rs. 14. 9 as. 4 p.

45. The fore-wheel of a carriage is 10 ft. in circumference and the hind-wheel is 16 ft. How many revolutions will one make more than the other in 100 miles ? [C. U. '30]

[Hints : A wheel goes a distance equal to its circumference in one revolution. Reduce 100 miles to feet and divide it by 10 ft. and 16 ft.] [Answer : 19800]

46. A leaky cistern can be filled in 5 hours with 30 pails of 3 gallons each, or in 3 hours with 20 pails of 4 gallons each, the pails being poured at equal intervals. Find how much the cistern holds and in what time the water would waste away ?

The leaky cistern is filled in 5 hours with 30×3 or 90 gallons of water, and it is filled in 3 hours with 20×4 or 80 gallons of water.

∴ (90 − 80) or 10 gallons of water are emptied in 2 hours.

∴ 1 hour is required to empty 5 gallons of water

∴ 5 hours " " " 25 " " "

∴ the cistern holds (90 − 25) or 65 gallons of water.

Again 5 gallons of water are emptied in 1 hour.

∴ 65 gallons of water of the whole cistern will be emptied in (65 ÷ 5) or 13 hours.

47. A sum of money is distributed among A, B and C in the proportion of 1, 2, 3 so that C gets Rs. 30 more than B. Find the sum distributed and the share of each.

[Ans. : The sum=Rs. 180. A's share=Rs. 30 ; B's share =Rs. 60 ; C's share=Rs. 90.]

48. The water in a leaky cistern will suffice for a family for 15 days if 30 gallons are consumed each day or for 20 days if 20 gallons are consumed each day. For how many days will the water last if 15 gallons are consumed each day ?

Suppose, x gallons of water leak out every day. In the first case $(30+x)$ gallons of water are required every day and in the second case $(20+x)$ gallons of water are requirded every day.

$\therefore$ $20(20+x)=15(30+x)$, $\therefore$ $x=10$.

$\therefore$ the cistern contains $20(20+x)$ or $20(20+10)$ or 600 gallons of water.

In the third case $(15+x)$ or 25 gallons of water are required every day. $\therefore$ 600 gallons of water will last for $(600 \div 25)$ or 24 days.

49. A got from B change for a rupee in pice coins. Owing to darkness B mistook some half-rupees for pice and gave as pice to A who thus received Rs. 3. 6 as. 9 p. How many half-rupees were there ?

[Answer : 5]

50. A bag contains a certain number of rupees, twice as many half-rupees, five times as many quarter-rupees, and eight times as many two-anna pieces, and the value of the whole sum in the bag is Rs 544. Find the number of two-anna pieces.

[Answer—1024] [G. U. '48]

51. From a tank $\frac{3}{4}$ full of water, 16 gallons are drawn and the tank is then found to be 25 gallons more than half-full. Find how many gallons it will hold. [Ans.—164 gallons] [G.U. '49]

52. The total expenses of a family when rice is at Rs. 10 per maund are Rs. 220. When rice is at Rs. 9. 12 as. per md., they are Rs. 215. 8 as. (other expenses remaining the same). Find the total expenses when rice is at Rs. 10. 8 as. per md.

[Answer—Rs. 229] [C. U. '49]

53. Of a regiment of soldiers $0{\cdot}0\dot{3}$ are killed in the first battle, $0{\cdot}175$ of the remainder in the second, $0{\cdot}2\dot{7}$ of the remainder in the third, and 870 are left. How many were in the regiment at first ?
[Answer—1500] [C. U. '36]

54. At certain company sells 64,850 copies of a newspaper per day on an average. If the price of a copy of the paper be 2as. and the profit is only 10% of this price, find the total annual profit (assuming the year to have 365 days). [D. B. '44]
[Answer—Rs. 295878. 2 as.]

55. I am thinking of a number. I subtract 87'04 from it and divide the remainder by 3'24. The quotient is 4. What is the number ? [Such sums are to be worked out from the end]
[Ans.—100] [D. B. '41]

56. A has a yearly salary of Rs. 2000 and spends $\frac{3}{5}$ of it, B has a monthly salary of Rs. 120 and spends $\frac{5}{8}$ of what he earns. What is the ratio of their savings at the end of the year ? [D. B. '44]

A spends $\frac{3}{5}$ of his salary in 1 year.

∴ A saves $(1-\frac{3}{5})$ or $\frac{2}{5}$ of his salary in 1 year.

∴ A's total savings in 1 year = Rs. $2000 \times \frac{2}{5}$ = Rs. 800.

Again, B saves $(1-\frac{5}{8})$ or $\frac{3}{8}$ of his salary in 1 year.

∴ B's total savings in 1 year = Rs. $120 \times \frac{3}{8} \times 12$ = Rs. 540.

∴ A's savings : B's savings = 800 : 540 = 40 : 27.

87. What day of the week was 18th February, 1925 A.D. ?
[Answer—Wednesday] [P. U. '34]

58. A, B and C play cricket. A's runs are to B's and B's runs are to C's as 3 : 2. They get together 342 runs. How many did each get ? (Answer—A 162, B 108, C 72) [D. B. '43]

59. Lightposts occur at regular intervals of 44 yds. along a road. If a man cycling along the road passes 9 posts each minute, what is his speed in miles per hour ? [D. B. '46]
(Answer—$13\frac{1}{2}$ miles.]

60. If the first day of the year 1942 was a Thursday, what day of the week must have been the first day of the twentieth century ? [Answer—Monday] [D. B. '42 Addl.]

61. Show that the product of any three consecutive integers is exactly divisible by 6. [D. B. '43]

Of the three consecutive integers one of them must be divisible by 3 and at least one must be an even number i.e. divisible by 2.

$\therefore$ the product of 3 such consecutive integers $=$ a certain number $\times$ a multiple of $2\times$ a multiple of $3=$ one number $\times$ multiple of $(2\times 3)=$ one number $\times$ multiple of 6.

$\therefore$ the product is exactly divisible by 6.

62. A merchant mixes 45 lbs. of tea at one price with 30 lbs. of tea at a dearer price. By selling the mixture at 4*s.* per lb. he gains 20% on his outlay. Find the price of each kind of tea, the difference in price being 1*s.* 8*d.* per lb. [D. B. '35]

Suppose, the price of the first kind of tea is x shillings per pound ; $\therefore$ the price of the second kind of tea is $(x+1\frac{2}{3})s.$ per pound. $\therefore$ the total cost price $=\{45\times x+30(x+1\frac{2}{3})\}s.$

$\therefore$ the total selling price $=\frac{120}{100}\{45x+30(x+\frac{5}{3})\}s.$, if he is to gain 20% on his outlay.

$\because$ $(45+30)$ or 75 lbs. of tea are sold at 4*s.* per lb.,

$\therefore$ the total selling price $=4s.\times 75=300s.$

$\therefore$ $\frac{120}{100}\{45x+30(x+\frac{5}{3})\}=300.$ Solving this equation, we get $x=\frac{8}{3}$. $\therefore$ the price of the first kind of tea is $\frac{8}{3}s.$ or 2*s.* 8*d.* per pound and the price of the second kind of tea is 4*s.* 4*d.* per pound.

*63. The diameter of a fore-wheel of a carriage is $3\frac{1}{2}$ dm. while that of a hind-wheel is $4\frac{2}{3}$ dm. How far will the carriage have travelled when the fore-wheel has made 200 revolutions more than the other ? (The ratio of the circumference of a circle to the diameter is $\frac{22}{7}$)

$\frac{\text{Circumference}}{\text{Diameter}}=\frac{22}{7}$, $\therefore$ circumference $=\frac{22}{7}\times$ diameter.

Here, the circumference of the fore-wheel $=\frac{22}{7}\times\frac{7}{2}$ dm. $=11$ dm. and the circumference of the hind-wheel $=\frac{22}{7}\times\frac{14}{3}$ dm. $=\frac{44}{3}$ dm.

L. C. M. of 11 dm. and $\frac{44}{3}$ dm. $=44$ dm.

The two wheels make complete number of revolutions when they go 44 dm. To go this distance the fore-wheel makes (44 dm. $\div$ 11 dm.) or 4 revolutions and the hind-wheel makes (44 dm. $\div\frac{44}{3}$ dm.) or 3 revolutions.

$\therefore$ the fore-wheel makes 1 revolution more than the hind-wheel when the carriage travels 44 dm.

$\therefore$ the required distance $=44$ dm. $\times 200=8800$ dm. $=880$ metres.

*64. Interest in a savings Bank is paid at the rate of $2\frac{1}{2}$ per cent per annum. Find the interest on an account of a year, if at the beginning of the year there is Rs. 560 in it, 4 months later Rs. 240 is taken out, and 3 months after this Rs. 140 is put in.

There was Rs. 560 in the bank for the first 4 months. Rs.(560 − 240) or Rs. 320 for the next 3 months and Rs.(320+140) or Rs, 560 for the last 5 months.

Now, the interest on Rs. 560 for 4 months=the interest on Rs. 560×4 or Rs. 2240 for 1 month.

The interest on Rs. 320 for 3 montos=the interest on Rs. 320×3 or Rs. 960 for 1 month,

and the interest on Rs. 460 for 5 months=the interest on Rs. 460×5 or Rs. 2300 for 1 month.

∴ the total interest=interest on Rs. (2240+960+2500) or Rs. 5500 for 1 month.

The interest on Rs. 100 for 1 year in the Bank=Rs. $2\frac{1}{2}$,

∴ the interest on Rs. 100 for 1 month=Re. $\frac{5}{2\times12}$

∴ the interest on Rs. 100 for 1 month $=$ Re. $\frac{5}{2\times12\times100}$

∴ " " " Re. 1 " " $=$ Rs. $\frac{5\times5500}{2\times12\times100}$

∴ " " " Rs. 5500 " $=$ Rs. $\frac{275}{24}$ = Rs. 11. 7as. 4p.

65. If 15 chairs and 2 tables cost Rs. 400, find the cost of 12 chairs and 3 tables, the cost of 10 chairs being equal to that of 4 tables. (Answer—Rs. 390) [C. U. '54]

66. A man lost as much by selling 20 chests of tea at Rs. 620 per chest as he gained by selling 25 chests at Rs. 692 per chest. What did each chest cost him? [Answer—Rs. 660] [W. B. S. F. '52]

67. A person buys 20 srs. of milk at 9 as. 6 p. per seer, How much water must he add to it so that he may gain Re. 1. 10 as. by selling the mixture at 9 as. per seer? [W. B. S. F. '52]

The cast price of 20 seers of milk=9 as. 6 p.$\times$20=190 as.

In order to gain Re. 1. 10 as. the total selling price of the mixture of milk and water=190 as.+Re. 1. 10 as.=216 as.

The selling price of 1 seer of the mixture=9 as.

∴ the quantity of mixture=(216÷9) seers=24 seers.

∴ the quantity of water added=24 seers − 20 seers=4 seers.

68. The grass growing uniformly in a meadow can be consumed by 30 oxen in 160 days. If the same can be consumed by 36 oxen in 120 days, how many will consume it in 90 days? [Vide Example 21. Answer 44] [G. U. '52]

69. A and B buy mangoes at 10 for 12 as.: A retails them at 9 for 12 as. and B at Re. 1. 1 a. per dozen. Compare the gains of A and B in selling an equal number of mangoes. [Ans.—8 : 13].

70. A can run a mile in 4 minutes 33 seconds and B in 4 mins. 40 secs. How many yards' start can A give B in a mile race to make a dead heat? [Answer—44 yards]

71. A can do a piece of work in 6 days, B in 8 days and C in 12 days, B and C work together for 2 days, and then C is replaced by A. Find when the work will be finished. [Answer—4 days]

72. Divide Rs. 2075 into two such sums that if the first be put out at simple interest for 5 years at $3\frac{1}{2}\%$ and the second for 4 years at 3%, the interest on the first sum shall be double that on the second. [Answer—Rs. 1200 and Rs. 875]

73. Three bells toll at intervals of 12, 15 and 28 seconds respectively. A man at a distance hears their sound only when *two* at least toll at the same time. If they first toll together, how many times will he hear their sound in 7 minutes?

[Answer—12 times]

74. A train 88 yds. long overtook a man walking along the line at the rate of 4 miles an hour, and passed him completely in 10 secs.; afterwards it overtook another man and passed him completey in 9 seconds. At what rate per hour was the second man walking?

(Answer—2 miles per hour. Vide Time and Work, Example 13]

75. A milkman bought a maund of milk for Rs. 20 and mixed with it a quantity of water. He sold the mixture at 6 as. 9 p. a seer and gained Re. 1. 1 a. 6p. How much water did he mix? [Ans.—19 seers] [W. B. S. F. '53]

76. A fruit-seller sold to one customer just one-fourth of the number of oranges he had and to another four-fifteenths of the remaining number. If he had still 33 left, find the number of oranges he originally had. [Answer—60 ; W. B. S. F. '53]

77. A farmer bought 96 oxen for Rs. 2400. He sold 36 at a profit of 15 per cent and 48 at a profit of 12 per cent. Two of the remainder died and he sold the rest at cost price. How much did he gain ? [W. B. S. F. '53]

The cost price of each ox = Rs. 2400 ÷ 96 = Rs. 25.

∴ the cost price of 36 oxen = Rs. 25 × 36 = Rs. 900.

The profit on Rs. 900 at 15% = Rs. (15 × 9) or Rs. 135.

Again, the cost price of 48 oxen = Rs. 25 × 48 = Rs. 1200.

∴ the profit on Rs. 1200 at $12\frac{1}{2}$% = Rs. $12\frac{1}{2} \times 12$ = Rs. 150.

∴ the total profit on selling 84 oxen = (Rs. 135 + Rs. 150) = Rs. 285.

Two oxen having died, there is a loss equal to the cost price of two oxen, *i.e.*, Rs. 25 × 2 or Rs. 50. ∴ the profit then stands at (Rs. 285 − Rs. 50) or Rs. 235. The remaining 10 oxen were sold at cost price, thereby making neither gain nor loss.

∴ the required total gain = Rs. 235.

78. The number of literates in India was 116 out of every thousand persons in 1911, and increased to 140 per thousand in 1921. In how many years more will this number be 992 per thousand, if the rate of increase continues to be the same ?

[Answer—355 years.] [C. U. '32]

79. A man buys Rs. 10 worth of oranges at 14 per rupee and Rs. 25 worth of oranges at 16 per rupee. He sells the whole lot at 15 oranges per rupee. Find his gain or loss. [E. B. S. B.'51]

[Ans. Profit Re. 1]

80. Two persons have Rs. 11. 8 *as.* between them. If one had Re. 1 more and the other 8 *as.* less, the former would have twice as much as the other. How much has each ? [Pat. U.'26]

If the first man have Re. 1 more and the second 8 *as.* less, then the two together have 8*as.* more than Rs. 11. 8*as.* *i.e.* Rs. 12.

Then the first would have twice as much as the second ;

∴ the first would then have Rs. 8 and the second Rs. 4.

∴ the first person has Rs. (8 − 1) or Rs. 7 and the second person has (Rs. 4 + 8 *as.*) or Rs. 4. 8 *as.*

81. Divide Rs. 4250 among A, B and C so that if their shares be diminished by Rs. 10, Rs. 15 and Rs. 25 respectively, the remainders shall be in the ratio of 2 : 5 : 7 ? [U. U. '52]

[**Hints :**—Rs. 10+Rs. 15+Rs. 25=Rs. 50.

If the shares be diminished by Rs. 50, there remains Rs. (4250 − 50) Rs. 4200.

Divide Rs. 4200 in the proportion of 2 : 5 : 7. 2+5+7=14 ;

If Rs. 4200 be divided into 14 equal parts, each part=Rs. 300.

∴ the three parts are Rs. 600, Rs. 1500 and Rs. 2100 respectively. ∴ A's share=Rs. (600+10)=Rs. 610 ; similarly, B's share=Rs. 1515 and C's share=Rs. 2125.]

82. Which is the better and why—an annual increment of Rs. 25 or a biennial increment of Rs. 50 ? [B. C.S. '34]

[Answer—the first]

83. A person has to pay income-tax at the rate of 9 pie in the rupee on the excess of his total income over Rs. 1500. If he pays Rs. 162 towards income-tax, find his total income and the average income-tax per rupee. [Ans. Rs. 4956. $6\frac{114}{413}$ p.]

84. The expenses of a family when rice sells at 2 seers a rupee are Rs. 250 a month and Rs. 244 when the price of rice is 20% less. What will be the expenses when the price of rice is $33\frac{1}{3}$% less ?

[W. B. S. F. '55 Addl.]

[**Hints :** When rice sells at 20% less, the total expenses diminish by Rs. (250 − 244) or Rs. 6. ∴ when rice sells at $33\frac{1}{3}$% less, the total expenses diminish by Rs. $\frac{6\times100}{20\times3}$ or Rs. 10.

∴ the required expenses=Rs. (250 − 10)=Rs. 240.]

STATISTICS

1. Statistics

Statistics is a science which deals with practical theories or decisions that are arrived at by comparison, analysis, interpretation, etc. of several quantitative or numerical data that we may collect on close observation.

Mere collection of data is not the sole object of statistics whose chief aim is moreover to find out a theory based on their analysis and interpretation. Statistics, therefore, involves collection of data and their tabulation, analysis and interpretation.

Statistics is highly necessary for ascertaining various facts and theories relating to the population of a state, public health, growth of industries and spread of education, etc., all based on a proper knowledge of statistical theories.

Many necessary theories about the daily attendance of school boys and their age, weight, height, scores in their examinations, etc. can be obtained with the help of statistics.

Statistics is also called a **science of averages**. An average is a value which is typical of a number of like things or data.

2. Collection of data and tabulation

No theory can be easily established from the data collected at random, unless they are properly arranged according to the scope of the enquiry. After the collection of data, the next step should, therefore, be their classification, i.e., the materials collected should be tabulated (i.e., exhibited in a table) according to the ascending or descending order of magnitude for analysis and interpretation.

Suppose the weights of 400 students of a school are taken at random and they are found to range from 24 Kg. to 56 Kg. (Here the weights are taken correct to the integer).

Now the following questionnaire may be put : (1) How many students have the greatest weight ? (2) What is the number of students having the smallest weight ? (3) What is the weight of the greatest number of students ? (4) What are the numbers of students having different weights ?

Now, from the random sample of weights collected we can neither answer the above questions nor form an idea of the weights of the whole 400 students.

In order to answer those questions we have to tabulate those weights (here these are data) in ascending or descending order of magnitude and then it will be easy to analyse them for solution of the questions.

In the above example every student is an *individual*, the whole number of 400 students is an *aggregate* and the weight is *character*. In such cases the characteristics of the aggregate can be derived from the collection of the values of the *character* of the individual.

3. Variable and variate

You know that any changeable value or measure is called *variable*. By a variable is meant any character which is capable of variation or difference in size or kind. The *quantity* or *character* whose values or measures vary is called *variate*. For example, the ages, heights, weights of students and their *scores* at an examination, temperatures recorded from the thermometer are variates. In many cases 'variable' and 'variate' are used in the same sense.

In the opinion of some, if with the changes in the value of one quantity, the value of another quantity also changes, the first quantity is called the *variate* and the second is called the *variable*. As for example, with the increase in height the weight of the body increases. Here height is the variate and weight is the variable.

If anybody takes *weight* to be the first object of measurement and from it determines the changes in height, then weight is regarded as the variate and height as the variable.

This variate is of two kinds—(1) *quantitative* variate and (2) *numerical* variate.

Quantitative variate. The *height* of the students of a school is a quantitative variate. So age, weight, etc. are also quantitative variates. A quantitative variate can have any value within two fixed limits. The difference of its values may be very small within those limits. Suppose that the limits of the age-variate

of a number of boys are 4—6 (years). Then the values of the variate may be 4 years, 4·1 years, 4·2 years, 4·3 years, etc. or $4\frac{1}{8}$ years, $4\frac{1}{4}$ years, etc. Such a quantitative variate is called *continuous variate.*

Numerical variate. The variate whose values are expressed only in integral numbers is called a numerical variate : e.g., the number of boys in a class, the number of trees in a garden, etc. are numerical variates. The values of such a variate can never be mixed fractions but shall always be integers. As for example, if the number of boys is said to be within the limits of 4 and 6, then their number may be 4, 5 or 6 ; but it can never be $4\frac{1}{2}$, $5\frac{1}{2}$ boys, etc. As the value of a numerical variate within two fixed limits is not continuous, the numerical variate is called a *discrete or discontinuous variate*.

4. Symbols

A *variable* is generally denoted by X or Y. Thus N denotes the number of data collected and S their sum or aggregate. The symbol Σ (called capital sigma) also denotes the total or the aggregate. Besides these, the Greek letters β (beta), γ (gamma), π (pi), ϕ (phi), μ (mu), σ (sigma), etc. are used as symbols.

5. Collection of statistical data and tabulation

Data or numerical facts are collected in various ways, e.g.

(1) with the help of individual observations (personal investigation) ;

(2) by circulation of questionnaires to different persons, companies or factories ;

(3) from the annual reports published by the Government or some other institutions ; etc.

If the collected data be tabulated unclassified or ungrouped, they are called *raw* data or *unclassified* data. Again, when the data are arranged in ascending or descending order of magnitude in a table, they are called *arrayed* data.

The collected data are thus tabulated (1) according to their numerical values, averages, ranges, (2) in ascending or descending

order of magnitude, or (3) in a particular way to serve the purpose of statistics.

If the number of data collected be very large, the different data are classified or grouped according to their characteristics and then tabulated at different intervals.

Suppose the quantities of paddy grown in a country are collected. They may be first classified, according to our need or purpose, into different kinds of paddy, such as Aus, Aman, etc. and then arranged in a table according to the quantity of each kind of paddy. If it is needed, those data may be again further divided district-wise.

Unclassified Data

Example 1. Below is given the table of scores in an examination of 60 students of a school.

Table No. 1

86	97	85	92	71	105
96	88	99	102	108	108
71	84	80	108	87	118
97	88	85	98	94	96
76	89	72	90	100	97
79	98	88	98	92	94
111	90	98	75	87	88
94	107	90	94	112	90
94	95	101	90	81	77
87	89	88	104	99	82

The very table shows that the data have been noted in an unclassified way. From this it is not easy to ascertain (*a*) the highest or the lowest score ; (*b*) how many students scored more than 76, how many less than 83 ; (*c*) how many students secured the same score ; etc.

So the data given in the above table are being arranged in ascending order of their magnitude. (Vide Table No. 2).

ARRAYED DATA

Table No. 2

71	83	88	92	96	102
71	83	88	93	96	103
72	83	89	93	97	104
75	84	89	93	97	105
76	85	90	94	97	107
77	85	90	94	98	108
79	86	90	94	99	108
80	87	90	94	99	111
81	87	90	94	100	112
82	88	92	95	101	113

From the table it can be at once said that the lowest score is 71 and the highest score is 113 and the scores being from 71 to 113 their range is (113 − 71) or 42.

But the questions such as—'how many students have scored below 100' Or 'how many students have scored from 85 to 96' cannot be easily answered even from the table No. 2. For this purpose the data should be arranged in a different way.

Frequency Distribution Table

In table No. 2, the 60 data have been arranged in an array, but they have not been grouped in intervals. When the number of different data is very large, it is disadvantageous to formulate a principle from them unless they are arranged in intervals.

In dividing the data into intervals, the values of the qualitative variates are generally grouped in intervals. In the Example 1 above, the values of the scores are quantitative variates and the numbers of those values are numerical variates ; so here the values of the scores are to be grouped in intervals. The number of values of the variate included in each interval is called the *frequency* of the interval.

The showing of the frequencies of the values of the variate is known as *Frequency Distribution.*

The table that contains such a frequency distribution is known as a *Frequency Distribution Table*.

The rule for drawing up a frequency distribution (Vide Example 1)

(1) First determine the range of the data (here scores of Examplel-1), *i.e.*, the difference of the highest and the lowest scores. In our example the range $= 113 - 71 = 42$.

(2) Next, choose the **number** and **size** of the intervals to be used in grouping the scores. Generally chosen intervals are 3, 5, or 10 units in length.

Here, the range is 42, so there will be 9 intervals, if an interval is to include 5 kinds of values. $42 \div 5 = 8\frac{2}{5}$.

$\therefore$ the number of intervals is 9, because it cannot be a fraction.

Here, the first interval is 70—74, the second interval is 75—79, etc, and the last interval is 110—114.

Every interval has its two limits, highest and lowest, and the portion between these two limits is called an interval (briefly i).

Method of Frequency Distribution

After determining the intervals, the frequency of each interval is to be noted. Frequency Distribution can be made whether the data are ungrouped or classified in an array. We show this method from Table No. 1 given in Example 1.

The frequency distribution may, of course, be more advantageously made from Table No. 2 ; but if the data be not so arranged as in Table No. 2, we can also make the frequency distribution even from the raw data. (Vide Table No. 3).

Make three columns in a table. The intervals of the scores should be written in the first column, the tallies of the scores in the second column and the number (frequency) of the students in the third column.

Write down the intervals of the scores $70-74$, $75-79$, etc. one below another in the first column. Now the first score 86 in Table No. 1 falls within the interval 85—89, place a vertical

tally beside that interval in the second column. For the second score 96 place also a vertical tally, beside the interval 95 – 99, in the second column. Thus go on marking the tallies for all the scores of the table. If the number of tallies in any interval be 5 or more than 5, every fifth tally should be placed cross-wise over the preceding 4 tallies, so that it may at once indicate that

FREQUENCY DISTRIBUTION

Table No. 3

Intervals	Tallies	Number of Students *f* (*frequency*)
70 — 74	///	3
75 — 79	////	4
80 — 84	~~////~~ //	7
85 — 89	~~////~~ ~~////~~	10
90 — 94	~~////~~ ~~////~~ ~~////~~	15
95 — 99	~~////~~ ////	9
100 — 104	~~////~~	5
105 — 109	////	4
110 — 114	///	3
	Total N =	60

there are 5 scores up to that tally. Keep a little space after every fifth tally. The number of tallies of each interval will indicate the frequency or number of scores (*i.e.* the number of students making the scores) in that interval. Write down those numbers representing the frequency in the third column.

Frequency Distribution is usually made in two columns in a Table where the second column (the column of tallies) is omitted. It is shown in Table No. 4.

FREQUENCY DISTRIBUTION

Table No. 4

(Intervals)	No. of students f (frequency)
70— 74	8
75— 79	4
80— 84	7
85— 89	10
90— 94	15
95— 99	9
100—104	5
105—109	4
110—114	8
N =	60

Number of intervals of Frequency Distribution

In the tables No. 3 and No. 4 of Example 1, the range or size of the intervals is taken to be 5 and the total number of intervals has been 9. If the range of each interval were 9, there would have been 5 intervals. There is no fixed rule regarding the size of the intervals in a frequency distribution (*i.e.*, into how many intervals the values of the variates should be divided). We may, therefore, choose any advantageous number of intervals. These intervals may be of equal or unequal size. It is, however, convenient to choose intervals of equal range. Otherwise, no comparative theory can be formulated. The range of each interval should be so determined that the number of intervals be not very large. It is better if the range of each interval be an odd number, such as 3, 5, 7 etc.

The choice of number of intervals depends on the number of values. The number of intervals is generally taken to be between 10 and 30. Suppose a variate has 63 different values. If 63 is divided by 10 and 30, the respective quotients are 6·3 and 2·1.

So it is advantageous to take 4 which is between 6 and 2 as the range of each interval. If the range of each interval be 4, then the number of intervals for the whole range of 63 will be 16.

In the frequency distribution of Example-1, 5 is taken as the range of each interval, ∴ the number of intervals is 9. See how the number of intervals is being determined according to the above rule. There are together 43 different values from 71 to 113. $43 \div 10 = 4 \cdot 3$, $43 \div 30 = 1 \cdot 43$.

So here 3, which is between the two quotients, is taken as the range of each interval. According to this the frequency distribution is shown in Table No. 5.

FREQUENCY DISTRIBUTION

Table No. 5 (From Example 1)

(Interval)	Frequency (f)
70— 72	3
73— 75	1
76— 78	2
79— 81	3
82— 84	5
85— 87	5
88— 90	10
91— 93	5
94— 96	8
97— 99	6
100—102	3
103—105	3
106—108	3
109—111	1
112—114	2
N =	60

Limits of an interval and midpoint

Suppose the age of each boy of a class is stated to be 12 years. This statement does not indicate that the ages of all the boys are equal and just 12 years. The real meaning of this statement is that their ages will range from 11·5 years to 12·5 years, *i.e.*, no one's age will be less than 11·5 years and more than 12·5 years, These 11·5 years and 12·5 years will be the limits of their ages. Actually though their individual ages are between the two limits 11·5 years and 12·5 years, the age of each boy is taken to be 12 years in aggregate.

Here notice that 12 years is just in the middle of 11·5 years and 12·5 years, *i.e.*. their average. This 12 years is said to be the midpoint of 11·5 years and 12·5 years.

Again, if the weights of a few pieces of stone range from 17 Kg. to 18 Kg., then the weights of the different pieces may be any weight from 17 Kg. and under 18 Kg. (briefly, 17 *a. u.* 18). Their weight is said to be 17·5 Kg. collectively or on the average. Thus if the minimum height of a few boys be 48—56 (in inches), then it will indicate that the interval is within the limits of 47·5—56·5 and the midpoint will be $\frac{47{\cdot}5+56{\cdot}5}{2}$ or 52 years.

If the collective weight of those pieces of stone is said to be 54·3 Kg. correct to one decimal place, then their weight may be any weight (54·3 − ·05) or 54·25 Kg. and under (54·3 + ·05) or 54·35 Kg.

If the value is given correct to one decimal place, the lower limit is diminished by ·05 and the higher limit increased by ·05. Again, if the value is given correct to the integer, the lower limit is diminished by ·5 and the higher limit increased by ·5.

It is advantageous to have the limits of the intervals correct to the integer. If the limits of the intervals be 48—52, 53—57, 58—62, etc. then there being 5 values in each interval, the third value of each interval will be the midpoint of the interval.

∴ the midpoints will be 50, 55, 60, etc. respectively.

If the values of the intervals are not correct to the integer and if it is said that the lowest integral value in a 3 score interval

is 60 Kg., then the intervals may be written as 60 and under 63, 63 and under 66, 66 and under 69, etc.

Again, they may also be written as 60 and above 60, 63 and above 63, 66 and above 66, etc.

Example 2. Frequency Distribution (pointing out the limits) of 324 studente of a school with regard to the percentages of their scores in an examination is shown in the following table (Vide Table No. 6). The scores are given correct to the integer.

Draw up a frequency distribution table doubling the size of each interval.

FREQUENCY DISTRIBUTION

Table No. 6

Intervals	No. of students (frequency) f
32 and under 37	14
37 „ 42	22
42 „ 47	28
47 „ 52	34
52 „ 57	56
57 „ 62	64
62 „ 67	77
67 „ 72	19
72 „ 77	10
N=	324

∵ in the above frequency distribution the size of each interval is 5, ∴ by the question we have to draw up here a frequency distribution table in which the size of each interval will be 10.

In the given table the total range is (77—32) or 45 and the length of each interval is 5 ; ∴ the number of the intervals is (45÷5) or 9. Now, if we are to take 10 values as the size of each interval, the number of intervals will be (45÷10) or 4½, *i.e.*, the last interval will consist of 5 scores. But as no student

has secured more than 77 scores, the last interval cannot be made of 10 scores on increasing the last score by 5 scores.

∴ the first interval should begin from 27, *i.e.*, from 5 more preceding values. Thus the first interval will be from 27 and under 37.

[N. B. Here percentage of scores are given ; the total range of the scores will be from 0 to 100.

∴ the first size of an interval may be taken as 27—37. No student has secured less than 32 scores. ∴ the number of students (*i. e.*, frequency) in the intervals '32 and under 37' as well as '27 and under 37' will remain the same.

In the new table the frequency (or the number of students) of the first interval will be 14. The number of students of the second interval will be the sum of the number of students in the second and third intervals of the original table, *i.e.* (22+28) or 50

Now, the required frequency distribution table is given below :

FREQUENCY DISTRIBUTION

(of 10 score interval)

Table No. 7

Score Intervals	Frequency(f) or No. of students
27 and under 37	14
37 " " 47	50
47 " " 57	90
57 " " 67	141
67 " " 77	29
N =	324

[N. B. (1) The intervals of the table No. 6 should not be written as 32—37, 37—42, 42—47, etc., for thus the same values (*e. g.*, 37, 42,...) are taken in both intervals and, therefore, the conclusions arrived at from the frequency distribution made from these raw or arrayed data may be wrong.

(2) Frequency distribution may be made with intervals of unequal size. It is not generally done, if it is not needed.]

To Find the interval midpoint

We have already discussed the interval midpoint. The formula for determining the interval midpoint is as follows :—

(1) Interval midpoint (when the limits are not given)

$$= \text{The first score of interval} + \frac{(\text{the highest score} - \text{the lowest score})}{2}$$

(2) Interval midpoint (when limits are given)

$$= \text{The lowest limit of the interval} + \frac{(\text{The highest limit} - \text{the lowest limit})}{2}.$$

Example 3. Find the midpoints of the intervals (i) 90 – 94 and (ii) 89·5 – 94·5.

Now, (i) the reqd. midpoint $= 90 + \frac{94-90}{2} = 90 + 2 = 92.$

(ii) The reqd. midpoint $= 89{\cdot}5 + \frac{94{\cdot}5 - 89{\cdot}5}{2} = 89{\cdot}6 + 2{\cdot}5 = 92.$

Cumulative Frequency Table

You have already noticed how the frequency of each row has been formed in the Table No. 7 from the Table No. 6.

It appears from the Table No. 6 that the number of students securing scores below 37 is 14 and the number of students securing scores below 42 is (14+22) or 36. For, the table shows that the number of students securing scores from 37 and under 42 is 22, and that of students whose scores are below 37 is 14. These 14 students have also secured scores less than 42. The number of students whose scores are below 42 will, therefore, be the sum of 14 and 22, *i.e.*, the sum of the second frequency and the previous first frequency in Table No. 6.

Again, the number of students securing scores below 47 is (14+22+28) or 64, *i.e.*, the total of the third frequency and the two previous frequencies and so on.

As this frequency is determined by successive addition, it is called *cumulative frequency*.

Thus cumulative Frequency Distribution is a form of frequency distribution in which each frequency is the total of the previous ones. The cumulative frequency table derived from Table No. 6 is shown below :

CUMULATIVE FREQUENCY TABLE

Table No. 8. (formed from table No. 6)

Score Interval	No. of students (Cumulative Frequency
below 37	14
" 42	36(i.e. 14+22)
" 47	64(=14+22+28. or=36+28)*
" 52	98(=14+22+28+34 or=64+34)
" 57	154(= 98+56)
" 62	218(=154+64)
" 67	295(=218+77)
" 72	314(=295+19)
" 77	324(=314+10)

*[**N. B.** To find the number of students in the third row of Table No. 8 from the table No. 6, you have to add together the three numbers of students of the first, second and third rows of Table No. 6. It may be done more easily. To find the number of students in the third row of Table No. 8 you have only to add the number of students in the third row of Table No. 6 (*i.e.* 28) to the number of students (*i.e.* 36) in the second row of Table No. 8. Similarly to find the frequency in the fourth row of Table No. 8, you have to add the frequency (*i.e.* 34) in the fourth row of Table No. 6 to the frequency (*i.e.* 64) in the third row of Table No. 8 and so on. Here notice that the number of students in the first row of the new Table No. 8 must be equal to that in the first row of Table No. 6 and the number of students in the last row of the new table is equal to the total given number.]

Exercise 1

1. Define and illustrate the following terms :
Variable, variate, range, size of interval, frequency.

2. What are meant by the *'limits'* and *'midpoint'* of an interval ?

3. State with suitable examples what is meant by raw data.

4. What is called an *'array'* ?

5. How is Frequency distribution made from arrayed data ?

6. Describe a few ways of collecting statistical data.

7. A list of scores secured by 100 students of a school in an examination is given below. From it draw up a table of arrayed data arranged according to the ascending order of the scores.

53	28	35	28	49	54	29	30	30	36
50	55	48	35	23	24	37	51	52	27
44	41	42	37	43	47	35	36	42	35
43	44	45	56	39	34	41	44	45	34
38	40	51	42	39	38	40	41	40	41
38	41	40	44	50	39	56	40	40	57
41	45	46	39	36	42	46	45	46	47
42	33	32	39	43	47	43	37	34	46
33	52	39	47	29	33	54	51	55	49
31	31	52	32	38	48	25	26	27	38

8. Answer the following questions from the arrayed data derived from the table of sum No 7 above :—

(i) How many students have secured more than 53 scores ?

(ii) How many students have scored more than 32 but less than 53 ?

9. Draw up a frequency distribution table from the data given in sum No. 7 above, the size of each interval being 5.

10. Draw up a frequency distribution table with 4-score intervals from the raw data in sum No. 7 above.

11. If 10 integral values of a variate are included in the interval 72·5 – 79·5, find those values.

12. The weights (in pounds) of the students of a class are arranged in regular intervals of 75 – 79, 80 – 84, etc. Find the limits and midpoints of these intervals.

13. Complete the following table :

Intervals	Limits of intervals	Midpoints
83 and under 88		
78 " " 83		
73 " " 78		

14. In the following table the heights of 50 students of a school are given in inches correct to one decimal place. Complete the table.

Intervals	Interval Limits	Midpoint	Frequency
50·8 and under 52·8		51·75	12
52·8 " " 54·8		53·75	18
54·8 " " 56·8		55·75	20

15. The weights of 100 students are given in kilograms in the table below :—

Weight (Kg.)	53	54	55	56	57	58	59	60	61	62	63
No. of students	3	4	2	5	10	18	17	15	14	6	6

Draw up a cumulative frequency distribution table from the weights with the intervals (1) 54 Kg. and below, 55 Kg. and below, etc. (2) 62 Kg. and above, 61 Kg. and above, etc.

16. The ages of 144 students of a school are given in integral number of years in the table below :

Age (year)	8	9	10	11	12	13	14	15	16	17	18	19
No. of students	12	10	8	10	4	7	15	15	18	30	12	3

Draw up a frequency distribution table grouping the ages into intervals 8 – 10, 11 – 13, etc.

Graphical representation

It is said that "one picture is worth a thousand words" *i.e.*, one vivid picture may sometimes make us understand a thing more easily than a thousand words.

It is for this reason that a diagram or a graph is often more revealing than the most carefully collected array of numbers.

It is often seen that our attention is not drawn to the important publications of numerical statistics by the Railway Company, Government or Municipality.

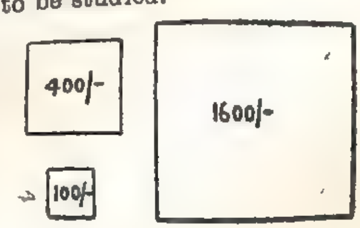
To attract more attention various facts and necessary information are now-a-days published through the medium of diagrams. Besides this, graphs are particularly necessary for comparative statistical treatment.

Use of different kinds of diagrams

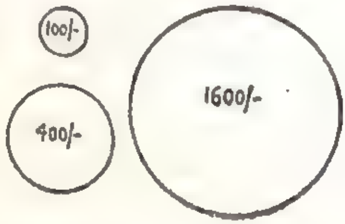
Comparative statistical data can be expressed with the help of various kinds of diagrams, such as, squares, rectangles, circles or Pai charts, etc.

Squares : If we want to make a comparative study of a few given data by square diagrams, we should draw the squares whose areas are in proportion to the data to be studied.

Suppose the annual incomes per capita of three countries are Rs. 100, Rs. 400 and Rs. 1600 respectively. The comparative squares of these are drawn here.



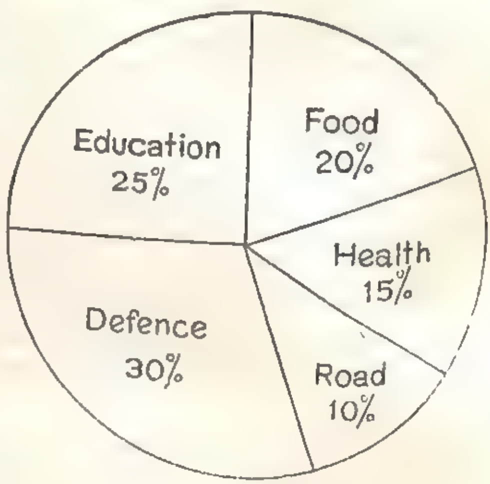
Circles : Comparative circles may also be drawn with regard to the above annual incomes.



Comparison may be made by drawing different circles whose areas are proportional to the given quantities. Again, they may be expressed by dividing the same circle into parts proportional to the given quantities.

Suppose that India Government spends 10% of its total income for construction of roads, 15% for health, 20% for procurement of food, 25% for education and 30% for defence of the country. These are expressed below by drawing a circle.

The area of the whole circle denotes the total expenditure and each part of it denotes expenditure under each different head. The areas of those parts are proportional to 10 : 15 : 20 : 25 : 30, *i.e.*, 2 : 3 : 4 : 5 : 6 :

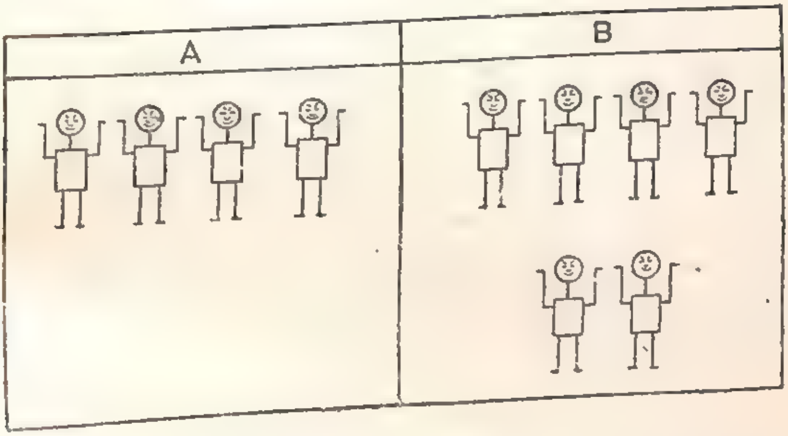


Similarly, those data can also be expressed by dividing squares or rectangles into parts proportional to those data.

This is called a circle diagram or a pie chart.

Representation by pictures. Suppose that the population of a country (A) is 40000 and that of another country (B) is 60000. The population of these two countries can easily be shown by means of pictures. First draw two columns A and B and then draw portraits of 4 men in the first and those of 6 men in the second column.

Now, if each man in the picture is taken to represent 10000 men, then it can be easily understood that the population of A is



40000 and that of B is 60000. These kinds of pictures are widely circulated by government or in the exhibition to explain statistical data to the public.

Graph. You know from Algebra how to draw a bar graph, column-graph and linear graph. Here two examples are given.

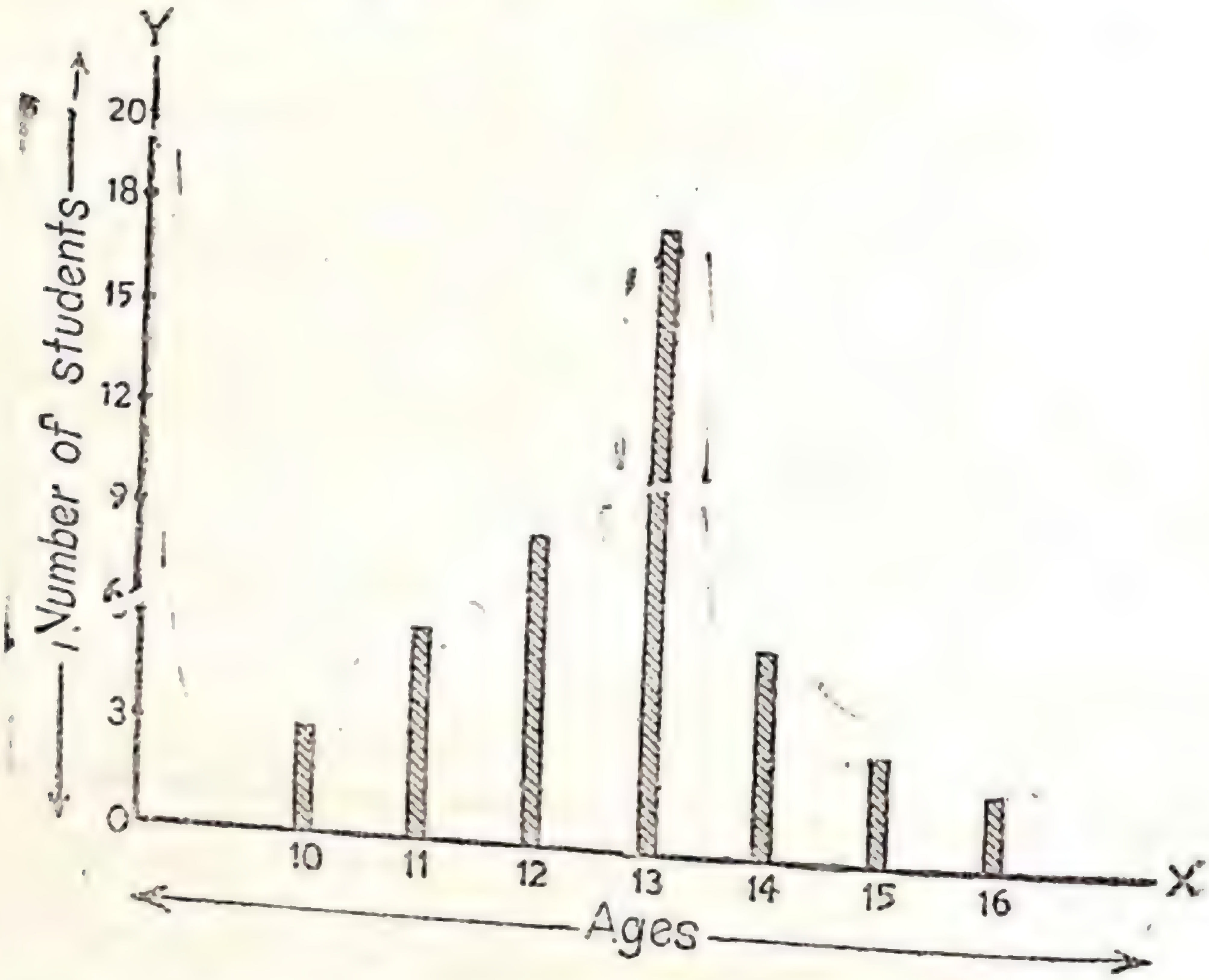
Example 1. The ages of 450 students of a school are given in the following table. Express it by means of a bar graph.

Age	10 yrs.	11 yrs.	12 yrs.	13 yrs.	14 yrs.	15 yrs.	16 yrs.
No. of students	80	60	80	130	50	80	20

Draw a horizontal line OX and a vertical line OY on your graph paper.

On choosing suitable units represent the ages along the horizontal line OX and the numbers of students along the vertical line OY. Then from each point denoting age draw a vertical line equal in length to the unit denoting the corresponding number of

students. The vertical lines thus drawn express the above table. Vide the diagram below.



[N. B. If the number of students be indicated along the horizontal line and the age along the vertical line, then a few horizontal lines will express the above data.]

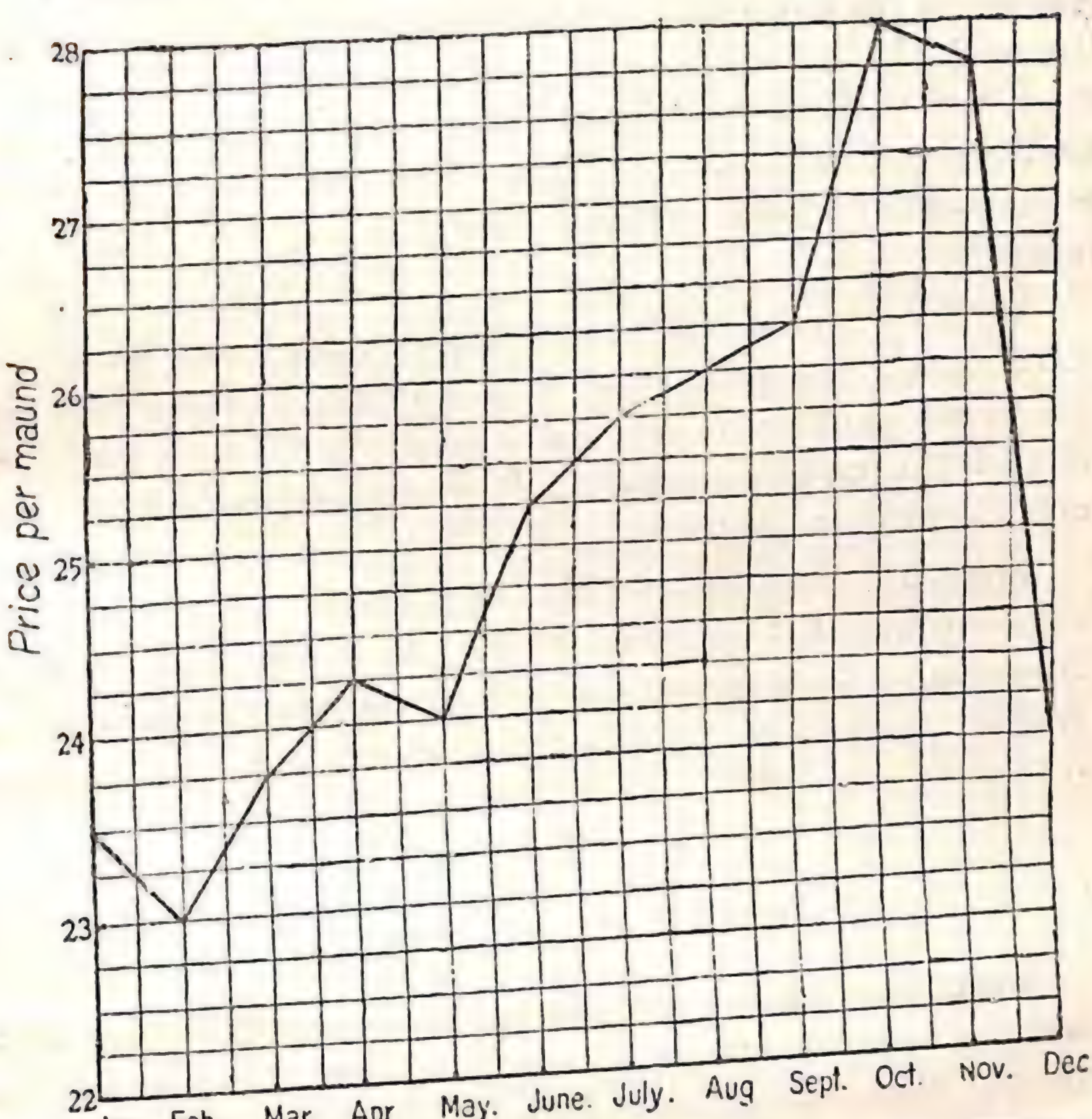
Linear graph. In some cases of economical statistics, etc. data collected at certain intervals are expressed by line graphs. (See the example below.)

Example 1. The average market prices of rice per maund in different months in 1957 are given in the table below. Express this by a line graph.

1957	Jan.	Feb	Mar.	Apr.	May	June.	July	Aug	Sept.	Oct.	Nov.	Dec.
Price of rice	28·50	28.00	28.75	24.25	24.00	25.25	25.75	26.00	26.25	28.00	27.75	28.75

Let two small divisions on the horizontal line in the graph paper represent 1 month and 4 such divisions along the vertical line represent 1 rupee. The graph line is drawn with the above points.

[See the diagram below]



Histogram and Frequency Polygon are used in statistics. Now, we shall discuss how to draw these graphs and to represent the frequency distribution graphically thereby.

Histogram or Column Diagram

Histogram is similar to the column graph you have learned in Algebra. As in Algebra two straight lines intersecting each other at right angles are taken to draw this graph. One of them is horizontal and called the base, and the other line is vertical and perpendicular to the former.

The st. lines XOX' and YOY' represent the horizontal and vertical straight lines respectively. The former is called the x-axis and the latter y-axis. The different values of x and y are

represented along x-axis and y-axis respectively. We have seen that one of the two variates in frequency distribution is quantitative and the other is numerical. In order to draw graphs with these two variates the different values of the quantitative variate are represented along the horizontal line (x-axis) and the values of the numerical variate are represented along the vertical line (y-axis).

You have seen in Algebra that the straight lines XOX′ and YOY′ divide the plane surface of the graph paper into four quadrants. The positive values of x are noted along the right hand side of the x-axis from the origin O and the negative values are noted along it on the left.

Similarly, the positive values of y are noted along the y-axis upwards and the negative values of y are noted along it downwards. In frequency distribution the values of variates are seldom negative, so in these graphs only the first quadrant is used.

Construction of Histogram

Example 1. Below is given the frequency distribution of the scores of a number of boys in an examination. Draw up a histogram of this frequency distribution.

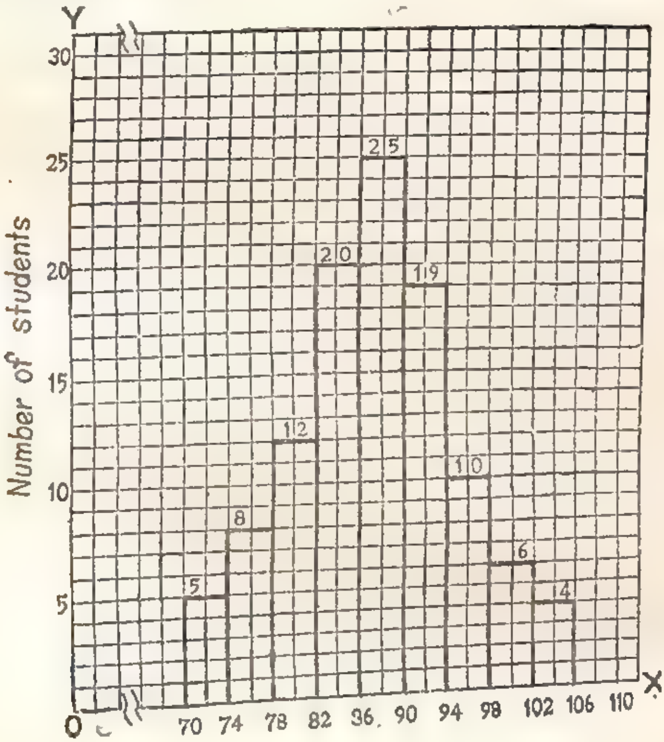
Score	70 a.u. 74	74 a.u. 78	78a.u 82	82a.u 86	86a.u 90	90a.u 94	94a.u 98	98 a.u. 102	102 a.u. 106
No. of students	5	8	12	20	25	19	10	6	4

Method : A horizontal st. line OX and a vertical st. line OY intersecting each other at O are taken. (See graph No. 1).

Here, as the two variates, scores and the number of students (frequency), have no neagative values, so the histogram must be in the first quadrant.

Now taking a suitable unit for the values, which correspond to the different score-intervals 70 – 74, 74 – 78, 78 – 82, 82 – 86, etc. are placed along the harizontal line OX.

Here one side of a small square on the graph paper represents two values of scores.



[Graph 1]

Again, taking any suitable unit to represent the number of students, the numbers 0, 5, 10, 15,...etc. are represented along the vertical line OY (*i.e.* *y*-axis).

Here one side of a small square on the graph paper represents one student.

Here the first interval of the given scores is '70 and under 74' and its corresponding frequency is 5.

$\therefore$ two perpendiculars, each 5 units in length, are erected from the marks 70 and 74 and the rectangle is completed to represent that the interval 70 – 74 has its frequency 5.

Thus here is drawn a rectangle whose base is equal to the length of the interval 70 – 74 and height is 5 units.

The next interval is '74 and under 78' and its frequency is 8, so to represent this interval and its frequency, a rectangle is

drawn whose base is the length of the interval 74 – 78 and whose height is 8 units. Thus nine rectangles are drawn corresponding to the nine intervals.

The total area of these 9 rectangles is the **histogram** of the given frequency distribution.

[**N. B.** Choose suitable units for measurement along the horizontal and vertical lines. The same length or different lengths may be taken as units along the two lines.

(2) Now notice in the diagram that the distance of the interval 70 – 74 from the origin O (as placed here) has not been correctly shown considering the unit of length chosen. If the correct distance from O is taken here, the diagram will be considerably large. So it is to be understood that we have drawn the vertical line OY closer to the interval 70 – 74. To signify this removal of the vertical line a portion of the line OX, between O and the interval 70 – 74 has been cut off with the sign ≀ ≀ .

A similar sign has also been put in the upper border-line parallel to OX.

(3) The first interval is sometimes marked even from the origin O.

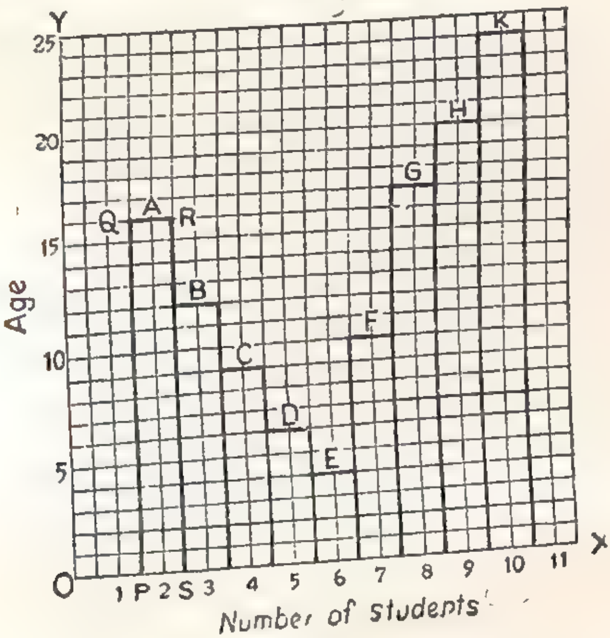
(4) In graph-1, as the intervals of scores are equal, the bases of the rectangles are also equal and therefore the histogram is symmetrical. If the lengths of the intervals are not equal, the histogram will not be symmetrical.

(5) The frequency (number of students) may be represented along the horizontal line and the scores along the vertical line according to convenience.]

Example 2. The following table shows the ages of a few boys (correct to the whole number of years). Draw a histogram corresponding to this table.

No. of students	2	8	4	5	7	8	9	10	6
Age in yrs.	16	12	9	6	10	17	20	24	5

[See graph 2] Here the graph is drawn taking two sides of a small square on the graph paper along the horizontal line to represent 1 student and one side of the square along the vertical line to represent 1 year. Now mark the numbers (of boys) 2, 3, 4, 5,.........upto 10 along OX and the numbers (of years) 5, 10, 15,.........etc. along OY. It is done so, as there is no age interval here.



[Graph 2]

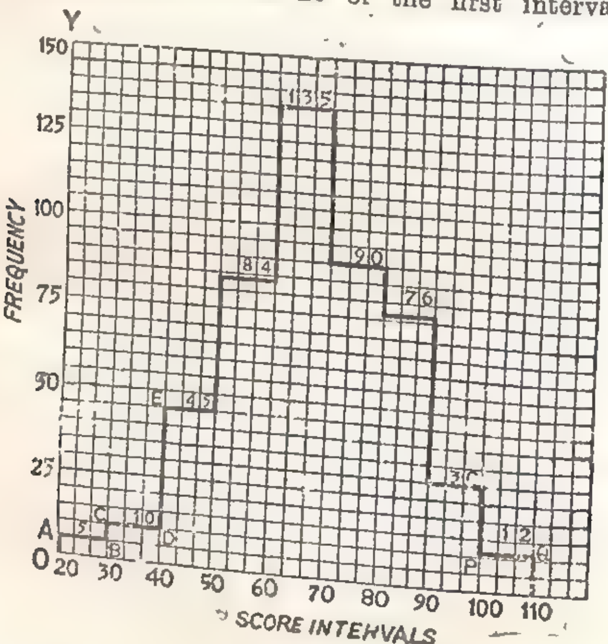
Now, from the marks 2, 3, 4, etc. the points whose heights are 16, 12, 9, etc. *i.e.*, the points having co-ordinates (2, 16) (3, 12), (4, 9),......, (10, 24) are plotted. Let these points be A, B, C, ...,K. Now, draw vertical lines on OX from P, the midpoint between the marks 1 and 2 and from S, the midpoint between the marks 2 and 3. Then through A draw a horizontal line to complete the rectangle PQRS. Thus vertical lines are drawn from the middle points of the marks 2 and 3, and 3 and 4 ; 3 and 4, and 4 and 5 ;......and finally 9 and 10, and 10 and 11. Now, complete the rectangles drawing horizontal lines through the points B, C, D,......, K. Nine rectangles are thus obtained

and the figure which is the sum of these 9 rectangles is the required **Histogram.**

Example 3. Represent the following frequency distribution graphically by a histogram.

Score intervals	*frequency*
20 – 29	5
30 – 39	10
40 – 49	45
50 – 59	84
60 – 69	135
70 – 79	90
80 – 89	76
90 – 99	35
100 – 109	12

Take the horizontal line OX and the vertical line OY. (See graph 3). Along OX mark off 20, 30, 40,......upto 100, taking 3 times the side of a small square to represent each score interval. Mark the least value 20 of the first interval at the



[Graph 5]

origin O. Mark off the frequencies 25, 50, 75, etc. along OY taking one side of a small square to represent 5 frequencies. Here, the frequency of the first interval is 5, so draw a perpendicular on OX from the mark 20 and make it equal to 1

side to represent the frequency 5. Let OA be the perpendicular. From A draw AB parallel to the horizontal line OX upto B, a point above the next mark 30. The frequency of the next interval is 10 ; so draw a vertical line BC from B and let the height of the point C from OX represent 10 scores. (*i.e.*, let BC represent 5 scores).

Then draw CD parallel to OX upto D, a point above the mark 40. Thus go on drawing lines upto the last interval and you get PQ there. Join the pt. Q to the pt. marking 110. Now, the figure obtained with the horizontal line as its base, is the required Histogram of the given frequency distribution.

[**N. B.** (1) As there are no scores less than 20, the score 20 has been marked at the origin O. It may also be marked a little way off from O as shown before.

(2) The limits of the interval 20 – 29 are 19·5 to 29·5. Thus taking the limits of the intervals, we should have plotted the marks 19·5, 29·5, 35·5,......etc. ; but we generally mark 20, 30, 40,...... etc, as it is convenient to do so.

(3) Here as the st. lines CB, ED, etc. have not been produced to the marks 30, 40, etc., so the rectangles on the intervals have not been completed. If they be completed, the area of the figure in graph-3 is equal to the areas of the 9 rectangles so formed. Histogram is generally drawn by completing the rectangles.

Frequency Polygon

The frequency distribution of statistical data can also be represented graphically by a **frequency polygon.**

Method of its construction. On the graph paper draw the horizontal line OX and the vertical line OY. The quantitative values of the variate are generally marked along the horizontal line and the numerical values along the vertical line. Taking suitable units of length for measurement along these two lines, mark the quantitative values along the line OX and the numerical values along the line OY. Then beginning from the first, plot each point taking a quantitative value and its frequency as the co-ordinates of the point. Then place on OX the value just

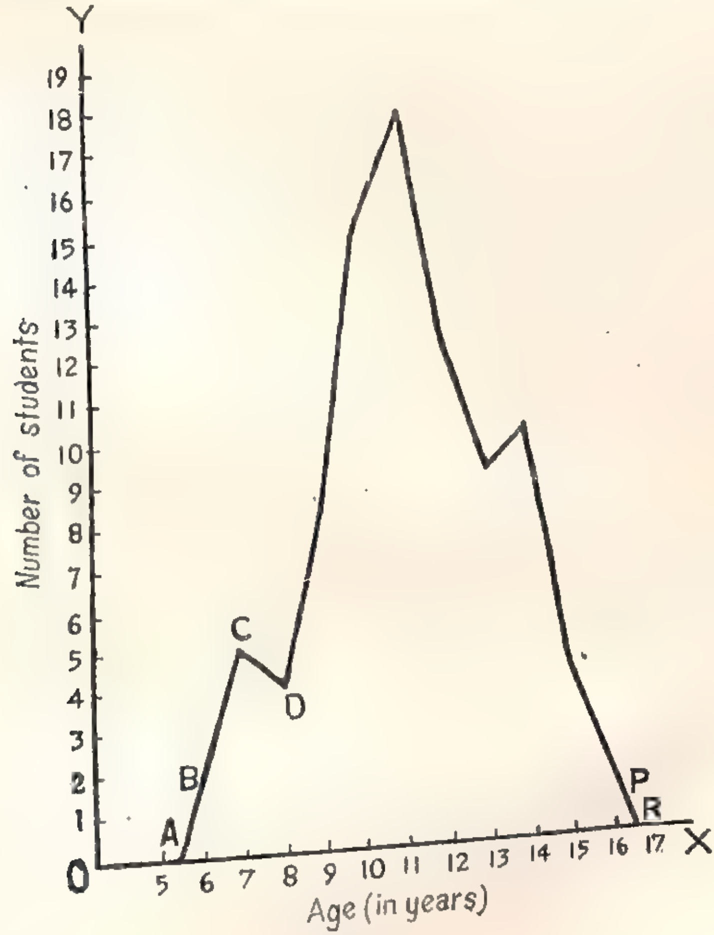
preceding the least quantitative value and find the mid-point of the length between these values. Similarly put the value next to the greatest quantitative value on OX and find the mid-point of the length between these values. Then join the points continuously one after another by drawing st. lines beginning from the first point to the last (*i.e.*, join the first point to the second point, the second point to the third point, and so on). Now, the polygon bounded by those straight lines and the horizontal line is the required **frequency polygon.**

Example 1. Below is given a table showing the different ages (correct to the whole number of years) of students of a school. Represent it graphically by a frequency polygon.

Age	6	7	8	9	10	11	12	13	14	15	16
No. of students	2	5	4	8	15	18	12	9	10	4	1

OX and OY are the horizontal and vertical lines on the graph paper (see graph 4). Let two sides of a small square measured along OX represent one year and similar two sides measured along OY represent 1 student. Then plot the points having co-ordinates (6, 2), (7, 5), (8, 4),......., (16, 1), each to represent an age and its frequency. Then put the values 5 and 17 on OX, one being the age just preceding the minimum age 6 and the other being the age next to the maximum age 16. Mark the mid-point of the length between the values 5 and 6 by A and the mid-point of the length between 16 and 17 by R. Now, join the points beginning from the first point A upto the last point R by drawing st. lines AB, BC,......,PR.

Thus the polygon bounded by these st. lines and the horizontal line AR is the required **frequency polygon.**



[Graph 4]

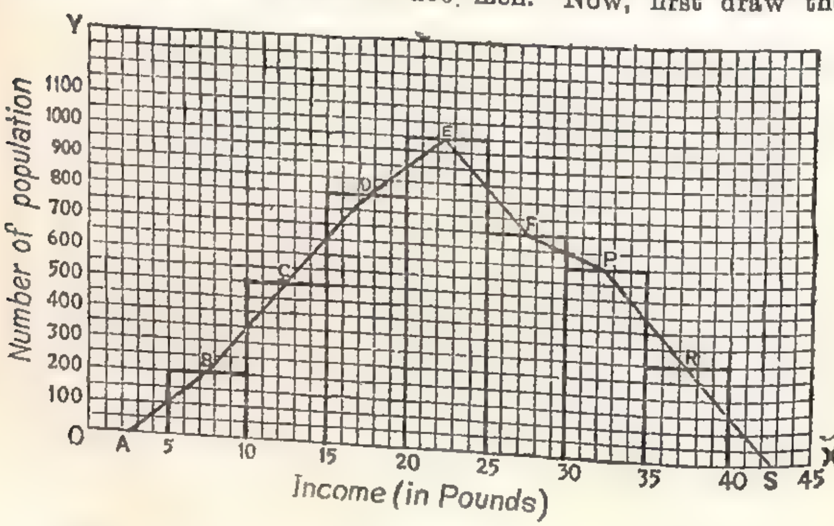
Construction of Histogram and Frequency polygon in the same diagram.

If the histogram is first drawn to represent a given frequency distribution and then middle points of its horizontal sides are joined in order by straight lines the frequency polygon of the distribution is obtained.

Example 2. The frequency distribution of monthly incomes of some men is given below. Draw in the same diagram both the histogram and the frequency polygon to represent it.

Income in Pound	£ 5 and under £ 10	£ 10 and under £ 15	£ 15 and under £ 20	£ 20 and under £ 25	£ 25 and under £ 30	£ 30 and under £ 35	£ 35 and under £ 40
No. of men	200	500	800	1000	700	600	800

The horizontal line OX and the vertical line OY are taken on the graph paper (Graph 5). Let one side of a small square measured along OX represent one pound and two such sides measured along OY represent 100 men. Now, first draw the



[Graph 5]

histogram of the frequency distribution according to the method previously shown. Then mark the middle points of its horizontal sides by B, C, D,...,R. Now mark the middle point of the interval just preceding the interval 5 – 10 by A and the middle point of the interval next to the interval 35 – 40 by S. Then join the points from A to S in order by the st. lines AB, BC,......PR, RS. The figure formed by these st. lines and the st. line AS is the required frequency polygon drawn in the same diagram.

[N. B. Notice in graph-5 that the areas of the histogram and the frequency polygon are equal. Here eight triangles included in the histogram are found to be outside the frequency polygon, but eight triangles outside the histogram have fallen within the frequency polygon. It can be proved geometrically that each internal triangle is equal in area to each external triangle attached to it. Consequently the histogram is equal in area to the frequency polygon. Sometimes to make a comparative analysis the histogram and the frequency polygon are drawn in the same diagram.

Frequency Curve

If in example-2 the two points A and S at the two ends of the intervals $5-10$ and $35-40$ are not taken and only the middle points from B to R of the sides of the histogram parallel to OX are joined in order, the curved line thus obtained is called a **Frequency curve.** It is not a closed curve.

Comparison of Frequency Distributions

In order to make a comparative study of two or more frequency distributions, it is necessary to draw their frequency curves in the same diagram.

Important statistical results are often derived from such comparative studies.

Ogive

The graph drawn to represent a cumulative frequency distribution is called an **Ogive.**

Example : Draw up an Ogive from the following frequency distribution of the scores of 65 students :

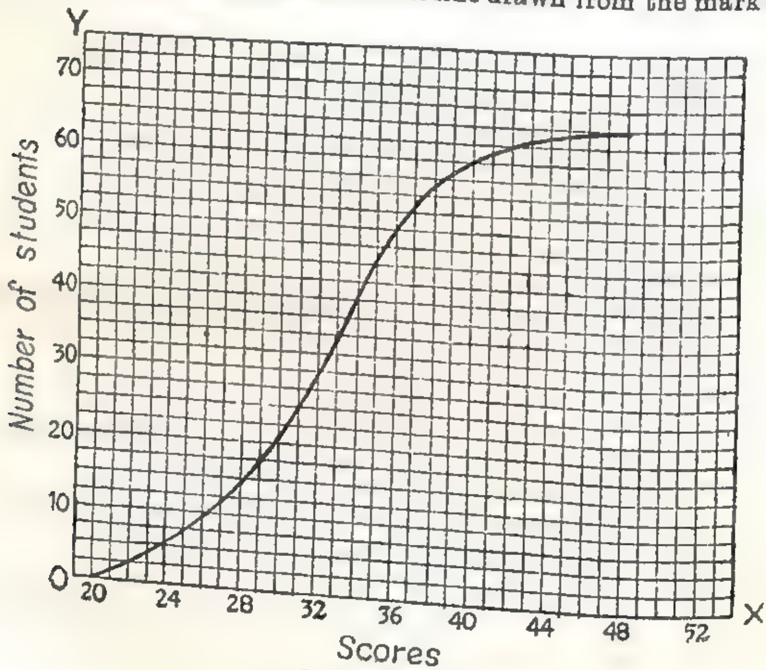
Scores p. c.	20 and under 24	24 and under 28	28 and under 32	32 and under 36	36 and under 40	40 and under 44	44 and under 48
Number of students	5	10	15	20	10	3	2

First the cumulative frequency distribution is to be made from the given frequency distribution and then a frequency curve

is to be drawn from the cumulative frequency distribution, which is given below..

Number p. c.	under 24	under 28	under 32	under 36	under 40	under 44	under 48
Cumulative frequency	5	15	30	50	60	63	65

[See graph 6] Now draw the horizontal line OX and the vertical line OY on the graph paper and choosing suitable units mark off the numbers 20, 24, 28, etc. along OX to represent the scores and 10, 20, 30, etc. along OY to represent the number of students. Here no student obtained scores below 20, so the first point is plotted 0 unit up the vertical line drawn from the mark 20,



[Graph 6]

i.e., on OX. The number of students obtaining scores under 24 is 5, so the second point is plotted 5 units up the vertical line from the mark 24. Thus the other points are plotted. Now joining the points in order beginning from the first point we obtain the required Ogive.

Exercise 2

1. The following table shows the percentage of successful candidates in the periodical examinations for four consecutive years in a school. Draw up a line graph to represent the average percentage of passes at different periods.

Years	1st term	2nd term	3rd term	final
1956	75%	72%	83%	91%
1957	62%	58%	79%	86%
1958	81%	70%	63%	76%
1959	59%	68%	74%	88%

2. 20% of the total population of the world are Americans, 25% Europeans, 35% Asians, 5% Africans and 15% Australians. Express the statement in pictures.

3. 40% of the students in a school are Hindu, 35% Mahomedan, and 25% Christain. Draw a pie chart to express this.

4. The numbers of students in the first six classes of a school are 70, 60, 50, 35, 45 and 40 respectively. Express this by vertical rectangles.

5. Draw up a column graph and a pie chart to represent the following data: The numbers of students present from Monday to Saturday in a School are 79, 85, 63, 48, 72 and 91.

6. The monthly incomes of labourers in a factory are given below. Draw a histogram and a frequency polygon to represent the table.

Income (in rupees)	30	35	40	45	50
No. of labourers.	10	15	20	12	8

7. The following table shows the percentages of marks obtained in an examination by the students of a class.

Marks Obtained	45%	50%	55%	60%	65%	70%	75%
No. of students	8	10	12	20	15	9	6

From the table draw up in the same diagram a histogram and a frequency polygon.

8. The following is the table of monthly salaries of 54 men.

Rs. 85 and under Rs. 89	Rs. 89 and under Rs. 93	Rs. 93 and under Rs. 97	Rs. 97 and under Rs. 101	Rs. 101and under Rs. 105	Rs. 105and under Rs. 109	Rs. 109and under Rs. 113
4	7	10	15	8	6	4

Draw up in the same diagram a histogram and a frequency polygon to represent the above statement.

9. Below is given the frequency distribution of the weight (in kilograms) of some students. Draw a frequency curve from it.

Weight (kilogram)	No. of students
24 and under 28	1
28 ,, ,, 32	3
32 ,, ,, 36	5
36 ,, ,, 40	9
40 ,, ,, 44	15
44 ,, ,, 48	10
48 ,, ,, 52	8
52 ,, ,, 56	4
56 ,, ,, 60	2

10. Draw up a cumulative frequency distribution from the frequency distribution of the above question 9 and draw an Ogive therefrom.

11. Draw the cumulative curve (Ogive) of the following cumulative distribution table of incomes :

Incomes less than (rupees)	Cumulative $f(X)$	Per cent of total number
200	10	·006
300	80	·044
400	640	·354
500	1170	·646
600	1450	·801
700	1600	·884
800	1710	·945
900	1747	·965
1000	1769	·977
1100	1785	·986
1200	1797	·993
1300	1805	·997
1400	1810	1·000

Find from the curve what the income is when the cumulative number of incomes is 905.

12. The following table shows the numbers of boys and girls of different ages of two schools. From it make a cumulative frequency distribution table and therefrom draw the Ogive.

Age (in years)	Under 10	Under 12	Under 14	Under 15
No. of boys	60	65	75	80
No. of girls	10	15	20	25

AVERAGE

You know what an *average* is. This *average* is very essential in statistics.

If the number of collected data be very large, it becomes difficult to derive a theory by examination or comparative analysis of each datum. So we study the characteristics of the

datum which is a representative of those data to form an idea of the characteristics of the other data. It is required to find an average to ascertain this representative datum. This average helps us to acquire knowledge of the aggregate from that of an individual.

Suppose, we are to form an idea of the height of boys aged 10 years of a country. For this if we have to collect the heights of all boys 10 years old of that country to form an idea of their height, it will require much time and hard labour. If we can somehow know the height of only one boy who is the representative sample of those boys, we can easily form a clearer idea about the height of all those boys.

Three kinds of averages of different values of a variate are generally used in statistics, viz., (1) Arithmetic mean or Mean, (2) Median and (3) Mode. Besides these, there are also two more kinds of average, viz., (4) Geometric mean and (5) Harmonic mean. These two averages are seldom used.

Arithmetic mean : Average is generally the Arithmetic mean. Arithmetic mean is briefly called *mean* or *average*. If the sum of several quantities of the same kind is divided by the number of the quantities, the quotient obtained is the average (**Arithmetic mean** or simply **mean**).

Suppose, you are to find the average of 15 quintals, 19 quintals and 20 quintals. Here 15 quintals, 19 quintals and 20 quintals are three quantities of the same kind. Their sum is $(15+19+20)$ quintals or 54 quintals. $\therefore$ the average of the three quantities is $(54 \text{ quintals} \div 3)$ or 18 quintals. Again, 18 quintals $\times 3 = 54$ quintals. $\therefore$ the average multiplied by the number of quantities is equal to the sum of the quantities.

Suppose, a man ran 17 kilometres in the first hour, 25 km. in the second hour and 30 km. in the third hour. The man ran $(17+25+30)$ or 72 km. in 3 hours. So his average speed is $\frac{17+25+30}{3}$ or 24 km. per hour. Here notice that the man runs in 3 hours at different speeds the same distance as he can run in 3 hours at the uniform average speed of 24 km. per hour. So here the 24 km. is a representative value or an average.

Formula for finding an Arithmetic mean : A formula can be deduced from the rule of finding an average. If a, b, c, etc. be N numbers of different quantities, their Arithmetic mean $=\frac{a+b+c+\ldots}{N}$.

Now, suppose that a, b, c, d,......etc. are the different values of the variate X. If N be the number of those values, S(X) the sum of those values and M_x or $\bar{X}$ (read as X-bar) their mean, then the formula stands thus : M_x or $\bar{X}=\frac{S(X)}{N}$......(1).

ΣX also denotes the sum of those values, so the formula may be written as $M_x=\frac{\Sigma X}{N}$.

Weighted mean

Arithmetic average is of two kinds, viz, simple average and weighted average.

We have discussed simple average above.

Weighted mean : In simple average the number of each kind of quantity is one, *e.g.*, *one* piece of cloth worth Rs. 10, *one* piece of cloth worth Rs. 15 and *one* piece of cloth worth Rs. 17. Here the number of each kind of cloth is only 1. But if it is required to find the average value of 3 pieces of cloth at Rs. 10 each, 5 pieces of cloth at Rs. 15 each and and 2 pieces at Rs. 17 each, then that average is called **weighted mean or weighted average.**

Here the word 'weight' does not mean actual *weight*, but it denotes the numerical strength of each kind of cloth.

Here, the price of 3 pieces of cloth at Rs. 10 per piece $=$Rs. 10×3, that of 5 pieces at Rs. 15 per piece$=$Rs. 15×5 and that of 2 pieces at Rs. 17 per piece$=$Rs. 17×2.

$\therefore$ the required weighted mean$=\frac{\text{Rs. } 10\times3+\text{Rs. } 15\times5+\text{Rs. } 17\times2}{3+5+2}$

$=$Rs. $\frac{30+75+34}{10}=$Rs. $\frac{139}{10}=$Rs. 13·9.

Formula : To find the formula for the weighted mean suppose the respective numbers (frequencies) of n different things are $f_1, f_2, f_3, \ldots\ldots, f_n$ respectively and their corresponding values

are $x_1, x_2, x_3, \ldots\ldots, x_n$ each respectively and let their required weighted mean be denoted by M_x.

$\therefore$ the formula will be

$$M_x = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots\ldots + f_n} = \frac{S(fx)}{S(f)}$$

$$= \frac{S(fx)}{N} \text{ or } \frac{\Sigma fx}{N} \ldots\ldots(2).$$

Application of the formula : If as a result of observation the number of data collected be very large, their average is determined from their frequency distribution table. In forming such tables of the variate its different kinds of values are to be put one below another in one column of the table and their corresponding frequencies side by side in another column. In the third column the product of each value and its frequency is to be noted one under another. The table thus prepared is called the **frequency distribution table.**

Supppose, the variate is X, the frequency of each of its values is f and the product of each value and its frequency is denoted by fX.

It is evident that here the sum of all f's [*i.e.*, S(f)] is the same as the total number of values *i.e.*, N. We have already said that S or Σ denotes the sum.

Now, if we divide the sum of the column fX, *i.e.*, S(fX) by the sum of the column f, *i.e.*, by N, the quotient obtained is the required average.

$$\therefore \quad M_x \text{ or } \overline{X} = \frac{S(fX)}{N} \ldots\ldots(3).$$

Example 1. Calculate the mean of the following observations by making a frequency distribution table :

18, 22, 20, 19, 24, 26, 18, 20, 19, 25, 24, 20, 24, 26, 25, 20, 25, 26, 19, 20, 25, 19, 25, 24.

Different values X	No. of values f	Product of both fX
18	2	36
19	4	76
20	5	100
22	1	22
24	4	96
25	5	125
26	3	78
	24	533

The frequency distribution table is shown aside.

From it we have

$S(fX) = 533$

$N \;\; = 24$

$\therefore \quad \overline{X} = 533 \div 24 = 22{\cdot}2.$

Example 2. Find the average rate of interest on the following outstanding loans: £ 1050 at 4%, £ 5324 at $3\frac{1}{2}$%, £ 1254 at 3% and £ 3780 at $3\frac{1}{4}$%.

Rate of interest %	Amount of loan £	Product £
4	1050	4200
$3\frac{1}{2}$	5324	18634
3	1254	3762
$3\frac{1}{4}$	3780	12285
	11408	38881

$\therefore$ the average rate of interest $=\frac{38881}{11408}\% = 3{\cdot}4\%$.

Example 3. There was an average rainfall of 2·75″ in a week in a certain country. The average rainfalls for the first six days were 2·25″, 3·5″, 2″, 2·5″, 3·25″, 4″ respectively. Find the rainfall on the seventh day ?

Suppose, there was rainfall of x inches on the seventh day.

$\therefore$ the average rainfall for 7 days

$$=\frac{2{\cdot}25+3{\cdot}5+2+2{\cdot}5+3{\cdot}25+4+x}{7}=\frac{17{\cdot}5+x}{7}$$

$\therefore$ here $\frac{17{\cdot}5+x}{7}=2{\cdot}75$, or, $17{\cdot}5+x=2{\cdot}75\times7=19{\cdot}25$,

$\therefore$ $x=19{\cdot}25-17{\cdot}5=1{\cdot}75$.

Hence, there was 1·75″ rainfall on the seventh day.

Miscellaneous Solutions

The method of finding out an Arithmetic mean by means of an assumed mean is shown here. In many cases it is easy to find out the average form an assumed mean.

Assumed mean : In many cases where the actual average is a mixed fraction, we take its nearest average for the sake of convenience. This adopted average is called *assumed mean.*

In the previous example-1, the actual average is 22·2. In this case we may take 22 as the assumed mean.

Deviation. The difference between each value of the quantity and the assumed mean is called **deviation.** Deviation is denoted briefly by d.

Example 4. Find the average of 434 and 443.

Here, the arithmetic mean (as calculated by the previous rule) $=(434+443)\div 2=438·5$.

[Alternative method] suppose, we take 438 between the two numbers as the assumed mean.

Now, $\left.\begin{array}{l} 434-438=-4 \\ 443-438=\ \ 5 \end{array}\right\}$ deviations from the assumed mean,

$\therefore$ the required arithmetic mean

=assumed mean+average of the two deviations

$=438+\dfrac{-4+5}{2}=438+·5=438·5.$

[**N. B.** The Algebraical sum of the deviations of several quantities from their real average is zero. In the above example the deviations of the two given quantities from the real average (438·5) is $-4·5$ and $+4·5$; $\therefore$ their sum$=-4·5+4·5=0$.]

Example 5. The following table shows the weights of students of 5 classes. Find their average weight from the table.

Weight (in. kg.)	35	38·25	39	40	42·5
No. of students	18	24	16	22	20

Suppose the assumed mean to be 39 kg.

1 Every weight (Kg.) x	2 Number of students f	3 Deviation (every wt.— assumed mean) d	4 Products of f and d fd	
			Positive	Negative
35	18	$35-39=-4$		-72
38·25	24	$38·25-39=-·75$		-18
39	16	$39-39=0$		
40	22	$40-39=1$	22	
41·5	20	$41·5-39=2·5$	50	
	Total = 100		72	-90 $+72$
				-18

$\therefore$ the required average $=(39+\frac{-18}{100})$ kg.

$=(39-·18)$ kg. $=38·82$ kg.

[N. B. In cases of a very large number of data, this method is adopted to find out the average.]

To find the arithmetic mean from another kind of table

Example 6. The following table shows the frequency distribution of the quantities of coal (in maunds) dug up daily for 124 days from a coal mine, their range being 94 maunds to 125 maunds. Find the arithmetic mean from the table.

Table No 9

Weight (in maunds)	frequency (f)
94— 97	2
98—101	5
102—105	12
106—109	14
110—113	20
114—117	16
118—121	30
122—125	25

Here, let 111·5 (from the interval 110 – 113) be the assumed mean. Now,

Interval	Frequency f	Mid point x	Mid point-assumed mean d	$f \times d$ Positive	$f \times d$ Negative
94– 97	2	95·5	–16		–32
98–101	5	99·5	–12		–60
102–105	12	103·5	– 8		–96
106–109	14	107·5	– 4		–56
110–113	20	111·5	0		
114–117	16	115·5	+ 4	+ 64	
118–121	30	119·5	+ 8	+240	
122–125	25	123·5	+12	+300	
	Total=124			+604 – 244 360	–244

∴ the reqd. mean $=111·5+\frac{360}{124}=111·5+2·9=114·4$ maunds (App.).

[**N. B.** **(Another method)** : In the above example multiply the average of the values in each interval by its frequency and note the product in another column. Then find the sum of all the products obtained from all the intervals. Now the required average is also obtained by dividing the sum of those products by the sum of all the frequencies. If the frequencies be $f_1, f_2, f_3, \ldots, f_n$ and the corresponding means of the intervals be $m_1, m_2, m_3, \ldots, m_n$ respectively, then

$$\text{M (mean)}=\frac{f_1m_1+f_2m_2+f_3m_3+\cdots+f_nm_n}{(f_1+f_2+f_3+\cdots+f_n=\text{N})}.$$

The above method is easier than this method.

Median

There is another kind of average called **Median.**

If some odd number of quantities be given and they be arranged in the ascending or descendng order of magnitude, the quantity lying just in the middle is called the **median** of the given quantities.

The number of quantities on either side of the median is the same. Suppose, you are to find the median of Rs. 7, Rs. 8, Rs. 9, Rs. 10 and Rs. 11.

Here, the total number of quantities is 5, $\therefore$ $\frac{5+1}{2}$th or 3rd quantity lying just in the middle is the median. Mean and median of some quantities may not be equal. If the given quantities are in arithmetical progression, their mean and median are equal. So the median cannot be called the true average.

If the number of given quantities be even, none of them can be taken lying just in the middle. In such a case two consecutive quantities in the middle are taken to be the middle terms (quantities). There are equal number of quantities on either side of them. The average of these two middle terms (quantities) is the *median* of all the quantities. Suppose, you are to find the median of 2″, 3″, 4″, 5″, 6″, 7″, 8″ and 9″. There are 8 quantities here. $\therefore$ there are two middle quantities. $\frac{8}{2}$th and $(\frac{8}{2}+1)$th, i.e., the 4th and the 5th quantities are the two middle quantities. Here they are 5″ and 6″.

$\therefore$ their average $= \frac{1}{2}(5'' + 6'') = 5\cdot5$ inches.

$\therefore$ the required median $= 5\cdot5$ inches.

The given quantities should be first arranged in order of their magnitude to find out their median.

Determination of the Median

(1) If the given quantities are not arranged in order, they should be first arranged in the ascending or descending order of their magnitudes.

If the number of the quantities (i.e. N) be odd, then the quantity just lying in the middle $\left(\text{i.e. } \frac{N+1}{2} \text{ th quantity}\right)$ will be the median.

Again, if the number of the quantities (i.e. N) be even, then the average of the two consecutive quantities just in the middle will be the median, i.e., in this case the average of $\frac{N}{2}$th and $\left(\frac{N}{2}+1\right)$th quantities is the required median.

Example 1. Find the median of the measures 7, 9, 11, 13, 15, 17, 19.

Here, N=7 (odd). $\therefore$ $\frac{N+1}{2}$th term is $\frac{7+1}{2}$ or 4th term. Here, the 4th term is 13. $\therefore$ the required median=13.

Example 2. Find the median of the measures 10, 13, 16, 19, 22, 25, 28, 31.

Here, the number of terms=8 (even),

$\therefore$ the middle term is not one in number. They are two in number, viz., the $\frac{N}{2}$th and $\left(\frac{N}{2}+1\right)$th. terms. Putting the value 8 for N we have the 4th. and 5th. terms as the middle terms. So here the average of the 4th and 5th terms will be the median.

$\therefore$ the required median$=\frac{19+22}{2}=20{\cdot}5$.

[N. B.. It is seen from the above two examples that the number of quantities greater than the median is just equal to the number of quantities less than the median. If the series of the quantities be symmetrical, then the mean and the median will be equal.

Example 3. Find the median of the following observations: 25, 18, 22, 20, 19, 24, 26, 18, 20, 19, 24, 20, 24, 26, 25, 20, 19, 25, 26, 20, 25, 19, 25, 24.

Now, arranging the numbers in ascending order of magnitude we have 18, 18, 19, 19, 19, 19, 20, 20, 20, 20, 20, 22, 24, 24, 24, 24, 25, 25, 25, 25, 25, 26, 26, 26.

$\because$ here the number of quantities=24.

$\therefore$ the average of the 12th and 13th quantities will be the required median.

$\therefore$ the required median$=\frac{22+24}{2}=23$.

Formula for finding the median

Suppose, M_d=median, f_c=cumulative frequency up to the interval preceding the interval containing the median,

f_1=frequency of the interval containing the median,

L=Lower limit of the interval,

i=interval and N=total frequency.

$\therefore$ the formula stands thus : $$M_d=L+\frac{\frac{N}{2}-f_c}{f_1}\times i.$$

Example 4. Suppose, we are finding the median from the table No. 9.

Here N=124, $\therefore$ $\frac{N}{2}=62.$

Here, the sum of the first five frequencies is 53, which is less than 62 ; but the sum of the first six frequencies is greater than 62. $\therefore$ the median exists in the 6th interval (*i.e.*, in the interval 114 − 117), the lower limit of which is 113·5 and frequency is 16.

$\therefore$ $L=113{\cdot}5,\ \frac{N}{2}=62,\ f_c=53,\ f_1=16,\ i=4.$

$\therefore$ the required $$M_d=L+\frac{\frac{N}{2}-f_c}{f_1}\times i=113{\cdot}5+\frac{62-53}{16}\times 4$$

$$=(113{\cdot}5+2{\cdot}25) \text{ maunds}=115{\cdot}75 \text{ maunds (App.).}$$

A median is not so dependable as a mean. It is being explained by an example.

Suppose, the series 3, 5, 7, 9, 11, 13, 15 is taken. It being symmetrically extended its median is the 4th term 9. Again, its mean is (sum of the terms÷7), *i.e.* 9. Now, if I take at random three other numbers greater than 9 in place of the given numbers greater than 9, then the mean will change, but not the median.

Suppose, we take the series 3, 5, 7, 9, 14, 16, 23 changing the three numbers following 9 of the above series. There being 7 terms, the median is the 4th term 9. $\therefore$ the median remains the same, though the values of the terms are changed. But their

average or mean is $(3+5+7+9+14+16+23)\div 7$, *i.e.*, 11. So it is said that a median is not so dependable.

Example. 5. Find the median from the following frequency distribution :

Interval	frequency
70—74	3
75—79	4
80—84	7
85—89	10
	24
90—94	15
95—99	8
100—104	6
105—109	4
110—114	3
	N=60

Here, the sum of frequencies $N=60$, $\frac{N}{2}=30$. Now, the sum of the frequencies of the first four intervals is 24, which is less than 30. But the sum of the first five frequencies is greater than 30. $\therefore$ the required median exists in the interval 90 − 94, the limits of which are 89·5 − 94·5. $\therefore$ the lower limit of this interval is 89·5. Here, $f_c=24$, $f_1=15$, $\frac{N}{2}=30$, $i=5$.

$$\therefore \text{ the required median} = 89{\cdot}5+\frac{\frac{N}{2}-f_c}{f_1}\times i = 89{\cdot}5+\frac{30-24}{15}\times 5$$
$$=89{\cdot}5+2=91{\cdot}5.$$

Example 6. The following table shows the frequencies of scores of 76 students in an examination. Find their median.

[D. U. 1961]

Scores	0—10	10—20	20—30	40—30	40—50
frequency	4	8	12	32	20

By the formula, the median $= L + \dfrac{\frac{N}{2} - f_0}{f_1} \times i$

Here $N = 76$, $\therefore$ $\dfrac{N}{2} = \dfrac{76}{2} = 38$; and $i = 10$.

Here the sum of the frequencies of the first three intervals is $(4+8+12)$ or 24, which is less than 38. Again, the sum of the frequencies of the first four intervals is 56, which is greater than 38. $\therefore$ the median exists in the fourth interval. The limits of that interval are $29{\cdot}5 - 40{\cdot}5$. Now, $L = 29{\cdot}5$, $f_0 = 24$, $f_1 = 32$.

$\therefore$ the required median $= 29{\cdot}5 + \frac{38-24}{32} \times 10 = 29{\cdot}5 + \frac{35}{8}$
$= 29{\cdot}5 + 4{\cdot}375 = 33{\cdot}875.$

Example 7. Find the median from the following frequency distribution table :

Interval	frequency
92·5—97·5	4
87·5—92·5	6
82·5—87·5	12
77·5—82·5	19
72·5—77·5	$\boxed{37}$
67·5—72·5	24
62·5—67·5	11
57·5—62·5	6
52·5—57·5	4
47·5—52·5	2
	N = 125

Here, $i = 5$, $N = 125$, $\therefore$ $\dfrac{N}{2} = 62{\cdot}5$.

The sum of the frequencies of the five intervals beginning from the bottom is 47, $\therefore$ the median exists in the interval $72{\cdot}5 - 77{\cdot}5$.

$\therefore$ the required median $= 72{\cdot}5 + \left(\dfrac{62{\cdot}5 - 47}{37}\right) \times 5 = 74{\cdot}6$ (App.).

Example 8. The monthly wages of the labourers of a factory are given below. Find the median of their incomes.

Rs. 20 and below Rs. 25	245	labourers.
Rs. 25 " " Rs. 30	322	"
Rs. 30 " " Rs. 35	525	"
Rs. 35 " " Rs. 40	275	"
Rs. 40 " " Rs. 45	230	"
	Here $N=1597$	

$\therefore$ the $\frac{1597+1}{2}$th. or 799th. term is the median. The number of labourers in the first two intervals$=567$, $\therefore$ the median exists in the interval $30-35$. The lowest income of that interval is Rs. 30, number of labourers is 525 and interval is 5.

$\therefore$ the required median$=$Rs. $30+$Rs. $\frac{799-567}{525}\times 5$

$=$Rs. $30+$Rs. $2\cdot2095=$Rs. $32\cdot2$ (approximately).

To find the median graphically

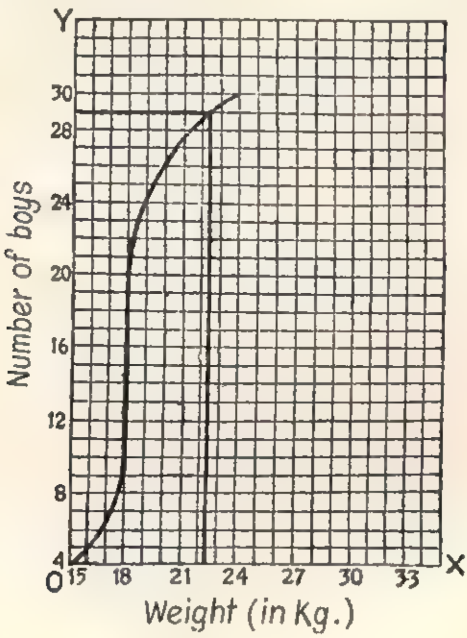
Example 9. Below is given the list of weights of 57 boys. From it draw up a cumulative frequency distribution table and then find the median of the given weights by drawing a graph.

Weight (in kg.)	No. of boys (f)
12 and below 15	4
15 " " 18	12
18 " " 21	12
21 " " 24	2
24 " " 27	15
27 " " 30	8
30 " " 33	4
Total$=$	57

Solution : The cumulative frequency distribution of the given table is drawn up below :—

Weight (in kg.)	Cumulative frequency
Under 15 kg.	4
" 18 "	16
" 21 "	28
" 24 "	30
" 27 "	45
" 30 "	53
" 33 "	57

Here, the total number of boys $=57$. $\therefore$ the middle number of the total boys is $\frac{57+1}{2}$ or 29 and the 29th term (the median) exists in the interval 21—24, so it will be enough to draw the graph only up to that interval. Now draw the graph taking the units of weight measured along the x-axis and those of the



(Graph 7)

numbers of boys measured along the y-axis. Here, the weight of the 29th boy, will be the required median. $\therefore$ find from the graph the value of x corresponding to the value 29 of y. From the graph it is seen that $x=22{\cdot}5$, when $y=29$. [See Graph 7]

$\therefore$ the required median $=22{\cdot}5$ kg.

Mode

There is another kind of average called **Mode.** When the given measures are arranged in order of magnitude, the most common, i.e., the oft-recurring measure in the series is the mode.

Example. Suppose the mode of the numbers 9, 9, 11, 12, 12, 12, 13, 17, 18 is to be found. Here it is seen that 12 occurs most often (here 3 times), so the mode of the given numbers is 12.

If, each score occurs only once, there will be no mode, there being no oft-recurring score.

See again that if the three numbers preceding 12 be other than 9, 9, 11 (e.g. 7, 8, 10) or if the three numbers following 12 be other than 13, 17, 18 (e.g. 15, 19, 20) their mode is still 12, because in both cases 12 occurs most frequently.

Method of determination of Mode

(1) The mode can be determined by arranging the given measures according to their order of magnitude by the above method.

(2) The mode can be obtained by grouping the given measures.

(3) The mode can also be found by drawing a graph from the frequency distribution. When this graph is drawn, the position of the mode will be known from the highest point of the curve, the abscissa of the highest point (i.e. the pt. whose ordinate is maximum) on the curve will give the mode.

Formula for determination of mode

(1) Mode $(M_o)=L+\dfrac{f_2}{f_1+f_2}\times i.$

Here, $L=$ the lower limit of the interval in which the mode occurs,

$f_1=$ the frequency of the interval preceding the interval in which the mode exists,

$f_2=$ the frequency of the interval following the mode-interval,

$i=$ interval.

(2) Mode=mean − 3 (mean − median)=3 median—2 mean.
In brief, $M_o = 3M_d - 2M$.
(3) For an *alternative method* see Example 4.

Example 1. Find the mode from the following table of ages of 74 students in a class :

Number of students	10	12	14	18	11	5	4
Age (in years)	8	9	10	11	12	14	15

The required mode will be the age corresponding to the greatest number of boys. It is seen from the table that one group shows the greatest number of students to be 18 and the age of the boys in that group is 11 years,

∴ the required mode of ages=11 years.

Example 2. Find the mode of the given values of the table No. 9.

We have already found the mean of the given values in the table to be 114·4 and their median to be 115·75.

∴ the required mode = 3 median − 2 mean

$$= 115{\cdot}75 \times 3 - 114{\cdot}4 \times 2 = 347{\cdot}25 - 228{\cdot}8$$
$$= 118{\cdot}45 \text{ maunds (nearly).}$$

Example 3. Find the mode from the following table :

Interval	frequency	Interval	frequency
9·5—10·5	12	12·5—13·5	30
10·5—11·5	25	13·5—14·5	15
11·5—12·5	36	14·5—15·5	1

Solution. Here the mode occurs in the interval 11·5 − 12·5, of which L (the lower limit)=11·5, Here, $f_1 = 25$, $f_2 = 30$ and $i = 1$.

∴ the required mode $= L + \dfrac{f_2}{f_1 + f_2} \times i = 11{\cdot}5 + \dfrac{30}{25+30} \times 1$

$$= 11{\cdot}5 + \tfrac{6}{11} = 12{\cdot}05 \text{ (approximately)}$$

Example 4. The following table records the heights of 55 students. Find the mode of their heights from this table.

No. of students	4	7	10	15	8	6	5
Height (inch)	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55	55 – 60

Alternative method. It is seen from the table that the interval 40 – 45 has the greatest number of boys, *i.e.*, 15.

∴ the required mode occurs in that interval. The heights of that interval range from 40 inches to 45 inches.

The number of students in that interval is 15, that of the interval just preceding it is 10 and that of the interval just following it is 8. $15-10=5$, $15-8=7$.

The required mode lies between 40″ and 45″. Suppose, the mode is M and it is greater than 40″ and less than 45″.

∴ the difference of 40 from M and the difference of M from 45 are as 5 : 7.

$\therefore \quad \frac{M-40}{45-M}=\frac{5}{7}$, or, $7M-280=225-5M$. or, $12M=505$,

$\therefore \quad M=42{\cdot}08$.

∴ the required mode $=42{\cdot}08$ inches (nearly).

Example 5. The following data were collected regarding the number of children in different age-groups in a town. Find the mode of their ages.

Ages in years	Number of children
1	300
2	855
3	855
4	365
5	135
6	275

The given data can be arranged in different groups as follows :

Age	Number			
1	300	1155		2010
2	855		1710	
3	855	1220		
4	365		500	775
5	135	410		
6	275			

In the highest frequency 1220 of the first grouping, the values are 3, 4 ;

In the highest frequency 1710 of the second grouping, the values are 2, 3 ;

In the highest frequency 2010 of the third grouping, the values are 1, 2, 3.

Here, we find that 3 is common to all the biggest groups.

$\therefore$ the required mode=3 years.

Example 6. The frequencies of the intervals 0 – 8, 8 – 16, 16 – 24, 24 – 32, 32 – 40 and 40 – 48 are 8, 7, 16, 24, 15 and 7 respsctively. Find their mean, median and mode.

(*i*) To find the mean let the assumed mean be 28.

Interval	Frequency f	Mid point x	Mid point - assumed mean α	$f\times\alpha$ Negative	$f\times\alpha$ Positive
0 – 8	8	4	–24	–192	
8–16	7	12	–16	–112	
16–24	16	20	– 8	–128	
24–32	24	28	0		
32–40	15	36	8		+120
40-48	7	44	16		+112
	Total =77			–432 +232	+232
				–200	

$\therefore$ the required mean$=28+\dfrac{-200}{77}=28-2{\cdot}6=25{\cdot}4.$

(*ii*) Here $\frac{N}{2}=\frac{77}{2}$; $i=8$.

Here the sum of the first three frequencies is less tan $\frac{77}{2}$, but the sum of the first four frequencies is greater than $\frac{77}{2}$.

$\therefore$ the median exists in the fourth interval.

$\therefore$ $L=23{\cdot}5, f_o=31, f_1=24.$

$\therefore$ the required median$=23{\cdot}5+\dfrac{\frac{77}{2}-31}{24}\times 8=23{\cdot}5+2{\cdot}5=26.$

(*iii*) Now, the required mode$=3$ median $-$ 2 mean
$=26\times 3-25{\cdot}4\times 2=78-50{\cdot}8=27{\cdot}2.$

Example 7. The intervals are $6{\cdot}5-7{\cdot}5$, $7{\cdot}5-8{\cdot}5$, $8{\cdot}5-9{\cdot}5$, $9{\cdot}5-10{\cdot}5$, $10{\cdot}5-11{\cdot}5$, $11{\cdot}5-12{\cdot}5$ and $12{\cdot}5-13{\cdot}5$ and their frequencies are 5, 12, 25, 48, 32, 6 and 1 respectively. Find their mean, median and mode.

(*i*) To find the mean let the assumed mean be 11.

Interval	Frequency f	Mid point x	Mid point - assumed mean d	$f\times d$ Positive	$f\times d$ Negative
6·5 − 7·5	5	7	−4		−20
7·5 − 8·5	12	8	−3		−36
8·5 − 9·5	25	9	−2		−50
9·5 − 10·5	48	10	−1		−48
10·5 − 11·5	32	11	0		
11·5 − 12·5	6	12	1	+6	
12·5 − 13·5	1	13	2	+2	−154
	N=129			+8	+ 8
					−146

Now, the required mean$=11+\dfrac{-146}{129}=11-1{\cdot}13=9{\cdot}87.$

(*ii*) Here $N=129$, $\therefore$ $\dfrac{N}{2}=\dfrac{129}{2}=64{\cdot}5$, $i=1$;

$\because$ the sum of the first three frequencies is 42,

$\therefore$ the median exists in the fourth interval.

$\therefore$ $L=9{\cdot}5$, $f_o=42$ and $f_1=48$.

$\therefore$ the required median$=L+\dfrac{\frac{N}{2}-f_o}{f_1}=9{\cdot}5+\dfrac{64{\cdot}5-42}{48}\times 1$

$=9{\cdot}5+\dfrac{22{\cdot}5}{48}=9{\cdot}5+{\cdot}47=9{\cdot}97.$

(*iii*) The required mode$=$mean $-$ 3 (mean $-$ median)
$=9{\cdot}87-3(9{\cdot}87-9{\cdot}97)=9{\cdot}87+{\cdot}3=10{\cdot}17.$

Determination of mode graphically

Like the determination of the median from the graph the mode also can be found graphically from the frequency distribution. First draw up the frequency distribution table of the given data. Then from it draw a graph which will be a curve. The abscissa of the highest point on the curve (graph) will be the mode.

[N. B. We have so long discussed three kinds of average or mean. Besides these there are two other kinds of mean, viz., Geometric mean and Harmonic mean, which being out of syllabus are not discussed here.]

Exercise 3

1. Find the means of the following :—
(*i*) 325, 927, 630, (*ii*) $13\frac{3}{4}$, $19\frac{3}{8}$ and $21\frac{7}{16}$, (*iii*) 107, 210·24, 90·06, 51·2, 112·75.

2. (*a*) Find the mean, the median and the mode of the following :—

21, 33, 27, 23, 24, 32, 28, 24, 27, 22, 27.

(*b*) In the following table are given the market prices of rice per maund in a week :

Week days	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Market price of rice per maund	28·4	28·7	29·1	29·5	28·9	28·8	27·9

Find the (1) m[illegible] the median and (3) the mode of the prices of rice per ma[illegible] he week.

3. What is the median of the first n natural numbers ?

4. If I buy 12 books at Rs. 15 per copy, 10 books at Rs. 12.50 per copy and 8 books at Rs. 13.25 per copy, what is the average price of each copy ?

5. Find the mean of 328, 332, 337 and 340 taking 334 as the assumed mean.

6. The following table gives the weekly incomes of 25 men. Find the mean of their weekly incomes from this table.

Income (in Rupees	19	20	21	22	23	24	25
Number of men (f)	1	3	5	7	6	2	1

7. Find the mean, the median and the mode of the scores from their frequency distribution table given below :—

x ...	f	x ...	f	x ...	f
90 – 93	2	110 – 113	24	130 – 133	18
94 – 97	4	114 – 117	27	134 – 137	13
98 – 101	7	118 – 121	35	138 – 141	6
102 – 105	12	122 – 125	26	142 – 145	5
106 – 109	19	126 – 129	21	146 – 149	2
				150 – 153	1

8. Below are given scores of 50 students in an examination :
41, 49, 41, 35, 35, 55, 50, 38, 49, 46, 28, 64, 48, 31, 45, 49, 66, 46, 21, 27, 50, 49, 26, 39, 52, 48, 44, 50, 49, 45, 61, 49, 38, 41, 50, 52, 32, 35, 44, 61, 39, 52, 47, 29, 48, 40, 44, 29, 50, 40.

(a) What is the mean of the first 25 scores ?
(b) Find the mean of the last 26 given scores.
(c) Find the median of the given scores.
(d) Find the mode of the given scores.

9. Find the Algebraical sum of the deviations of the numbers 25, 29, 31, 37, 43 from their mean.

10. The following table records the age-groups of the children in a town. Find from it the mode of their ages.

Age (year)	1	2	3	4	5	6
No. of children	300	855	855	365	135	275

11. The frequency distribution of properties for which taxes ranging from £15 to £55 (correct to the nearest pound) may be levied. Find from this the median of the taxes levied.

Leviable taxes				Number of properties
		Not exceeding	£15	276
above	£15	but not exceeding	£20	843
"	£20	" "	£25	1007
"	£25	" "	£30	1984
"	£30	" "	£35	3010
"	£35	" "	£40	2476
"	£40	" "	£45	1540
"	£45	" "	£50	781
"	£50	" "	£55	352

12. Two horses run in consecutive 5 hours 18, 19, 20, 16, 13 Km. and 26, 16, 11, 27, 5 Km. respectively. Find from the means of their speeds which horse runs faster.

13. Find the median and the mode of the percentage scroes in an examination of 115 boys from the following table ;

Mark	Between 53 – 55	Between 55 – 57	Between 57 – 59	Between 59 – 61	Between 61 – 63
No. of students	15	25	35	23	17

14. The intervals of heights of some boys correct to the nearest inches are 35 – 37, 38 – 40, 41 – 43, 44 – 46, 47 – 49, 50 – 52 and their frequencies are 2, 3, 5, 10, 7, 2 respectively.

Draw an Ogive of the heights and determine the mode therefrom.

15. Percentages of deaths in some towns in a year are given below. Find from them the average percentage of death rate of each town.

Rate p.c. of death	5	6·5	7·5	9	10	11·5	12	17
No. of towns	18	12	15	8	6	7	14	16

16. The following table gives the scores of 33 students in an examination. Find the median of the scores graphically.

Marks obtained	Between 55 – 60	Between 60 – 65	Between 65 – 70	Between 70 – 75	Between 75 – 80
No. of students	4	6	15	5	3

Scatter or Dispersion

We have already said that the mean is the *representative* of different data, i.e., we can form a general idea of each datum to some extent from the mean. But the mean does not help us form the complete idea thereof.

Scatter or Dispersion :

There may be no equal scatter or dispersion or symmetry among the collected mass of data or different values of any variate. Two different series of data may have their means equal ; but the deviations of the values from their mean in one series may not be equal to those in the other series. Unless the data of each of the two series be equally scattered, the mean deviations of the two series cannot be equal. So the mean deviations of the two series of data depend on the dispersion of their values.

Example : Suppose, the daily incomes of Ram are Rs. 60, Rs. 63, Rs. 65, Rs. 72, Rs. 89, Rs. 100 and Rs. 76 respectively in a week and those of Hari are Rs. 73, Rs. 65, Rs. 70, Rs. 62, Rs. 112, Rs. 80 and Rs. 63 respectively.

Here the average daily income of Ram

$=\frac{1}{7}$ (Rs. 60+Rs. 63+Rs. 65+Rs. 72+Rs. 89+Rs. 100+Rs. 76)

=Rs. 75, and the average daily income of Hari

$=\frac{1}{7}$ (73+65+70+62+112+80+63) rupees=Rs. 75.

So we find that though the average daily incomes of Ram and Hari are equal, the ranges or dispersions of their daily incomes are not similar or equal and hence the deviations of their daily incomes from their averages also are not equal.

Range : Range represents the difference between the greatest value and the least value of a variate. In the above example the range of Ram's income=Rs. (100 – 60)=Rs. 40, and the range of Hari's income=Rs. (112 – 62)=Rs. 50. Here notice that no correct comparison of their incomes can be made merely from the knowledge of their ranges.

Deviation

The difference of each value of a variate from the mean or average of its different values is known as **deviation.**

It has already been shown that though two groups of data may have equal means, yet they may not have equal deviations. In the same way it may so happen that though they may have equal deviations, yet they may not have equal means.

In Statistics the determination of this deviation is essential to determine the accuracy of the mean (*i.e.*, to find if the extent of error is as little as possible). To determine this we have to note how the different values of any variate are scattered about their mean.

Deviation can be determined by taking the differences of the values from their mean, median or mode ; but it is convenient to find it from Arithmetic average or mean.

This deviation is of two kinds—(*i*) **Mean deviation** or **Average deviation** and (*ii*) **Standard deviation.**

Mean deviation

Mean deviation. Mean deviation is the arithmetic mean of all the deviations (independent of signs) from the average (mean, median or mode) of the given values.

Determination of mean deviation

First find the mean of the given data and then note down the differences of each datum from the mean. Then find the sum of those differences (ignoring their positive or negative signs). Then the quotient obtained from the division of the sum of those differences by the total number of data (N) will give the required mean deviation or **mean variation.** It is also called **average variation.**

It has been already said that mean deviation can be calculated in the same way by writing down the differences of each datum from the median or mode of all the given data.

Example 1. Find the mean deviation of the following measures :

24, 20, 22, 23, 21, 19, 22, 23, 20, 23, 22, 20, 22, 25, 21, 22, 21, 24, 23, 21, 22, 24, 22, 21, 23.

Here, the sum of the measures=550 and the total number of measures=25.

$\therefore$ their mean$=\frac{550}{25}=22$.

Now, the differences of the different measures from the mean are 2, -2, 0, 1, -1, -3, 0, 1, -2, 1, 0, -2, 0, 3, -1, 0, -1, 2, 1, -1, 0, 2, 0, -1, 1 respectively.

The sum of those differences (ignoring the positive or negative signs)=28.

Here N=25. $\therefore$ the required mean$=\frac{28}{25}=1{\cdot}12$.

[Here the mean deviation can also be determined from the median and the mode.]

Example 2. The scores of 4 boys in an examination are 5,7,9 and 11 respectively. Find the mean deviation from it. [D.U. '61]

We prepare the following table :

Score (1)	No. of boys (2)	Assumed mean (3)	Deviations from mean 9 taken positively (4)	Total deviation (2)×(4)
5 7 9 11	1 1 1 1	9	9 − 5=4 9 − 7=2 9 − 9=0 11 − 9=2	4×1=4 2×1=2 0×1=0 2×1=2
Total	(N)=4			8

Here, the total number of boys (N)=4, and the total deviation=8

$\therefore$ the required mean deviation$=\frac{8}{4}=2$.

Example 3. The frequency distribution of scores of 125 students in an examination is given below. Find the mean deviation from their median.

Scores	92	87	82	77	72	67	62	57	52	47	
No. of students	4	6	12	19	37	24	11	6	4	2	N=125

Let the assumed median be 72. Then the calculation may be made as given below :

Score (1)	Frequency (No. of students) (2)	Assumed median (3)	Deviations from the median free from signs (4)	Product of frequency and deviation (2)×(4)
92	4	72	92 − 72=20	20× 4= 80
87	6		87 − 72=15	15× 6= 90
82	12		82 − 72=10	10×12=120
77	19		77 − 72= 5	5×19= 95
72	37		72 − 72= 0	0×37= 0
67	24		72 − 67= 5	5×24=120
62	11		72 − 62=10	10×11=110
57	6		72 − 57=15	15× 6= 90
52	4		72 − 52=20	20× 4= 80
47	2		72 − 47=25	25× 2= 50
Total	N=125			835

∴ the required mean deviation$=\frac{835}{125}=6{\cdot}68$.

Standard Deviation

Standard deviation. The standard deviation (SD) is a measure of variability calculated around the mean.

It is the most stable measure of variability and commonly used in research problems.

The Greek letter σ (sigma) is the symbol for the SD (standard deviation).

The SD is computed in the following way :—

(*i*) First find the mean of the given scores.

(*ii*) Find the squares of all the deviations of the scores from the mean.

(*iii*) Divide the sum of those squares by the total number of scores (N).

(*iv*) Then find the square root of the quotient. This square root will be the required standard deviation.

Formula

(*i*) If M be the arithmetic mean of N numbers $a_1, a_2, \ldots, a_n$ then the standard deviation

$$\sigma = \sqrt{\left[\frac{(a_1 - M^2) + (a_2 - M)^2 + \cdots\cdots + (a_n - M)^2}{N}\right]}.$$

(2) Suppose, the different scores are denoted by X, their mean by M, any sum by S and the total number of scores by N.

Now $(X-M)$ will denote the deviation of each datum from the mean ; $\therefore$ the sum of the squares of all the deviations will be $S(X-M)^2$.

$\therefore$ the formula stands thus : $\sigma = \sqrt{\dfrac{S(X-M)^2}{N}}$ ......(6)

[**N. B.** The square of standard deviation is called variance.

$\therefore \quad \sigma^2 = \dfrac{S(X-M)^2}{N}$].

Example 1. Find the standard deviation of the following items : 4, 6, 9, 13, 19, 22, 23, 24, 25, 29, 32, 34.

Solution :

(Items) X	(Deviations from the mean) X − M	(Squares of deviations) $(X-M)^2$
4	−16	256
6	−14	196
9	−11	121
13	− 7	49
19	− 1	1
22	2	4
23	3	9
24	6	36
25	5	25
29	9	81
32	12	144
34	14	196
S(X)=240		1118

Here $N=12$, $\therefore\ M=\frac{240}{12}=20$. $\therefore\ S(X-M)^2=1118$,

$$\therefore\ \sigma=\sqrt{\frac{S(X-M)^2}{N}}=\sqrt{\frac{1118}{12}}=\sqrt{93.1\dot{6}}=9.65 \text{ (App.).}$$

If the value of N be very large and the mean be not an integral number, then it will be difficult to find the standard deviation by the above method. In that case the following formula can be applied.

Formula : $(\sigma)^2=\frac{S(X^2)}{N}-M^2$...(7), i.e., $\sigma=\sqrt{\frac{S(X^2)}{N}-M^2}$...(8)

Example 4. Calculate the standard deviation of the following observations : 1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15.

Solution

X	X^2
1	1
3	9
4	16
5	25
6	36
7	49
8	64
10	100
11	121
12	144
14	196
15	225
96	986

Here,

$S(X)=96$

$N=12$

$\therefore\ M=96\div12=8$

$$\therefore\ \sigma^2=\frac{S(X^2)}{N}-M^2$$

$$=\frac{986}{12}-8^2$$

$$=82.1\dot{6}-64=18.1\dot{6}$$

$$\therefore\ \sigma=\sqrt{18.1\dot{6}}=4.26.$$

[N.B. To find σ from a frequency distribution table adopt the formula : $\sigma^2=\frac{S(fX^2)}{N}-M^2$.]

Example 3. Find the standard deviation from the following frequency distribution of the ages of some boys.

Age (in years) (X)	1	2	3	4	5	6	7
Number of boys (f)	2	5	4	10	11	6	2

By adding 3 more columns of X^2, fX and fX^2 to the given table, we have

X	X^2	f	fX	fX^2
1	1	2	2	2
2	4	5	10	20
3	9	4	12	36
4	16	10	40	160
5	25	11	55	275
6	36	6	36	216
7	49	2	14	98
		N=40	169	807

Here $N = S(f) = 40$, $S(fX) = 169$, $S(fX^2) = 807$

$\therefore \quad M = \dfrac{S(fX)}{N} = \dfrac{169}{40}$.

$\therefore \quad \sigma^2 = \dfrac{S(fX^2)}{N} - M^2 = \dfrac{807}{40} - \left(\dfrac{169}{40}\right)^2 = 20{\cdot}175 - 17{\cdot}8506 = 2{\cdot}3244.$

$\therefore \quad \sigma = \sqrt{2{\cdot}3244} = 1{\cdot}524.$

Example 2. Find the standard deviation from the following frequency distribution of weekly incomes of 90 labourers.

Weekly income (in rupees)	5 – 7	8 – 10	11 – 13	14 – 16	17 – 19	20 – 22	23 – 25
No. of labourers	6	12	20	25	15	8	4

Solution : We get from the given table

Income (correct to rupees)	Frequency (f)	Midpoint (of interval X	X^2	fX	fX^2
5 – 7	6	6	36	36	216
8 – 10	12	9	81	108	972
11 – 13	20	12	144	240	2880
14 – 16	25	15	225	375	5625
17 – 19	15	18	324	270	4860
20 – 22	8	21	441	168	3528
23 – 25	4	24	576	96	2304
Total =	90			1293	20385

Now, $S(fX^2)=20385$, $N=90$, $M=\frac{fX}{N}=\frac{1293}{90}$

$\therefore$ σ (standard deviation)$=\sqrt{\frac{S(fX^2)}{N}-M^2}=\sqrt{\frac{20385}{90}-\left(\frac{1293}{90}\right)^2}$

$=\sqrt{226{\cdot}5-(14{\cdot}36)^2}=\sqrt{226{\cdot}5-206{\cdot}401}=\sqrt{20{\cdot}099}=4{\cdot}48\ldots$

Example 5. Find the standard deviation from the following frequency distribution. [D. U. 1921]

Interval	0 – 4	4 – 8	8 – 12	12 – 16
Frequency	4	8	2	1

Solution from the given table :

Interval	frequency f	midpoint X	X^2	fX	fX^2
0 – 4	4	2	4	8	16
4 – 8	8	6	36	48	288
8 – 12	2	10	100	20	200
12 – 16	1	14	196	14	196
Total =	15			90	700

Now, $S(fX^2)=700$, $N=15$, $M=\frac{fX}{N}=\frac{90}{15}=6$

$\therefore$ the required $\sigma=\sqrt{\frac{S(fX^2)}{N}-M^2}$

$=\sqrt{\frac{700}{15}-36}=\sqrt{46{\cdot}6-36}=\sqrt{10{\cdot}6}=3{\cdot}27$ (App.)

Example 6. Find the standard deviation from the following frequency distribution of the ages of 50 teachers of a school.

Age (in years)	46 – 48	43 – 45	40 – 42	37 – 39	34 – 36	31 – 33
Number of teachers	5	3	18	15	4	5

[Another method] solution :—

Age	Frequency f	Mid point x	Deviation of mid-point from mean(d)	d^2	fd^2
46-48	5	47	−7·5	56·25	281·25
43-45	3	44	−4·5	20·25	60·75
40-42	18	41	−1·5	2·25	40·5
37-39	15	38	1·5	2·25	33·75
34-36	4	35	4·5	20·25	81·00
31-33	5	32	7·5	56·25	281·25
Total	=50				778·5

Here in the first two columns ages and frequencies are written one under another. In the third column are noted the mid-points of the age-intervals one under another. Now, the mean of the ages is 39·5 as calculated from the mid-points and frequencies. In the fourth column are given the deviations of each mid-point from the mean. In the fith column the squares of these deviations and in the sixth column $f \times d^2$ (i.e. the product of the second and the fifth columns) are shown. The sum of the last column=778·5.

Now, the required standard deviation

$$= \sqrt{778·5 \div 50} = \sqrt{15·57} = 3·95 \text{ (approximately).}$$

[N. B. Here the mean is obtained by dividing the sum of the products of frequencies and mid-points by the sum (50) of the frequencies.]

Exercise 4

1. Find the mean deviation of the values 9, 8, 7, 6, 5 of a variate.

2. Find the mean deviation about the mean from the numbers 62, 68, 74, 76, 88 and 94.

3. The intervals of scores of some boys, correct to the nearest integer are 46—48, 49—51, 52—54, 55—57, 58—60 and the numbers of boys in the intervals 5, 8, 15, 10 and 4 respectively. Find the mean deviation of their scores.

4. Find the mean deviation from the following frequency distribution table :

Interval	Frequency
92·5—102·5	4
82·5—92·5	11
72·5—82·5	32
62·5—72·5	25
52·5—62·5	15
42·5—52·5	8
32·5—42·5	5

5. Find the standard deviation of the measures 5, 6, 7, 8, 9.

6. The intervals of the weights of a few boys (correct to kilograms) are 21 – 23, 24 – 26, 27 – 29, 30 – 32, 33 – 35 and 36 – 38 and the numbers of boys in those intervals are 2, 5, 17, 10, 8 and 1 respectively.

Find the standard deviation of their weights.

7. The temperatures for 10 days of a country are respectively 86°, 93°, 73°, 66°, 88°, 96°, 80°, 70°, 95° and 63°. Find the standard deviation of the temperatures about their mean.

8. A table of rainfall (correct to inches for a month in a country is given below. From this table find the standard deviation of rainfall.

Rainfall (in inches)	1	2	3	4	5	6
Number of days.	2	6	12	7	2	1

9. The weights of a few bags of wheat (in quintals) are given below. Find the standard deviation of the weights about the median which is taken to be 16·5 quintals :

Weight (quintal)	7·5	12·5	17·5	22·5	27·5
No. of bags	5	9	11	6	4

10. The scores of 905 students are given below ; find from the distribution the M_d and σ.

X	60·5	70·5	80·5	90·5	100·5	110·5	120·5	130·5	140·5
$f(X)$	3	21	78	182	305	209	81	21	5

Exercise 5

1. Prove that the sum of the deviations of 1, 3 and 7 from their mean is zero.

2. In a certain mathematical test 2 students got 55, 3 students 62, 5 students 68, 7 students 73, 5 students 75, 4 students 81, 3 students 85 and 1 student 91 scores. Find the mean of their scores.

3. The mean of the numbers $x_1, x_2, x_3, \ldots\ldots$etc is $\bar{x}$. If a constant k be added to each of them prove that the mean of the new set of numbers is $\bar{x}+k$.

4. The means of two sets of measurements are $\bar{x}_1=8$ and $\bar{x}_2=5$ respectively. If there are twice as many measurements in the first set as in the second, find the grand mean of the two sets.

5. Compute the variance of the three measures 1, 6, 8. If 7 be added to each, what is the variance of the new set of measures?

[The square of the standard deviation is called variance]

6. In a certain examination 5 students obtained 65 marks each, 10 students 70 marks each, 12 students 80 marks each and 3 students 90 marks each. Find the mean and the standard deviation of the scores.

7. The weights of a man taken every two hours are found to be 142·5, 142·5, 143, 144, 143, 144, 142·5 and 143·5 pounds. Find the mean and the standard deviation of those weights.

8. Show that the two deviations from the mean of two series of scores are equal, but opposite in sign.

9. There are 5 lengths each 1 cm., 3 lengths each 2 cm. and 1 length of 3 cm. Find their $\bar{x}$ and σ.

10. If half of a few measurements have the value 1 and half have the value 3, find the variance and standard deviation.

11. In a certain village 3 families have no land, 20 families have 1 acre each, 15 families 2 acres each and 2 families 3 acres each. Find the standard deviation of land per family.

12. Find the mean and standard deviation of the items of the following table :

Items X	1	2	3	4	5	6	7	8	9
Frequency f	7	11	16	17	26	31	11	1	1

ALGEBRA

FORMULAS

Learn the following formulas and commit them to memory.

1. $(a+b)^2 = a^2+b^2+2ab$;

Corollary : $a^2+b^2=(a+b)^2-2ab$.

2. $(a-b)^2=a^2+b^2-2ab$;

Corollary : (*i*) $a^2+b^2=(a-b)^2+2ab$;

(*ii*) $(a+b)^2=(a-b)^2+4ab$; (*iii*) $(a-b)^2=(a+b)^2-4ab$.

3. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ca+2bc$.

Corollary : (*i*) $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$.

(*ii*) $ab+ac+bc=\dfrac{(a+b+c)^2-(a^2+b^2+c^2)}{2}$

(*iii*) $a^2+b^2+c^2-ab-ac-bc$
$=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$.

4. $(a+b)^3=a^3+b^3+3ab(a+b)$
$=a^3+3a^2b+3ab^2+b^3$.

Corollary : $a^3+b^3=(a+b)^3-3ab(a+b)$.

5. $(a-b)^3=a^3-b^3-3ab(a-b)$
$=a^3-3a^2b+3ab^2-b^3$.

Corollary : $a^3-b^3=(a-b)^3+3ab(a-b)$.

6. $a^2-b^2=(a+b)(a-b)$

7. $a^3+b^3=(a+b)(a^2-ab+b^2)$

8. $a^3-b^3=(a-b)(a^2+ab+b^2)$

9. $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$;
or $=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

10. $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$;
and $a^3+b^3+c^3=(a+b+c)^3-3(a+b)(b+c)(c+a)$

11. $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$

12. $bc(b-c)+ca(c-a)+ab(a-b)=-(a-b)(b-c)(c-a)$

13. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a)$

14. $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$=-(a-b)(b-c)(c-a)(a+b+c)$.

Multiplication & Division

The general method of working out multiplication and division is known to you. In certain cases the multiplication or division can be easily done with the help of formulas. This is illustrated in the following examples.

Examples (1)

1. Multiply $4a^2+6ab+9b^2$ by $2a-3b$.

The required product $=(2a-3b)(4a^2+6ab+9b^2)$

$$=(2a-3b)\{(2a)^2+2a.3b+(3b)^2\}$$
$$=(2a)^3-(3b)^3=8a^3-27b^3.$$

[See Formula No. 8]

2. Find the continued product of $x+y$, $x-y$, x^2+y^2.

The reqd. continued product $=(x+y)(x-y)(x^2+y^2)$

$$=(x^2-y^2)(x^2+y^2)=(x^2)^2-(y^2)^2$$
$$=x^4-y^4.$$ [See Formula No. 6]

3. Find the continued product of

$$a+b+c,\ a+b-c,\ a-b+c,\ b+c-a.$$

The reqd. continued product

$$=\{(a+b)+c\}\{(a+b)-c\}\{c+(a-b)\}\{c-(a-b)\}$$
$$=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$$
$$=(a^2+b^2+2ab-c^2)(c^2-a^2-b^2+2ab)$$
$$=\{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}$$
$$=(2ab)^2-(a^2+b^2-c^2)^2$$
$$=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2$$
$$=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4.$$

4. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$a^3+b^3+c^3-3abc$$
$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

$\therefore$ the reqd. quotient $=\dfrac{(a+b+c)(a^2+b^2+c^2-ab-ac-bc)}{(a+b+c)}$

$$=a^2+b^2+c^2-ab-bc-ca.$$

Exercise 1

1. Multiply $x^2+y^2+z^2-xy-yz-zx$ by $x+y+z$.
2. Find the continued product of $1-x$, $1+x$, $1+x^2$, $1+x^4$.

3. Multiply $a^2-ab+a+1$ by $a+b-1$. [C. U. 1917]

4. Find the continued product of
a^2-ab+b^2, a^2+ab+b^2 and $a^4-a^2b^2+b^4$. [P. U. '26]

5. Multiply $a^2+b^2-ab+a+b+1$ by $a+b-1$. [D. B. '34]

6. Multiply $1-a+2a^2-3a^4$ by $3a-5+2a^2$. [C. U. 1918]

7. Find the continued product of x^2+x+1, x^2-x+1 and x^4-x^2+1. [C. U. 1911, '26 ; D. B. 1932]

8. Divide $4x^4+11x^3+27x^2+17x+5$ by x^2+2x+5. [D. B. 1924]

9. Divide $x^4-y^4+a^4+2a^2x^2$ by $x^2-y^2+a^2$. [C. U. 1915]

10. Divide $a+a^5+a^6$ by a^2+a+1. [C. U. 1918]

11. Divide a^4-6a-4 by $a-2$. [C. U. 1917]

12. Divide $a^3+b^3-c^3+3abc$ by $a+b-c$. [C. U. 1933]

13. Multiply $2x^2-3xy+y^2$ and $2x^2+3xy+y^2$ and divide the product by $2x^2-xy-y^2$. [M. U. 1918]

14. Divide a^6-b^6 by a^2-ab+b^2. [D. B. 1922]

15. Divide $x^3+y^3-1+3xy$ by $x+y-1$. [D. B. 1927]

16. Divide $x^5-y^5+\dfrac{y^{10}}{x^5}$ by $x-y+\dfrac{y^2}{x}$. [D. B. 1930]

17. Divide $a^3-b^3+c^3+3abc$ by $a-b+c$. [D. B. 1933]

18. For what value of a is $4x^3-12x^2+(a+4)x-3$ divisible by $2x-3$?

Application of the formulas

Examples (2)

[Formulas 1, 2, 3]

1. If $x+y=5$ and $xy=6$, find the value of $x-y$.
$(x-y)^2=(x+y)^2-4xy=5^2-4.6=25-24=1$
$\therefore\quad x-y=\sqrt{1}=\pm1.$

2. If $a-b=4$ and $ab=21$, find the value of $a+b$.
$(a+b)^2=(a-b)^2+4ab=4^2+4.21=16+84=100,$
$\therefore\quad a+b=\sqrt{100}=\pm10.$

3. If $x+\frac{1}{x}=3$, find the value of $x^2+\frac{1}{x^2}$. [C. U. 1931]

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2.x.\frac{1}{x}=(3)^2-2=9-2=7.$$

[Another method] : $\because$ $x+\frac{1}{x}=3$, $\therefore$ $\left(x+\frac{1}{x}\right)^2=(3)^2$.

or, $x^2+\frac{1}{x^2}+2x.\frac{1}{x}=9$, $\therefore$ $x^2+\frac{1}{x^2}=9-2=7$.

4. If $a+\frac{1}{a}=p$, express $a^2+\frac{1}{a^2}$ in terms of p.

$$a^2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)^2-2.a.\frac{1}{a}=(p)^2-2=p^2-2.$$

5. If $a+b=8$ and $ab=15$, find the values of a and b.

$(a-b)^2=(a+b)^2-4ab=8^2-4\times15=64-60=4$, $\therefore a-b=\pm2$.

Now, $a+b=8$
$a-b=2$
(Adding) $2a=10$,
$\therefore$ $a=5$, $\therefore$ $b=8-5=3$.

Again, $a+b=8$
$a-b=-2$
(Adding) $2a=6$, $\therefore$ $a=3$,
$\therefore$ $b=8-3=5$.

Hence, $a=5$, $b=3$; or, $a=3$, $b=5$.

6. If $x=a+\frac{1}{a}$ and $y=a-\frac{1}{a}$, find the value of

$x^4+y^4-2x^2y^2$. [C. U. 1944]

$$x^4+y^4-2x^2y^2=(x^2-y^2)^2=\{(x+y)(x-y)\}^2.$$
$$=(x+y)^2(x-y)^2$$ [Vide Formula 6]

$$=\left(a+\frac{1}{a}+a-\frac{1}{a}\right)^2\left(a+\frac{1}{a}-a+\frac{1}{a}\right)^2=(2a)^2\times\left(\frac{2}{a}\right)^2=4a^2\times\frac{4}{a^2}=16.$$

7. If $a+b=5$ and $ab=6$, find the value of a^2-b^2.

[Here the value of $(a+b)$ is given. If we can find the value of $(a-b)$, the value of a^2-b^2 can be ascertained.]

$$(a-b)^2=(a+b)^2-4ab=5^2-4\times6=1, \quad \therefore \quad a-b=\pm1.$$

Now, $a^2-b^2=(a+b)(a-b)=5\times\pm1=\pm5$,

8. Find the value of $2{\cdot}69\times2{\cdot}69+2{\cdot}62\times2{\cdot}69+1{\cdot}31\times1{\cdot}31$.

The given expression $=(2{\cdot}69)^2+2\times1{\cdot}31\times2{\cdot}69+(1{\cdot}31)^2$

[$\because$ $2{\cdot}62=2\times1{\cdot}31$]

$$=(2{\cdot}69+1{\cdot}31)^2=(4)^2=16.$$

9. If $x = \cdot29$ and $y = \cdot14$, find the value of $4x^2 + 9y^2 + 12xy$.

$4x^2 + 9y^2 + 12xy = (2x)^2 + (3y)^2 + 2.2x.3y$

$= (2x + 3y)^2 = (\cdot29 \times 2 + \cdot14 \times 3)^2 = (\cdot58 + \cdot42)^2 = (1)^2 = 1.$

10. If $x + \frac{1}{x} = \sqrt{2}$, show that $x^2 + \frac{1}{x^2} = 0$.

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x} = (\sqrt{2})^2 - 2 = 2 - 2 = 0.$$

11. Find the value of $x^2 - y^2 + z^2 + 2xz$,
when $x = b - c$, $y = c - a$, $z = a - b$. [C. U. 1922]

Here, $x + y + z = b - c + c - a + a - b = 0.$

Now, $x^2 - y^2 + z^2 + 2xz = x^2 + z^2 + 2xz - y^2 = (x + z)^2 - (y)^2$

$= (x + z + y)(x + z - y) = 0 \times (x + z - y) = 0.$

12. If $a + b + c = 5$ and $ab + bc + ca = 8$,
find the value of $a^2 + b^2 + c^2$.

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= 5^2 - 2 \times 8 = 25 - 16 = 9.$

13. If $a + b + c = 15$ and $a^2 + b^2 + c^2 = 77$,
find the value of $ab + bc + ca$. [C. U. 1945]

$\because \quad a + b + c = 15, \quad \therefore \quad (a + b + c)^2 = (15)^2,$

or, $\quad a^2 + b^2 + c^2 + 2(ab + bc + ca) = 225,$

or, $\quad 77 + 2(ab + bc + ca) = 225,$

or, $\quad 2(ab + bc + ca) = 225 - 77 = 148,$

$\therefore \quad ab + bc + ca = \frac{148}{2} = 74.$

14. Find the value of $x + y + z$,
when $x^2 + y^2 + z^2 = 7$ and $xy + yz + zx = 9$.

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 7 + 2 \times 9 = 25.$

$\therefore \quad x + y + z = \sqrt{25} = \pm 5.$

15. If $x + y + z = 2$ and $xy + yz + zx = 1$,
find the value of $(x + y)^2 + (y + z)^2 + (z + x)^2$. [C. U. '28]

$(x + y)^2 + (y + z)^2 + (z + x)^2 = 2(x^2 + y^2 + z^2 + xy + yz + zx)$

$= 2\{(x + y + z)^2 - (xy + yz + zx)\}$

$= 2(2^2 - 1) = 2(4 - 1) = 2 \times 3 = 6.$

16. If $x+\frac{1}{x}=2$, find the value of $x^4+\frac{1}{x^4}$. [D. B. 1936]

$\because \quad x+\frac{1}{x}=2, \qquad \therefore \quad \left(x+\frac{1}{x}\right)^2=2^2,$

or, $x^2+\frac{1}{x^2}+2x.\frac{1}{x}=4, \qquad$ or, $x^2+\frac{1}{x^2}=4-2=2,$

$\therefore \quad \left(x^2+\frac{1}{x^2}\right)^2=(2)^2, \qquad$ or, $\quad x^4+\frac{1}{x^4}+2x^2\times\frac{1}{x^2}=4,$

or, $x^4+\frac{1}{x^4}+2=4, \qquad \therefore \quad x^4+\frac{1}{x^4}=4-2=2.$

17. Express $x^2+2xy-z^2-2yz$ as the difference of two squares. [C. U. 1943]

The given expression $=x^2+2xy+y^2-y^2-z^2-2yz$
$=(x^2+y^2+2xy)-(y^2+z^2+2yz)=(x+y)^2-(y+z)^2.$

[Note that if x^2 and y^2 be added to $2xy$ and if y^2 and z^2 be added to $2yz$, there will be perfect squares. But there is not a single y^2 in the expression, $\therefore$ $+y^2-y^2$ is added to the expression. This addition does not change the value of the expression, but it forms two squares.]

18. Express xy as the difference of two squares.

$\because \quad x^2+y^2+2xy=(x+y)^2$

and $\quad x^2+y^2-2xy=(x-y)^2$

$\therefore$ (Subtracting) $4xy=(x+y)^2-(x-y)^2$

$\therefore \quad xy=\frac{(x+y)^2}{4}-\frac{(x-y)^2}{4}=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2.$

19. Express $(a+3b)(a+2b)$ as the difference of two squares.

Suppose, $a+3b=x$ and $a+2b=y$,

$\therefore \quad (a+3b)(a+2b)=xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$ [Vide Ex. 18]

$=\left(\frac{a+3b+a+2b}{2}\right)^2-\left(\frac{a+3b-a-2b}{2}\right)^2$

[substituting the values of x and y]

$=\left(\frac{2a+5b}{2}\right)^2-\left(\frac{b}{2}\right)^2.$

20. Find the value of $a^2+b^2+c^2-ab-ac-bc$, when $a=x+y$, $b=x-y$, $c=x+2y$. [C. U. '14 ; D. B. '31]

$a^2+b^2+c^2-ab-ac-bc=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$

[Vide Corollary, Formula 3]

$=\frac{1}{2}\{(x+y-x+y)^2+(x-y-x-2y)^2+(x+2y-x-y)^2\}$

$=\frac{1}{2}\{(2y)^2+(-3y)^2+(y)^2\}=\frac{1}{2}(4y^2+9y^2+y^2)=\frac{1}{2}\times 14y^2=7y^2$.

[Formulas 4, 5]

21. Find the value of $8x^3+36x^2y+54xy^2+27y^3$, when $x=19$ and $y=-12$.

$8x^3+36x^2y+54xy^2+27y^3$

$=(2x)^3+3.(2x)^2.3y+3.2x.(3y)^2+(3y)^3$

$=(2x+3y)^3=(2\times 19+3\times -12)^3=(38-36)^3=(2)^3=8$.

22. If $a+b=3$ and $ab=2$, find the value of a^3+b^3. [D. B. 1934]

$a^3+b^3=(a+b)^3-3ab(a+b)=(3)^3-3\times 2\times 3=27-18=9$.

23. If $x-\frac{1}{x}=1$, find the value of $x^3-\frac{1}{x^3}$. [C. U. 1929]

$x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)^3+3x.\frac{1}{x}\left(x-\frac{1}{x}\right)=(1)^3+3\times 1=1+3=4$.

24. If $a+b=4$, find the value of a^3+b^3+12ab.

$\because$ $a+b=4$, $\therefore$ $(a+b)^3=4^3$,

or, $a^3+b^3+3ab(a+b)=64$,

or, $a^3+b^3+3ab\times 4=64$. $\therefore$ $a^3+b^3+12ab=64$.

25. If $x-y=3$, show that $x^3-y^3-9xy=27$.

$\because$ $x-y=3$, $\therefore$ $(x-y)^3=(3)^3$,

or, $x^3-y^3-3xy(x-y)=27$,

or, $x^3-y^3-3xy\times 3=27$, $\therefore$ $x^3-y^3-9xy=27$.

26. If $x+\frac{1}{x}=p$, express $x^3+\frac{1}{x^3}$ in terms of p. [C. U. '26]

$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3x.\frac{1}{x}\left(x+\frac{1}{x}\right)=p^3-3p$.

27. If $x+\frac{1}{x}=\sqrt{3}$, find the value of $x^3+\frac{1}{x^3}$.

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3x.\frac{1}{x}\left(x+\frac{1}{x}\right)$$

$$=\left(x+\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)-3\left(x+\frac{1}{x}\right)$$

$$=(\sqrt{3})^2\left(x+\frac{1}{x}\right)-3\left(x+\frac{1}{x}\right)=3\left(x+\frac{1}{x}\right)-3\left(x+\frac{1}{x}\right)=0.$$

28. If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$.

[C. U. 1924, '45 ; D. B. '27 ; P. U. 1924]

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3.a.\frac{1}{a}\left(a+\frac{1}{a}\right)$$

$$=\left(a+\frac{1}{a}\right)^2\left(a+\frac{1}{a}\right)-3\left(a+\frac{1}{a}\right)=3\left(a+\frac{1}{a}\right)-3\left(a+\frac{1}{a}\right)=0.$$

29. Simplify :—

$(a+b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2-(a-b)^3$. [C.U.1920]

Suppose, $x=a+b$, and $y=a-b$.

Now, the given expression $=x^3-3x^2y+3xy^2-y^3=(x-y)^3$

$$=(a+b-a+b)^3=(2b)^3=8b^3.$$

30. If $x+y=2$, $x^2+y^2=4$, find the value of x^3+y^3.

$\because\ x+y=2,\quad \therefore\ x^2+y^2+2xy=4,$

or, $4+2xy=4$. $(\because\ x^2+y^2=4)$,

or, $2xy=4-4=0$, $\therefore\ xy=0$.

Now, $x^3+y^3=(x+y)^3-3xy(x+y)=(2)^3-3\times0\times2=8-0=8$.

31. If $x+y=5$ and $xy=7$, find the value of $x^3+y^3+4(x-y)^2$.

$x^3+y^3+4(x-y)^2=\{(x+y)^3-3xy(x+y)\}+4\{(x+y)^2-4xy\}$

$=\{(5)^3-3.7.5\}+4(5^2-4.7)=125-105+4(25-28)$

$=125-105-12=8$.

32. If $2x-\frac{2}{x}=3$, prove that $8\left(x^3-\frac{1}{x^3}\right)=63$. [D. B. '29]

$$\because\ \left(2x-\frac{2}{x}\right)=3,\quad \therefore\ \left(2x-\frac{2}{x}\right)^3=3^3$$

or, $(2x)^3-\left(\frac{2}{x}\right)^3-3.2x.\frac{2}{x}\left(2x-\frac{2}{x}\right)=27,$

or, $8x^3-\frac{8}{x^3}-12\times3=27$. $\left[\ \because\ 2x-\frac{2}{x}=3\ \right]$

or, $8x^3-\frac{8}{x^3}=27+36=63$, $\therefore\ 8\left(x^3+\frac{1}{x^3}\right)=63$.

33. If $x+y=a$, $x^2+y^2=b^2$ and $x^3+y^3=c^3$, prove that $a^3+2c^3=3ab^2$. [C. U. 1943]

$a^3+2c^3=(x+y)^3+2(x^3+y^3)$

$=x^3+y^3+3xy(x+y)+2(x^3+y^3)=3(x^3+y^3)+3xy(x+y)$

$=3(x+y)(x^2-xy+y^2)+3xy(x+y)$ [Vide Formula 7]

$=3(x+y)(x^2-xy+y^2+xy)=3a(x^2+y^2)=3ab^2$.

34. Find the value of $x^3+y^3+z^3-3xyz$, when $x=y=333$, $z=334$.

$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$=\frac{1}{2}(333+333+334)\times\{(333-333)^2+(333-334)^2+(334-333)^2\}$

$=\frac{1}{2}\times1000\{0+(-1)^2+(1)^2\}$

$=\frac{1}{2}\times1000\{1+1\}=\frac{1}{2}\times1000\times2=1000$.

35. Find the value of $\frac{x}{x^2+x+1}$, when $x+\frac{1}{x}=5$. [C. U. '48 Supl.]

$\because$ $x+\frac{1}{x}=5$, $\therefore$ $x^2+1=5x$ [multiplying both sides by x]

Now, $\frac{x}{x^2+x+1}=\frac{x}{(x^2+1)+x}=\frac{x}{5x+x}=\frac{x}{6x}=\frac{1}{6}$.

36. If $a+b+c=9$, $ab+bc+ca=26$ and $a^3+b^3+c^3=138$, find the value of abc. [U. U. 49]

$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=(9)^2-2\times26=29$.

Now, $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$,

or, $138-3abc=9(29-26)=27$,

or, $-3abc=27-138=-111$, $\therefore$ $abc=\frac{-111}{-3}=37$.

Exercise 2

1. Find the value of $(x-y)^2$, when $x+y=3$ and $xy=2$. [C. U. 1943]

2. Find the value of $1{\cdot}79\times1{\cdot}79+2{\cdot}42\times1{\cdot}79+1{\cdot}21\times1{\cdot}21$. [C. U. '30]

3. If $x+\frac{1}{x}=3$, find the value of $x^2+\frac{1}{x^2}$. [C. U. '31]

4. If $a-\frac{1}{a}=4$, find the value of $a^2+\frac{1}{a^2}$.

5. If $a+b=5$ and $ab=6$, find the value of $a-b$.

6. If $x-y=3$ and $xy=4$, find the value of $x+y$.

7. If $a^2+b^2+c^2=9$ and $ab+bc+ca=8$, find the value of $a+b+c$. [C. U. 1924 ; D. B. 1927]

8. (i) If $x+y+z=13$, and $xy+yz+zx=50$, find the value of $x^2+y^2+z^2$. [C. U. 1911]

(ii) Find the value of $xy+yz+zx$, when $x+y+z=9$ and $x^2+y^2+z^2=31$. [W.B.S. F. '52]

9. If $x+\frac{1}{x}=4$, find the values of $x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$. [D. B. 1948]

10. If $x=b-c$, $y=c-a$, $z=a-b$, find the value of $x^2+y^2-z^2+2xy$. [C. U. 1913]

11. If $a+\frac{1}{a}=5$, find the value of $a^3+\left(\frac{1}{a}\right)^3$. [A. U. 1932]

12. If $a+b=3$, shew that $a^3+b^3+9ab=27$. [C. U. 1927]

13. If $a+b=5$ and $ab=6$, find the value of a^3+b^3. [D. B. '24]

14. If $x-y=5$, find the value of x^3-y^3-15xy.

15. If $x-\frac{1}{x}=p$, prove that $x^3-\frac{1}{x^3}=p^3+3p$. [C. U. '10, '30]

16. Find the value of $x^2+y^2-z^2+2xy$, when $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$. [C. U. 1919]

17. (i) If $p-\frac{1}{p}=2$, find the numerical value of $p^4+\frac{1}{p^4}$. [B. U. 1931]

(ii) If $a+\frac{1}{a}=2$, find the value of $a^3+\frac{1}{a^3}$.

18. Evaluate $27x^3-54x^2y+36xy^2-8y^3$, when $x=33$, $y=49$.

19. If $p=3+\frac{1}{p}$, prove that $p^4=119-\frac{1}{p^4}$. [B. U. 1930]

20. If $x-\frac{1}{x}=c$, express $x^3-\frac{1}{x^3}$ in terms of c.

21. If $a=z+y$, $b=z-x$, $c=x-y$; find the value of
$$a^2+b^2+c^2-2ab-2ac+2bc.$$

22. Express ab as the difference of two squares. [C. U.]

23. Fxpress $x^2-y^2+2xz+2yz$ as the difference of two squares.

24. Express $(a+2b)(3a+2c)$ as the difference of two squares.

25. Simplify $0·75\times0·75+0·25\times0·25+2\times0·75\times0·25$. [C. U. 1940]

26. If $a+b+c=5$ and $a^2+b^2+c^2=13$, find the value of $ab+bc+ca$. [C. U. 1940]

27. Express $a^2+2ab-2bc-c^2$ as the difference of two squares. [C. U. 1946]

28. Express $(a^2+b^2)(c^2+d^2)$ as the sum of two squares. [C. U. 1950]

29. If $x-2$ is a factor of $120x^3-167x^2-ax+56$, find the numerical value of a. [D. B. 1925]

30. If $x+y=\sqrt{3}$, $x-y=\sqrt{2}$, find the value of $8xy(x^2+y^2)$.

[*Hints* : Squaring $x^2+y^2+2xy=3\ldots(i)$, $x^2+y^2-2xy=2\ldots(ii)$; From $(i)+(ii)$ we get $2(x^2+y^2)=5$; Again, from $(i)-(ii)$ we get $4xy=1$, $\therefore$ $8xy(x^2+y^2)=4xy.2(x^2+y^2)=5.1=5.$]

FACTORS

$a^2-b^2=(a+b)(a-b)$. [Formula 6]

[This formula is applicable in factorizing the expressions in which two perfect squares are connected with the minus sign. So you should first see if the terms of the expression may be arranged as the difference of two squares.]

Examples (3)

1. Factorize a^2-b^2.

$a^2-b^2=a^2+ab-ab-b^2=a(a+b)-b(a+b)=(a+b)(a-b)$.

2. Resolve into factors $81-x^4$.

$$81-x^4=(9)^2-(x^2)^2=(9+x^2)(9-x^2)$$
$$=(9+x^2)\{(3)^2-(x)^2\}=(9+x^2)(3+x)(3-x).$$

3. Factorize $2ab-a^2-b^2+c^2$. [C. U. '39 Supl.]

The given expression $=c^2-(a^2+b^2-2ab)=(c)^2-(a-b)^2$

$=\{c+(a-b)\}\{c-(a-b)\}=(c+a-b)(c-a+b)$.

4. Factorize $x^2-2xy+2yz-z^2$.

The given expression $=x^2-2xy+y^2-y^2+2yz-z^2$

$=(x^2-2xy+y^2)-(y^2-2yz+z^2)=(x-y)^2-(y-z)^2$

$=(x-y+y-z)(x-y-y+z)=(x-z)(x-2y+z)$.

5. Factorize a^4+4b^4. [C. U. 1922]

[N. B. Here $a^4=(a^2)^2$ and $4b^4=(2b^2)^2$, so they are perfect squares with the sign '+' in the middle. ∴ you may doubt if the formula will be applicable here or not. Now see how it has been arranged with the sign '−' between two squares.]

The given Exp. $=(a^2)^2+(2b^2)^2+2.a^2.2b^2-4a^2b^2$

$=(a^2+2b^2)^2-(2ab)^2=(a^2+2b^2+2ab)(a^2+2b^2-2ab)$.

6. Resolve into factors a^4+a^2+1.

The given Exp. $=(a^2)^2+2.a^2.1+(1)^2-a^2=(a^2+1)^2-(a)^2$

$=(a^2+1+a)(a^2+1-a)=(a^2+a+1)(a^2-a+1)$.

7. Resolve into factors $a^4+a^2b^2+b^4$.

The given Exp. $=(a^2)^2+2.a^2.b^2+(b^2)^2-a^2b^2$

$=(a^2+b^2)^2-(ab)^2=(a^2+b^2+ab)(a^2+b^2-ab)$.

8. Factorize a^8+a^4+1.

The given Exp. $=(a^4)^2+2a^4.1+(1)^2-a^4=(a^4+1)^2-(a^2)^2$

$=(a^4+a^2+1)(a^4-a^2+1)=\{(a^2)^2+2.a^2.1+(1)^2-a^2\}$

$\times(a^4-a^2+1)=\{(a^2+1)^2-(a)^2\}(a^4-a^2+1)$

$=(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$.

9. Factorize $x^4-16x^2y^2+36y^4$. [B. U. '31]

The given Exp. $=(x^2)^2-2.x^2.6y^2+(6y^2)^2-4x^2y^2$

$=(x^2-6y^2)^2-(2xy)^2=(x^2+2xy-6y^2)(x^2-2xy-6y^2)$.

10. Factorize $a^4-23a^2b^2+b^4$.

The given Exp. $=(a^2)^2+2.a^2.b^2+(b^2)^2-25a^2b^2$

$=(a^2+b^2)^2-(5ab)^2=(a^2+5ab+b^2)(a^2-5ab+b^2)$.

11. Factorize $4a^2+b^2-c^2-d^2+4ab+2cd$. [D. B. 1923]

The given Exp. $=(4a^2+b^2+4ab)-(c^2+d^2-2cd)$

$=(2a+b)^2-(c-d)^2=(2a+b+c-d)(2a+b-c+d)$.

12. Factorize $a^2b^2-a^2-b^2+1$.

The given Exp. $=a^2(b^2-1)-(b^2-1)=(a^2-1)(b^2-1)$

$=(a+1)(a-1)(b+1)(b-1)$.

13. Factorize $(a^2-b^2)(x^2-y^2)+4abxy$. [P. U. 1925,'33]

The given Exp. $=a^2x^2-a^2y^2-b^2x^2+b^2y^2+2abxy+2abxy$

$=(a^2x^2+b^2y^2+2abxy)-(a^2y^2+b^2x^2-2abxy)$

$=(ax+by)^2-(ay-bx)^2=(ax+by+ay-bx)(ax+by-ay+bx)$.

14. Factorize $a^2-b^2-c^2-2bc+a-b-c$.

The given Exp. $=a^2-(b^2+c^2+2bc)+a-b-c$

$=(a)^2-(b+c)^2+(a-b-c)$

$=(a+b+c)(a-b-c)+(a-b-c)=(a-b-c)(a+b+c+1)$.

15. Factorize $a^4+2a^3b-2ab^3-b^4$. [C. U. 1911]

The given Exp. $=a^4-b^4+2a^3b-2ab^3$

$=(a^2+b^2)(a^2-b^2)+2ab(a^2-b^2)$

$=(a^2-b^2)(a^2+b^2+2ab)=(a+b)(a-b)(a+b)^2=(a-b)(a+b)^3$.

16. Factorize $4x^4+1$ and hence find the two factors of 40001.

$4x^4+1=(2x^2)^2+(1)^2+2.2x^2.1-4x^2=(2x^2+1)^2-(2x)^2$

$=(2x^2+2x+1)(2x^2-2x+1)$.

Now, $40001=40000+1=4\times10000+1=4\times10^4+1$

$=4x^4+1$ [putting x for 10]

$=(2x^2+2x+1)(2x^2-2x+1)$

$=(2\times10^2+2\times10+1)(2\times10^2-2\times10+1)$,

$=(200+20+1)(200-20+1)=221\times181$.

17. Simplify $\dfrac{1{\cdot}73\times1{\cdot}73-{\cdot}27\times{\cdot}27}{1{\cdot}73-{\cdot}27}$.

Suppose, $1{\cdot}73=a$, ${\cdot}27=b$.

Now, the given Exp. $=\dfrac{a\times a-b\times b}{a-b}=\dfrac{a^2-b^2}{a-b}=\dfrac{(a+b)(a-b)}{(a-b)}$

$=a+b=1{\cdot}73+{\cdot}27=2$.

18. Resolve into factors $3a^2-b^2-c^2-2ab-2bc-2ca$.

The given Exp. $=4a^2-a^2-b^2-c^2-2ab-2bc-2ca$

$=4a^2-(a^2+b^2+c^2+2ab+2ac+2bc)$

$=(2a)^2-(a+b+c)^2=(2a+a+b+c)(2a-a-b-c)$

$=(3a+b+c)(a-b-c)$.

19. Resolve into factors
$2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$.

The given Exp. $=4a^2b^2-2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$
$=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2)$
$=(2ab)^2-(a^2+b^2-c^2)^2$
$=(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)$
$=\{(a+b)^2-(c)^2\}\{c^2-(a^2+b^2-2ab)\}$
$=(a+b+c)(a+b-c)\{(c)^2-(a-b)^2\}$
$=(a+b+c)(a+b-c)(c+a-b)(c-a+b)$.

[Formulas 7, 8]

20. Factorize (1) a^3+b^3 and (2) a^3-b^3.

(1) $a^3+b^3=(a+b)^3-3ab(a+b)=(a+b)\{(a+b)^2-3ab\}$
$=(a+b)(a^2+b^2+2ab-3ab)=(a+b)(a^2-ab+b^2)$.

(2) $a^3-b^3=(a-b)^3+3ab(a-b)=(a-b)\{(a-b)^2+3ab\}$
$=(a-b)(a^2+b^2-2ab+3ab)=(a-b)(a^2+ab+b^2)$.

21. Factorize a^3+27b^3.
The exp. $=(a)^3+(3b)^3=(a+3b)\{(a)^2-a.3b+(3b)^2\}$
$=(a+3b)(a^2-3ab+9b^2)$.

22. $2a^3-128b^3=2(a^3-64b^3)=2\{(a)^3-(4b)^3\}$
$=2(a-4b)(a^2+4ab+16b^2)$.

[**N. B.** In factorization first find out the common factors of the terms, if there be any and then ascertain what formula is to be applied. See the above sum.]

23. Resolve into factors a^6-b^6.
The given Exp. $=(a^3)^2-(b^3)^2=(a^3+b^3)(a^3-b^3)$
$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

24. Factorize $8(a+b)^3-c^3$.
The given Exp. $=\{2(a+b)\}^3-(c)^3=(2a+2b)^3-(c)^3$
$=(2a+2b-c)\{(2a+2b)^2+(2a+2b)c+c^2\}$
$=(2a+2b-c)(4a^2+4b^2+8ab+2ac+2bc+c^2)$.

25. Divide $a^6+\dfrac{b^6}{27}$ by $a^2+ab+\dfrac{a^2}{3}$. [C. U. 1930]

$$a^6+\frac{b^6}{27}=(a^2)^3+\left(\frac{b^2}{3}\right)^3=\left(a^2+\frac{b^2}{3}\right)\left(a^4-\frac{a^2b^2}{3}+\frac{b^4}{9}\right)$$
$$=\left(a^2+\frac{b^2}{3}\right)\left\{(a^2)^2+2a^2.\ \frac{b^2}{3}+\left(\frac{b^2}{3}\right)^2-a^2b^2\right\}$$

$$=\left(a^2+\frac{b^2}{3}\right)\left\{\left(a^2+\frac{b^2}{3}\right)^2-(ab)^2\right\}$$

$$=\left(a^2+\frac{b^2}{3}\right)\left(a^2+ab+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right)$$

$\therefore$ the required quotient

$$=\frac{\left(a^2+\frac{b^2}{3}\right)\left(a^2+ab+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right)}{\left(a^2+ab+\frac{b^2}{3}\right)}$$

$$=\left(a^2+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right).$$

26. Shew that $(ax+by)^3+(bx+ay)^3$ is divisible by $a+b$ and also by $x+y$. [C. U. '21, '26]

$$(ax+by)^3+(bx+ay)^3$$
$$=(ax+by+bx+ay)\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}$$
$$=\{a(x+y)+b(x+y)\}\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}$$
$$=(a+b)(x+y)\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}$$

Now, because both $a+b$ and $x+y$ are factors of the given expression,

$\therefore$ the expression is divisible by $a+b$ and $x+y$.

27. Factorize $x^3-y^3+3y^2-3y+1$.

The given Exp. $=x^3-(y^3-3y^2+3y-1)=(x)^3-(y-1)^3$

$$=(x-y+1)\{x^2+x(y-1)+(y-1)^2\}$$
$$=(x-y+1)(x^2+xy-x+y^2-2y+1)$$
$$=(x-y+1)(x^2+y^2+1+xy-x-2y).$$

[Formula 9]

28. Resolve into factors $a^3+b^3+c^3-3abc$.

The given Exp. $=(a+b)^3-3ab(a+b)+c^3-3abc$

$$=(a+b)^3+(c)^3-3ab(a+b)-3abc$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c)$$
$$=(a+b+c)(a^2+b^2+2ab-ac-bc+c^2-3ab)$$
$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

29. Resolve into factors $8x^3-y^3+z^3+6xyz$.

The given Exp. $=(2x)^3+(-y)^3+(z)^3-3(2x).(-y).(z)$

$=(2x-y+z)\{(2x)^2+(-y)^2+(z)^2-(2x)(-y)-(-y)(z)-(z)(2x)\}$

$=(2x-y+z)(4x^2+y^2+z^2+2xy+yz-2xz)$.

[**N. B.** It is factorized with the help of the formula. It is better to adopt the full method as shown in the sum 28.]

30. Resolve into factors x^6+8x^3+27.

The given Exp. $=x^6-x^3+27+9x^3$

$=(x^2)^3+(-x)^3+(3)^3-3x^2.(-x).3$

$=(x^2-x+3)(x^4+x^2+9+x^3-3x^2+3x)$

$=(x^2-x-3)(x^4+x^3-2x^2+3x+9)$.

[Miscellaneous]

31. Resolve into factors $x^2+4x-21$. [C. U. 1916]

[**N. B.** The rule of factorizing the expressions of this form is as follows :—Find out two numbers whose product is -21 (i. e., the term free from x) and sum is $+4$ (i. e., coefficient of x). Here 7 and -3 are those numbers, so $7x$ and $-3x$ are to be written in place of $4x$.]

$x^2+4x-21=x^2+7x-3x-21=x(x+7)-3(x+7)$

$=(x+7)(x-3)$.

32. Factorize a^2-5a-6.

The given Exp. $=a^2-6a+a-6=a(a-6)+1(a-6)$

$=(a-6)(a+1)$.

33. Factorize $6x^2+x-15$. [C. U. 1936]

[**N. B.** Here 6 (i. e., coefficient of x^2) being multiplied by -15 gives the product -90. Now find out two numbers whose product is -90 and sum $+1$ (i.e. coefficient of x)]

The given Exp. $=6x^2+10x-9x-15=2x(3x+5)-3(3x+5)$

$=(3x+5)(2x-3)$.

34. Factorize $6-5a+a^2$.

The given Exp. $=6-2a-3a+a^2=2(3-a)-a(3-a)$

$=(3-a)(2-a)$.

35. Factorize $3(2x^2-1)-7x$. [D. B. 1931]

The given Exp. $=6x^2-7x-3=6x^2-9x+2x-3$

$=3x(2x-3)+1(2x-3)=(2x-3)(3x+1)$.

36. Factorize $x^2+x-(a+1)(a+2)$.

[N. B. Here notice that the product of $(a+2)$ and $-(a+1)$ is $-(a+1)(a+2)$ and their sum is $+1$ (i.e. the coefficient of x) ; so write $(a+2)x-(a+1)x$ in place of x.]

The given Exp. $=x^2+(a+2)x-(a+1)x-(a+1)(a+2)$

$=x(x+a+2)-(a+1)(x+a+2)$

$=(x+a+2)(x-a-1)$.

37. Factorize $x^2-\left(a+\frac{1}{a}\right)x+1$.

The given expression $=x^2-ax-\frac{x}{a}+1$

$=x(x-a)-\frac{1}{a}(x-a)=(x-a)\left(x-\frac{1}{a}\right)$.

38. Factorize $(a+b)^2-5a-5b+6$. [A. U. 1926]

The given Exp. $=(a+b)^2-5(a+b)+6=x^2-5x+6$

[putting x for $a+b$]

$=x^2-3x-2x+6=x(x-3)-2(x-3)=(x-3)(x-2)$

$=(a+b-3)(a+b-2)$ [Substituting the value of x]

39. Factorize $(x^2-6x)^2-8(x^2-6x+8)-64$. [B. U. '26]

The given Exp. $=a^2-8(a+8)-64$ [putting a for x^2-6x]

$=a^2-8a-64-64=a^2-8a-128=a^2-16a+8a-128$

$=a(a-16)+8(a-16)=(a-16)(a+8)$

$=(x^2-6x-16)(x^2-6x+8)$ [Substituting the value of a]

$=(x^2-8x+2x-16)(x^2-4x-2x+8)$

$=\{x(x-8)+2(x-8)\}\{x(x-4)-2(x-4)\}$

$=(x-8)(x+2)(x-2)(x-4)$.

40. Factorize $(x+1)(x+3)(x+5)(x+7)+15$.

[C. U. 1941 ; M. U. 1926]

The given Exp. $=\{(x+1)(x+7)\}\{(x+3)(x+5)\}+15$

$=(x^2+8x+7)(x^2+8x+15)+15$

$=(a+7)(a+15)+15$ [let $x^2+8x=a$] $=a^2+22a+105+15$

$=a^2+22a+120=a^2+12a+10a+120=a(a+12)+10(a+12)$

$=(a+10)(a+12)=(x^2+8x+10)(x^2+8x+12)$
$=(x^2+8x+10)(x^2+6x+2x+12)$
$=(x^2+8x+10)\{x(x+6)+2(x+6)\}$
$=(x^2+8x+10)(x+6)(x+2).$

[N. B. : Here the four brackets have not been multiplied all together, but separately two at a time. Here two groups are so chosen that the sum of the expressions in each may be the same.]

41. Factorize $(x+1)(x+3)(x-4)(x-6)+24$. [D. B. '22]

The given Exp. $=\{(x+1)(x-4)\}\{(x+3)(x-6)\}+24$
$=(x^2-3x-4)(x^2-3x-18)+24$
$=(a-4)(a-18)+24$ [putting a for x^2-3x]
$=a^2-22a+96=a^2-16a-6a+96=a(a-16)-6(a-16)$
$=(a-6)(a-16)=(x^2-3x-6)(x^2-3x-16).$

42. Factorize x^3-3x+2. [C. U. 1930 ; D. B. 1929]

[N. B. First find the value of x for which the value of the expression is zero (0). Here the value of the expression is zero, if the value of x is $+1$; so one of its factors will be $x-1$. Thus if the value of the expression be zero when the value of x is -1, one of its factors would be $x+1$.]

$x^3-3x+2=x^3-x^2+x^2-x-2x+2$
$=x^2(x-1)+x(x-1)-2(x-1)$
$=(x-1)(x^2+x-2)=(x-1)(x^2+2x-x-2)$
$=(x-1)\{x(x+2)-1(x+2)\}$
$=(x-1)(x+2)(x-1)=(x-1)^2(x+2).$

[N. B. Here we find that the value of the expression is 0, if the value of x be $+1$. So one of the factors must be $x-1$. Now the boys often commit mistakes in arranging the terms of the expression. You may ordinarily divide the expression by $x-1$. Here the quotient is x^2+x-2. Now if you write that factor $(x-1)$ with each term of the quotient in the form of multiplication it will be equal to the expression.]

43. Factorize x^3-x^2-7x-2.

Here if the value of x be -2, the value of the expression is 0. So $x+2$ is one of the factors.

$\therefore$ the given Exp. $=x^3+2x^2-3x^2-6x-x-2$
$=x^2(x+2)-3x(x+2)-1(x+2)=(x+2)(x^2-3x-1).$

44. Factorize $x^3-6x^2+11x-6$. [A. U. 1921]

Here if the value of x be 1, the value of the expression is 0.

$\therefore$ $x-1$ is one of its factors.

$\therefore$ $x^3-6x^2+11x-6=x^3-x^2-5x^2+5x+6x-6$

$=x^2(x-1)-5x(x-1)+6(x-1)$

$=(x-1)(x^2-5x+6)=(x-1)(x^2-3x-2x+6)$

$=(x-1)\{x(x-3)-2(x-3)\}=(x-1)(x-3)(x-2).$

45. Factorize $8a^3+4a-3$.

The given Exp.$=8a^3-1+4a-2=\{(2a)^3-(1)^3\}+2(2a-1)$

$=(2a-1)(4a^2+2a+1)+2(2a-1)$

$=(2a-1)(4a^2+2a+1+2)=(2a-1)(4a^2+2a+3).$

[Factors of Cyclic Order]

46. Factorize $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
[C. U. 1928, '40, '45 ; D. B. '24 ; G. U. '48]

The given Exp.$=a^2(b-c)+b^2c-ab^2+ac^2-bc^2$

$=a^2(b-c)-a(b^2-c^2)+bc(b-c)$

$=(b-c)\{a^2-a(b+c)+bc\}=(b-c)(a^2-ab-ac+bc)$

$=(b-c)\{a(a-b)-c(a-b)\}=(b-c)(a-b)(a-c)$

$=-(a-b)(b-c)(c-a).$

[**N. B.** The given expression may be arranged in descending powers of any of the letters. At the time of splitting the expression retain the first bracket and break the rest so as to arrange it again in such a way that each pair of brackets contains $(b-c)$ as in the first term. Again, the answer may be given as $(a-b)(b-c)(a-c)$; but the general rule is that the factors of cyclic order should be given in cyclic order. The cyclic order will be $(a-b)(b-c)(c-a)$. In the answer, $(a-c)$ is not in cyclic order which should be $(c-a)$. Now $a-c=-c+a=-(c-a)$. So $-(c-a)$ may be written in place of $a-c$. The negative sign is to be placed first of all.]

47. Factorize $bc(b-c)+ca(c-a)+ab(a-b)$. [P. U. '30]

The given Exp.$=bc(b-c)+c^2a-ca^2+a^2b-ab^2$

$=bc(b-c)+a^2b-a^2c-ab^2+ac^2$

$=bc(b-c)+a^2(b-c)-a(b^2-c^2)=(b-c)(bc+a^2-ab-ac)$

$=(b-c)\{a(a-c)-b(a-c)\}=(b-c)(a-b)(a-c)$

$=-(a-b)(b-c)(c-a).$

48. Factorize $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$. [D. B. '30]

The given Exp. $=a(b^2-c^2)+bc^2-a^2b+a^2c-b^2c$

$=a(b^2-c^2)-a^2(b-c)-bc(b-c)$

$=(b-c)(ab+ac-a^2-bc)=(b-c)\{a(c-a)-b(c-a)\}$

$=(b-c)(c-a)(a-b)=(a-b)(b-c)(c-a)$.

49. Factorize $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

The given Exp. $=a^3(b-c)+b^3c-ab^3+ac^3-bc^3$

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$

$=a^3(b-c)-a(b-c)(b^2+bc+c^2)+bc(b+c)(b-c)$

$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$

$=(b-c)(a^3-ab^2-abc-ac^2+b^2c+bc^2)$

$=(b-c)(a^3-ab^2-abc+b^2c-ac^2+bc^2)$

$=(b-c)\{a(a^2-b^2)-bc(a-b)-c^2(a-b)\}$

$=(b-c)(a-b)(a^2+ab-bc-c^2)$

$=(a-b)(b-c)\{(a+c)(a-c)+b(a-c)\}$

$=(a-b)(b-c)(a-c)(a+c+b)$

$=-(a-b)(b-c)(c-a)(a+b+c)$.

50. Factorize $(b-c)^3+(c-a)^3+(a-b)^3$. [C. U. '29, '39]

Suppose, $x=b-c$, $y=c-a$, $z=a-b$,

then $x+y+z=b-c+c-a+a-b=0$.

Now, $(b-c)^3+(c-a)^3+(a-b)^3$

$=x^3+y^3+z^3=x^3+y^3+z^3-3xyz+3xyz$

$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$=0\times(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$=0+3xyz=3xyz=3(b-c)(c-a)(a-b)$.

51. Resolve into factors

$a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$. [C. U. '32, '35]

The given Exp.

$=(a^2b+a^2c+abc)+(b^2c+ab^2+abc)+(c^2a+c^2b+abc)$

$=a(ab+ac+bc)+b(bc+ab+ac)+c(ac+bc+ab)$

$=(ab+ac+bc)(a+b+c)$.

[Here $3abc=abc+abc+abc$; so one abc is taken with each term.]

52. Factorize $bc(b+c)+ca(c+a)+ab(a+b)+3abc$.

[C. U. 1939]

The given Exp.

$$=bc(b+c)+abc+ca(c+a)+abc+ab(a+b)+abc$$
$$=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$$
$$=(a+b+c)(bc+ca+ab).$$

53. Factorize $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$.

The given Exp.

$$=(a^2b+a^2c+abc)+(ab^2+abc+b^2c)+(abc+ac^2+bc^2)$$
$$=(a+b+c)(ab+bc+ca).$$ [Vide Ex. 51]

54. Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

[C. U. '33, '50]

The given Exp. $=a^2(b+c)+b^2c+ab^2+ac^2+bc^2+2abc$
$$=a^2(b+c)+(ab^2+ac^2+2abc)+(b^2c+bc^2)$$
$$=a^2(b+c)+a(b+c)^2+bc(b+c)=(b+c)(a^2+ab+ac+bc)$$
$$=(b+c)\{a(a+b)+c(a+b)\}=(b+c)(c+a)(a+b).$$

55. Factorize $bc(b+c)+ca(c+a)+ab(a+b)+2abc$.

The given Exp. $=bc(b+c)+ac^2+a^2c+a^2b+ab^2+2abc$
$$=bc(b+c)+(a^2b+a^2c)+(ab^2+ac^2+2abc)$$
$$=bc(b+c)+a^2(b+c)+a(b+c)^2=(b+c)(bc+a^2+ab+ac)$$
$$=(b+c)\{a(a+b)+c(a+b)\}=(b+c)(c+a)(a+b).$$

56. Factorize $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc$.

The given Exp. $=a(b^2+c^2)+2abc+a^2b+a^2c+b^2c+bc^2$
$$=a(b^2+c^2+2bc)+a^2(b+c)+bc(b+c)$$
$$=a(b+c)^2+a^2(b+c)+bc(b+c)$$
$$=(b+c)(ab+ac+a^2+bc)=(b+c)(c+a)(a+b).$$

57. Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3+c^3$.

The given Exp. $=\{a^2(b+c)+a^3\}+\{b^2(c+a)+b^3\}+\{c^2(a+b)+c^3\}$
$$=a^2(b+c+a)+b^2(c+a+b)+c^2(a+b+c)$$
$$=(a+b+c)(a^2+b^2+c^2).$$

58. Factorize $(x+y+z)^3-x^3-y^3-z^3$.

The given Exp. $=x^3+y^3+z^3+3(x+y)(y+z)(z+x)-x^3-y^3-z^3$
$$=3(x+y)(y+z)(z+x).$$ [Vide formula 10]

59. Factorize $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc$.

[A. U. 1922]

The given Exp. $=a(b^2-2bc+c^2)+b(c^2-2ca+a^2)$
$+c(a^2-2ab+b^2)+9abc$
$=b^2(c+a)+c^2(a+b)+a^2(b+c)-2abc-2abc-2abc+9abc$
$=b^2(c+a)+c^2(a+b)+a^2(b+c)+3abc$
$=(a+b+c)(ab+bc+ca)$. [Vide Ex. 51]

60. Factorize $(a+b+c)(ab+bc+ca)-abc$.

The given Exp. $=\{a+(b+c)\}\{a(b+c)+bc\}-abc$
$=a^2(b+c)+a(b+c)^2+bc(b+c)+abc-abc$
$=(b+c)(a^2+ab+ac+bc)=(b+c)(c+a)(a+b)$.

61. Factorize $(b+c)(c+a)(a+b)+abc$.

The given Exp. $=a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+2abc+abc$
$=(a^2b+ab^2+abc)+(b^2c+bc^2+abc)+(a^2c+ac^2+abc)$
$=ab(a+b+c)+bc(b+c+a)+ca(a+c+b)$
$=(a+b+c)(ab+bc+ca)$.

62. Factorize $(3x-2y)^3-(2x-3y)^3-(x+y)^3$.

Suppose, $a=2x-3y$, $b=x+y$; $\therefore$ $a+b=3x-2y$.

$\therefore$ the given Exp. $=(a+b)^3-a^3-b^3=a^3+b^3+3ab(a+b)-a^3-b^3$
$=3ab(a+b)=3(2x-3y)(x+y)(3x-2y)$.

63. Factorize $x^3+(x-1)^3+(1-2x)^3$.

[D. B. 1940]

Suppose, $a=x$, $b=x-1$, $c=1-2x$,

$\therefore$ $a+b+c=x+x-1+1-2x=0$.

Now, the given Exp. $=a^3+b^3+c^3=3abc$ [$\because$ $a+b+c=0$]
$=3x(x-1)(1-2x)$ [putting the values of a, b, c]

64. Factorize $a(b-c)x^2+b(c-a)x+c(a-b)$.
[Vide Ex. 33]

[D. B. 1933]

The given Exp. $=(ab-ac)x^2+(bc-ab)x+(ac-bc)$
$=(ab-ac)x^2-(ab-ac)x-(ac-bc)x+(ac-bc)$
[$\because$ $-(ab-ac)-(ac-bc)=(bc-ab)$]
$=(ab-ac)x(x-1)-(ac-bc)(x-1)$
$=(x-1)\{(ab-ac)x-(ac-bc)\}=(x-1)(abx-acx-ac+bc)$.

65. Factorize $8(a+b+c)^3-(a+b)^3-(b-c)^3-(c+a)^3$.

The given Exp.$=(2a+2b+2c)^3-(a+b)^3-(b+c)^3-(c+a)^3$

$=\{(a+b)+(b+c)+(c+a)\}^3-(a+b)^3-(b+c)^3-(c+a)^3$

$=(a+b)^3+(b+c)^3+(c+a)^3+3(a+b+b+c)$

$\times(b+c+c+a)(c+a+a+b)-(a+b)^3-(b+c)^3-(c+a)^3$

$=3(a+2b+c)(a+b+2c)(2a+b+c)$.

66. Factorize $(2b-a)^3+(2a-b)^3-(a+b)^3$. [B. U. 1900]

Suppose, $x=2b-a$, $y=2a-b$. $\therefore\ x+y=a+b$.

Now, the given Exp.$=x^3+y^3-(x+y)^3$

$=x^3+y^3-\{x^3+y^3+3xy(x+y)\}$

$=x^3+y^3-x^3-y^3-3xy(x+y)=-3xy(x+y)$

$=-3(2b-a)(2a-b)(a+b)$

$=3(a-2b)(2a-b)(a+b)$ [$\because\ -(2b-a)=(a-2b)$]

67. Factorize $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$. [I. P. S. 1936]

Suppose, $x=a+b-2c$, $y=b+c-2a$, $z=c+a-2b$

Then $x+y+z=a+b-2c+b+c-2a+c+a-2b=0$

$\therefore$ the given Exp.$=x^3+y^3+z^3=3xyz$ [$\because\ x+y+z=0$]

$=3(a+b-2c)(b+c-2a)(c+a-2b)$.

68. Factorize $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.

The given Exp.$=(ab-ac)^3+(bc-ab)^3+(ac-bc)^3$

$=3(ab-ac)(bc-ab)(ac-bc)$ [$\because\ ab-ac+bc-ab+ac-bc=0$]

$=3abc(b-c)(c-a)(a-b)$.

69. Factorize $2a^2+2a-3ab-b+b^2$. [D. B. 1949]

The given Exp.$=(2a^2-3ab+b^2)+2a-b=(2a^2-2ab-ab+b^2)$
$+1(2a-b)=(2a-b)(a-b)+1(2a-b)=(2a-b)(a-b+1)$.

70. Factorize $a^4+a^3-10a^2+a+1$.

[To find out the factors of the expression in which the coefficients of terms equidistant from the beginning and the end are equal, each pair of such equidistant terms is to be taken together.]

The given Exp.$=(a^4+1)+(a^3+a)-10a^2$

$=(a^2+1)^2-2a^2+a(a^2+1)-10a^2=(a^2+1)^2+a(a^2+1)-12a^2$

$=x^2+ax-12a^2$ [putting x for a^2+1] $=a^2+4ax-3ax-12a^2$

$=(x+4a)(x-3a)=(a^2+1+4a)(a^2+1-3a)$.

Exercise 3

Resolve into factors :—

1. $16x^4-81y^4$. [C. U. '21] 2. x^4-y^4. [C. U. '26]
3. $m^4+m^2n^2+n^4$. [C. U. '33] 4. x^4+4. [C. U. '34]
5. $x^4+x^2y^2+y^4$. [C. U. '35]
6. $4x^2-4xy-2yz-z^2$ [C.U. '35] 7. (i) $a^2-b^2+2bc-c^2$.
7. (ii) $a^2-b^2-c^2+d^2-2(ad-bc)$. [W. B. S. F. '53]
8. $4x^4+81$. [C. U. '37] 9. x^4+2x^2+9. [D. B. '30]
10. x^4+64. [W. B. S. F. '52 ; B. U. 1920 ; D. B. '31]
11. a^8-16b^8. [A. U. '23] 12. x^4-8x^2+4. [B. U. '24]
13. x^4+x^2+1. [C. U. 1920, '24 ; D. B. '34, '35]
14. $4b^2c^2-(b^2+c^2-a^2)^2$. [B. U. '23]
15. x^3-27. [C. U. '29] 16. x^3+64y^3. [C. U. '23]
17. (i) x^6-729. [D. B. '36] (ii) $8a^3+27b^3$.
18. x^2-x-6. [C. U. 1924 ; D. B. '27]
19. (i) $x^2-3x-28$. [C. U. '28] (ii) $12+x-20x^2$.
20. $125x^5y^2-27x^2y^5$. [A. U. '33]
21. $x^2-2x-35$. [C. U. '20] 22. $6-a-12a^2$. [C. U. '30]
23. $x^2-y^2-6xa+2ya+8a^2$. [D. B. '48]
24. $12x^2+65x+77$. [D. B. '34]
25. $21x^2+40xy-21y^2$. [D. B. '38]
26. $8a^4+2a^2-45$. [A.U. '27] 27. a^6+32a^3-64. [P.U. '26]
28. $a^3+b^3-8c^3+6abc$.
29. $a^3-3a-2a^2+4$.
30. $x^3+16x^2+13x-30$. [C. U. '39]
31. $x^3+9x^2+26x+24$. [D. B. '36]
32. $(x^2-4x)(x^2-4x-1)-20$. [D. B. '38]
33. $(ax+by)^2+(bx-ay)^2$. [B. U. '12]
34. $2(ab+cd)-a^2-b^2+c^2+d^2$. [P. U. '26]
35. $(a+b)^2-10(a^2-b^2)-56(a-b)^2$. [C. U. '09]
36. $x(x-1)(x-2)-3x+3$. [B. U. '34]
37. $a(a-1)(a-2)(a-3)-120$. [D. B. '35]
38. $(x+1)(x+3)(x-4)(x-6)+13$.
39. $(x+y+z)(xy+yz+zx)-xyz$. [D. B. '32]

40. $x^3 - y^3 - 1 - 3xy$.

41. $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$. [M. U. '27]

42. $b^2c^2(b - c) + c^2a^2(c - a) + a^2b^2(a - b)$.

43. $a(b+c)^2 + b(c+a)^2 + c(a+b)^2 - 4abc$.

44. $(2x - 3y)^3 + (3y - z)^3 + (z - 2x)^3$.

45. $(x+1)(x+2)(x+3)(x+4) - 3$. [C. U. 1946]

46. $x^2 + 2x - (a+1)(a+3)$. [Hints. $2x = (a+3)x - (a+1)x$]

47. $x(2x+1)(x-2)(2x-3) - 63$.

[Make $x(2x-3)$ one group and $(2x+1)(x-2)$ another group]

48. $(a+2b-c)^3 - (a+b)^3 - (b-c)^3$.

[Hints. $(a+2b-c)^3 = \{(a+b)+(b-c)\}^3$]

49. $4(ad - bc)^2 - (a^2 + d^2 - b^2 - c^2)^2$.

[Hints. $4(ad - bc)^2 = (2ad - 2bc)^2$]

50. $(1 - c^2)(1+a)^2 - (1 - a^2)(1 - c)^2$. [C. U. 1881]

51. (i) $4(x^2+2x+5)^2 + 17(x^2+2x+5)(x^2+6x) + 4(x^2+6x)^2$

(ii) $a^4 + 6a^3 + 4a^2 - 15a + 6$. [C. U. 1948]

52. $x^4 + x^3 + 2x^2 + x + 1$. [C. U. 1948]

53. $x^2(b - c) + b^2(c - x) + c^2(x - b)$. [D. B. '27]

H. C. F. and L. C. M.

Highest Common Factor [H. C. F.]

Examples (4)

1. Find the H. C. F. of $4x^2(x^2 - y^2)$, $6x^3(x^3 - y^3)$ and $2xy(x - y)$.

$$4x^2(x^2 - y^2) = 2^2.x^2.(x+y)(x-y)$$
$$6x^3(x^3 - y^3) = 2.3.x^3(x-y)(x^2+xy+y^2)$$
$$\text{and } 2xy(x-y) = 2.x.y.(x-y).$$

$\therefore$ the required H. C. F. $= 2.x.(x-y) = 2x(x-y)$.

[N. B. The rule for finding out H. C. F. by factorization : First find out the factors of each expression. (If there be 4, write 2^2 instead of 2×2). Now the product of all the common factors that can be found in these expressions will be the H. C. F. Here '2' is a factor of the three given expressions and there are different powers of 2. In the H. C. F. the lowest power is to be taken.]

2. (Other method :) Find the H. C. F. of x^3-3x^2+x-3 and x^4+6x^2+5. [C. U. '22]

$$
\begin{array}{r|l}
x^3-3x^2+x-3 & x^4 \qquad +6x^2 \qquad +5 \;(\, x+3 \\
 & \underline{x^4-3x^3+\;x^2-3x} \\
 & \quad 3x^3+5x^2+3x+5 \\
 & \quad \underline{3x^3-9x^2+3x-9} \\
 & {}^{*}14 \;|\; \underline{14x^2+14} \\
 & \qquad\qquad x^2+1
\end{array}
$$

$$
\begin{array}{r|l}
x^2+1 & x^3-3x^2+x-3 \;(\, x-3 \\
 & \underline{x^3 \qquad\quad +x} \\
 & \quad -3x^2 \quad -3 \\
 & \quad \underline{-3x^2 \quad -3}
\end{array}
$$

$\therefore$ The reqd. H. C. F. $=x^2+1$.

[**N. B.** When in the operation of division, the power of x in the first term of the remainder becomes less than that of the first term of the divisor, then divide the divisor by the remainder, then the second divisor by the second remainder and so on, until there is no remainder. The last divisor is the H. C. F. required.

Again, take out the common factors of the remainders, if there be any. If there be no other common factors, but the sign '$-$' at the beginning, you should take out -1 as a common factor. *Here the divisor has been divided by x^2+1 after taking out 14 as a common factor of $14x^2+14$. Take out the common factors only when you are to change the operation of division, but never take out common factors, even if there be any, so long as an ordinary work of division is possible.]

3. Find the H. C. F. of $3x^4+15x^3+5x^2+10x+2$ and $2x^4+9x^3+14x+3$. [D. B. 1926]

$$
\begin{array}{r|l}
2x^4+9x^3+14x+3 & 3x^4+15x^3+\;5x^2+10x+2 \;(\\
 & \underline{2^{*}} \\
 & 6x^4+30x^3+10x^2+20x+4 \;(\, 3 \\
 & \underline{6x^4+27x^3 \qquad\qquad +42x+9} \\
 & \qquad 3x^3+10x^2-22x-5
\end{array}
$$

[*Here $3x^4$ of the dividend is not divisible by $2x^4$ of the divisor ; so the whole dividend has been multiplied by 2 to make it divisible. You should multiply only by an integer and not by x.]

$$
\begin{array}{r|l}
3x^3+10x^2-22x-5 & 2x^4+9x^3\qquad+14x+3\,(\\
 & 3 \\ \hline
 & 6x^4+27x^3\qquad+42x+9\,(2x \\
 & 6x^4+20x^3-44x^2-10x \\ \hline
 & \qquad 7x^3+44x^2+52x+9 \\
 & \qquad 3 \\ \hline
 & \qquad 21x^3+132x^2+156x+27\,(7 \\
 & \qquad 21x^3+\ \ 70x^2-154x-35 \\ \hline
 & \qquad\qquad 62\ |\ 62x^2+310x+62 \\ \hline
 & \qquad\qquad\qquad x^2+\ \ 5x+1
\end{array}
$$

$$
\begin{array}{r|l}
x^2+5x+1 & 3x^3+10x^2-22x-5\,(3x-5 \\
 & 3x^3+15x^2+\ \ 3x \\ \hline
 & \quad -\ 5x^2-25x-5 \\
 & \quad -\ 5x^2-25x-5 \\ \hline
\end{array}
$$

$\therefore$ The reqd. H. C. F.$=x^2+5x+1$.

4. Find the H. C. F. of $2x^5-10x^3-4x^2$ and $6x^4-15x^3+3x$.

[**N. B.** If there be any common factors in the terms of any expression, first take them out. Here both the expressions have their common factors. Take out the common factors of each expression and find the H. C. F. of those common factors of the two given expressions. Then find the H. C. F. of the two expressions which are left after taking out the common factors of the given expressions. The product of these two H. C. F.'s will be the H. C. F. required.]

$$2x^5-10x^3-4x^2=2x^2(x^3-5x-2)$$
$$6x^4-15x^3+3x=3x(2x^3-5x^2+1)$$

Now, the H. C. F. of $2x^2$ and $3x=x$.

$$
\begin{array}{r|l}
x^3-5x-2 & 2x^3-5x^2\qquad+1\,(2 \\
 & 2x^3\qquad-10x-4 \\ \hline
-5 & |\ -5x^2+10x+5 \\ \hline
\end{array}
$$

$$
\begin{array}{r|l}
x^2-2x-1 & x^3\qquad\quad-5x-2\,(x+2 \\
 & x^3-2x^2-\ \ x \\ \hline
 & \qquad 2x^2-4x-2 \\
 & \qquad 2x^2-4x-2 \\ \hline
\end{array}
$$

$\therefore$ The reqd. H. C. F.$=x(x^2-2x-1)$.

[(1) Sometimes boys do not mention what the H. C. F. is after completing the operation. No credit is given unless the particular H. C. F. is separately mentioned.

(2) In finding out the H. C. F. of more than two expressions, first find out the H. C. F. of any two expressions. Then find out the H. C. F. of that H. C. F. and the third expression and so on. The last H. C. F. will be the H. C. F. required.]

5. If $(x+a)$ be the H. C. F. of x^2+px+q and $x^2+p'x+q'$, show that $(p-p')a=q-q'$. [C. U. 1941]

$\because$ $x+a$ is the H. C. F. of the two given expressions,

$\therefore$ it is a factor of the two expressions. So if the value of $(x+a)$ be 0, the value of the two expressions will also be 0.

Now, if $x=-a$, $x+a=0$. The value of the two expressions will, therefore, be 0, if the value of x is taken to be $-a$.

Therefore, $a^2-ap+q=0$

and $a^2-ap'+q'=0$

(subtractiug) $-ap+ap'+q-q'=0$, or, $-a(p-p')=-(q-q')$

$\therefore$ $a(p-p')=q-q'$.

Lowest Common Multiple [L. C. M.]

[N. B. (1). To find the L. C. M. of two or more expressions we have to resolve each expression into all elementary factors and then the product of all kinds of factors in their highest powers will be the L. C. M. required, whether the factors be common factors or not.

(2). To find the L. C. M. of any number of expressions if any expression be in the form $(x+a)^2$ or $(a+b)^3$, do not expand it.

(3). The L. C. M. of two expressions whose factors are not obvious by inspection can be found as follows :

To find the L. C. M. of any two expressions we have to divide one of them by their H. C. F. and multiply the quotient by the other expression.

(4). To find the L. C. M. of any number of expressions A, B, C, D, etc., we have first to find the L. C. M. of A and B, then to find the L. C. M. of the result and C, and so on ; the last result thus obtained is the L. C. M. required.]

1. Find the L. C. M. of $4(a^3-b^3)$, $6a^2(a^2-b^2)$, $8(a+b)^2$ and $a^3-2a^2b+ab^2$.

$4(a^3-b^3)=2^2(a-b)(a^2+ab+b^2)$;

$6a^2(a^2-b^2)=2\times3\times a^2(a+b)(a-b)$;

$8(a+b)^2=2^3(a+b)^2$;

and $a^3-2a^2b+ab^2=a(a^2-2ab+b^2)=a(a-b)^2$.

$\therefore$ the reqd. L. C. M.

$=2^3\times3\times a^2\times(a-b)^2(a+b)^2(a^2+ab+b^2)$

$=24a^2(a-b)^2(a+b)^2(a^2+ab+b^2)$.

2. Find the L. C. M. of $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$. [C. U. 1926]

$$6x^2-x-1=6x^2-3x+2x-1=3x(2x-1)+1(2x-1)$$
$$=(2x-1)(3x+1);$$
$$3x^2+7x+2=3x^2+6x+x+2=3x(x+2)+1(x+2)$$
$$=(x+2)(3x+1);$$
$$2x^2+3x-2=2x^2+4x-x-2=2x(x+2)-1(x+2)$$
$$=(x+2)(2x-1);$$

$\therefore$ the reqd. L. C. M. $=(x+2)(2x-1)(3x+1)$.

3. Find the L. C. M. of x^3-2x+1 and x^3+2x^2-1.

[D. B. 1930]

$$x^3-2x+1=x^3-x^2+x^2-x-x+1$$
$$=x^2(x-1)+x(x-1)-1(x-1)=(x-1)(x^2+x-1),$$
$$x^3+2x^2-1=x^3+x^2+x^2+x-x-1$$
$$=x^2(x+1)+x(x+1)-1(x+1)=(x+1)(x^2+x-1),$$

$\therefore$ the reqd. L. C. M. $=(x+1)(x-1)(x^2+x-1)$.

4. Find the L. C. M. of $x^3-16x+24$ and $2x^3-5x^2+4$.

[C. U. 1933]

[Another method with the help of the H. C. F. is shown here.]

$$\begin{array}{r|l|l}
x^3-16x+24 & 2x^3-5x^2+4 & 2 \\
 & 2x^3-32x+48 & \\ \hline
- & -5x^2+32x-44 & \\ \hline
5x^2-32x+44 & x^3-16x+24 & \\
 & 5 & \\ \hline
 & 5x^3-80x+120 & x \\
 & 5x^3-32x^2+44x & \\ \hline
 & 32x^2-124x+120 & \\
 & 5 & \\ \hline
 & 160x^2-620x+600 & 32 \\
 & 160x^2-1024x+1408 & \\ \hline
404 & 404x-808 & \\ \hline
 & x-2 &
\end{array}$$

$$\begin{array}{r|l|l}
x-2 & 5x^2-32x+44 & 5x-22 \\
 & 5x^2-10x & \\ \hline
 & -22x+44 & \\
 & -22x+44 & \\ \hline
\end{array}$$

$\therefore$ H. C. F. $=x-2$.

$$\therefore \text{ the reqd. L. C. M.}=\frac{(x^3-16x+24)(2x^3-5x^2+4)}{(x-2)}$$
$$=(x^2+2x-12)(2x^3-5x^2+4).$$

26. a^3-1 and a^7-1. [D. B. 1941]

27. $3x^3-5x^2+5x-2$ and $2x^4-2x^3+3x^2-x+1$. [P. U. '30]

28. $4x^4-12x^2-8x$ and $6x^4-24x^3+30x^2-12x$.

29. If $x-a$ be the H. C. F. of bx^2+cx+d and $b_1x^2+c_1x+d$, prove that $a(b-b_1)=c_1-c$.

Find the L. C. M. of :—

30. x^2-4, x^2-x-2 and x^2+x-2. [C. U. 1910]

31. $2x^2-x-1$, $2x^2+3x+1$ and x^2-1. [C. U. 1912]

32. $x^2-(a-c)x-ac$ and $x^2-(a+c)x+ac$. [C. U. 1928]

33. x^2-3x+2, x^2-4x+3 and x^2-5x+6. [C. U. 1922 ; D. B. 1929]

34. a^2+6a+8, a^2+5a+6 and a^3+4a^2+4a+3. [C. U. '34]

35. $x^2-12x+35$, x^2-8x+7 and x^3-5x^2-x+5. [C. U. '35]

36. x^2-3x+2, x^3-4x and $x^4+x^3-6x^2$. [C. U. 1936]

37. a^2-3a+2, $(a-1)^2$ and a^4-1. [C. U. 1938]

38. $a^2-b^2-c^2+2bc$, $(a+b-c)^2$, $a^2-b^2+c^2+2ac$. [C. U. '40]

39. $8x^3+27$, $16x^4+36x^2+81$ and $7x^2-5x-6$. [D. B. 1926]

[Here the last expression has no factors.]

40. (i) $4x^2+8x-12$, $9x^2-9x-54$ and $6x^4-30x^2+24$. [D. B. 1939]

(ii) $x^2(x^2-4)$ and $x^4+2x^3-8x^2$. [W. B. S. F. '52]

41. $x-2$, x^3+2 and x^2+x+1.

42. The H. C. F. and L. C. M. of two expressions are $x+1$, and $x^3+6x^2+11x+6$ and one of the expressions is x^2+3x+2, find the other.

43. The H. C. F. and L. C. M. of two expressions of the second degree are $(x-1)$ and $(x-1)(x+1)(x+3)$ respectively. Find the expressions. [D. B. 1937 Addl.]

FRACTIONS

Examples [5]

1. Reduce $\dfrac{2x^2-x-6}{2x^2-3x-9}$ to its lowest terms.

[**N. B.** In such examples where the numerator and the denominator cannot be factorised by inspection, the reduction of fraction is effected by dividing both the numerator and the denominator by their highest common factor, which should be actually worked out.]

Here the H. C. F. of $2x^2-x-6$ and $2x^2-3x-9=2x+3$.

$\therefore$ the required lowest terms$=\dfrac{(2x^2-x-6)\div(2x+3)}{(2x^2-3x-9)\div(2x+3)}=\dfrac{x-2}{x-3}$.

2. Simplify :—

$$\frac{1}{x^2-8x+15}+\frac{1}{x^2-4x+3}-\frac{2}{x^2-6x+5}. \qquad \text{[C. U. 1920]}$$

The given Exp.$=\dfrac{1}{(x-5)(x-3)}+\dfrac{1}{(x-3)(x-1)}-\dfrac{2}{(x-5)(x-1)}$

$$=\frac{(x-1)+(x-5)-2(x-3)}{(x-1)(x-3)(x-5)}=\frac{x-1+x-5-2x+6}{(x-1)(x-3)(x-5)}$$

$$=\frac{0}{(x-1)(x-3)(x-5)}=0.$$

3. Simplify :—

$$\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}. \qquad \text{[C. U. 1939 (Sup.)]}$$

The given Exp.

$$=\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^2(x-y)}{y(x^2-y^2)}=\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^2}{y(x+y)}$$

$$=\frac{(x+y)^2-xy-x^2}{y(x+y)}=\frac{x^2+2xy+y^2-xy-x^2}{y(x+y)}=\frac{xy+y^2}{y(x+y)}$$

$$=\frac{y(x+y)}{y(x+y)}=1.$$

4. Simplify :—

$$\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}-\dfrac{b}{a+b}}\div\dfrac{\dfrac{a+b}{a-b}-\dfrac{b-a}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}. \qquad \text{[C. U. 1934]}$$

$$\text{The given fraction}=\dfrac{\dfrac{a^2+ab-a^2+ab}{(a-b)(a+b)}}{\dfrac{ab+b^2-ab+b^2}{(a-b)(a+b)}}\div\dfrac{\dfrac{(a+b)^2+(a-b)^2*}{(a-b)(a+b)}}{\dfrac{(a+b)^2-(a-b)^2}{(a-b)(a+b)}}$$

$$\left[*-\frac{b-a}{a+b}=-\frac{-(a+b)}{a+b}=+\frac{a-b}{a+b}\right]$$

$$=\dfrac{\dfrac{2ab}{a^2-b^2}}{\dfrac{2b^2}{a^2-b^2}}\div\dfrac{\dfrac{2(a^2+b^2)}{a^2-b^2}}{\dfrac{4ab}{a^2-b^2}}$$

$$=\left(\frac{2ab}{a^2-b^2}\times\frac{a^2-b^2}{2b^2}\right)\div\left\{\frac{2(a^2+b^2)}{a^2-b^2}\times\frac{a^2-b^2}{4ab}\right\}$$

$$=\frac{a}{b}\div\frac{(a^2+b^2)}{2ab}=\frac{a}{b}\times\frac{2ab}{a^2+b^2}=\frac{2a^2}{a^2+b^2}.$$

5. Simplify :—

$$\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)\div\left(\frac{1}{\dfrac{x}{y}-\dfrac{y}{x}}-\frac{1}{\dfrac{x}{y}+\dfrac{y}{x}}\right).$$

[C. U. 1913]

The given fraction

$$=\left\{\frac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}\right\}\div\left(\frac{1}{\dfrac{x^2-y^2}{xy}}-\frac{1}{\dfrac{x^2+y^2}{xy}}\right)$$

$$=\frac{4xy}{x^2-y^2}\div\left(\frac{xy}{x^2-y^2}-\frac{xy}{x^2+y^2}\right)$$

$$=\frac{4xy}{x^2-y^2}\div\left\{\frac{x^3y+xy^3-x^3y+xy^3}{(x^2-y^2)(x^2+y^2)}\right\}$$

$$=\frac{4xy}{x^2-y^2}\div\frac{2xy^3}{(x^2-y^2)(x^2+y^2)}$$

$$=\frac{4xy}{(x^2-y^2)}\times\frac{(x^2-y^2)(x^2+y^2)}{2xy^3}=\frac{2(x^2+y^2)}{y^2}.$$

6. Simplify : $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}$. [A. U. 1912]

The given fraction

$$=\frac{1}{1+\dfrac{1}{1+\dfrac{1}{\dfrac{x+1}{x}}}}=\frac{1}{1+\dfrac{1}{1+\dfrac{x}{x+1}}}=\frac{1}{1+\dfrac{1}{\dfrac{x+1+x}{x+1}}}=\frac{1}{1+\dfrac{1}{\dfrac{2x+1}{x+1}}}$$

$$=\frac{1}{1+\dfrac{x+1}{2x+1}}=\frac{1}{\dfrac{2x+1+x+1}{2x+1}}=\frac{1}{\dfrac{3x+2}{2x+1}}=\frac{2x+1}{3x+2}.$$

6 (*a*). Simplify : $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}}$. [C. U. 1946, '48, '49]

$\left[\text{Ans.}=\dfrac{x^5}{x^4-x^2+1}\right]$

7. Simplify :—$\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$. [C. U. '21]

$$\text{The given Exp.}=\frac{\dfrac{b+c+a}{a(b+c)}}{\dfrac{b+c-a}{a(b+c)}}\left\{\frac{2bc+b^2+c^2-a^2}{2bc}\right\}$$

$$=\frac{(b+c+a)\times a(b+c)}{a(b+c)\times(b+c-a)}\left\{\frac{(b+c)^2-a^2}{2bc}\right\}$$

$$=\frac{(b+c+a)}{(b+c-a)}\times\frac{(b+c+a)(b+c-a)}{2bc}=\frac{(a+b+c)^2}{2bc}.$$

8. Simplify :—

$$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}.$$ [C. U. 1937]

$$\text{The given Exp.}=\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}$$

$$+\frac{(c+a-b)(c-a+b)}{(b+c-a)(b+c+a)}=\frac{a+b-c}{a+b+c}+\frac{b+c-a}{a+b+c}+\frac{c+a-b}{a+b+c}$$

$$=\frac{a+b-c+b+c-a+c+a-b}{a+b+c}=\frac{a+b+c}{a+b+c}=1.$$

$$=\frac{1}{c}(b^2+c^2-a^2+c^2+a^2-b^2)+\frac{1}{b}(b^2+c^2-a^2+a^2+b^2-c^2)$$
$$+\frac{1}{a}(c^2+a^2-b^2+a^2+b^2-c^2)$$
$$=\frac{1}{c}\times 2c^2+\frac{1}{b}\times 2b^2+\frac{1}{a}\times 2a^2=2c+2b+2a=2(a+b+c).$$

[N. B. We may write $\frac{b+c}{bc}=\frac{b}{bc}+\frac{c}{bc}=\frac{1}{c}+\frac{1}{b}$. Thus, the three terms have been split up. In two places $\frac{1}{c}$ is a multiplying factor and those two products with $\frac{1}{c}$ have been placed side by side. Similarly, the other products also have been written,]

15. Simplify ;—

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}.$$

[N. B. To simplify fractions involving cyclic order, first arrange the denominators in cyclic order. $(a-b)$ $(b-c)$ $(c-a)$ is in cyclic order. Here $(a-c)$ in the denominator of the first term is not in cyclic order. We may write $(a-c)=-c+a$ $=-(c-a)$. The factor which is not in cyclic order should thus be made to tally with the cyclic order. The negative sign should be placed at the beginning.]

The given Exp.

$$=\frac{a}{-(a-b)(c-a)}+\frac{b}{-(a-b)(b-c)}+\frac{c}{-(c-a)(b-c)}$$
$$=\frac{a(b-c)+b(c-a)+c(a-b)}{-(a-b)(b-c)(c-a)}=\frac{0}{-(a-b)(b-c)(c-a)}=0.$$

16. $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}$$

[D. B. 1932, 1948 ; A. U. 1925]

$$=\frac{a^2}{-(a-b)(c-a)}+\frac{b^2}{-(a-b)(b-c)}+\frac{c^2}{-(c-a)(b-c)}$$
$$=\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{-(a-b)(b-c)(c-a)}$$
$$=\frac{-(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)}=1.$$

[Vide formula 11]

17. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}$

$=\dfrac{a^3}{-(a-b)(c-a)}+\dfrac{b^3}{-(a-b)(b-c)}+\dfrac{c^3}{-(c-a)(b-c)}$

$=\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{-(a-b)(b-c)(c-a)}$

$=\dfrac{-(a-b)(b-c)(c-a)(a+b+c)}{-(a-b)(b-c)(c-a)}=a+b+c.$ [Vide formula 14]

18. $\dfrac{b^2c^2}{(a-b)(a-c)}+\dfrac{c^2a^2}{(b-a)(b-c)}+\dfrac{a^2b^2}{(c-b)(c-a)}.$ [C. U. '31]

The given Exp.

$=\dfrac{b^2c^2}{-(a-b)(c-a)}+\dfrac{c^2a^2}{-(a-b)(b-c)}+\dfrac{a^2b^2}{-(b-c)(c-a)}$

$=\dfrac{b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)}{-(a-b)(b-c)(c-a)}$

$=\dfrac{-(a-b)(b-c)(c-a)(ab+ac+bc)}{-(a-b)(b-c)(c-a)}=ab+ac+bc.$

19. $\dfrac{1}{bc(b-a)(c-a)}+\dfrac{1}{ca(c-b)(a-b)}+\dfrac{1}{ab(a-c)(b-c)}$

$=\dfrac{1}{-bc(a-b)(c-a)}+\dfrac{1}{-ca(b-c)(a-b)}+\dfrac{1}{-ab(c-a)(b-c)}$

$=\dfrac{a(b-c)+b(c-a)+c(a-b)}{-abc(a-b)(b-c)(c-a)}$

$=\dfrac{0}{-abc(a-b)(b-c)(c-a)}=0.$

20. $\dfrac{a(b+c)}{(a-b)(a-c)}+\dfrac{b(c+a)}{(b-a)(b-c)}+\dfrac{c(a+b)}{(c-a)(c-b)}$ [C. U. 1923]

$=\dfrac{a(b+c)}{-(a-b)(c-a)}+\dfrac{b(c+a)}{-(a-b)(b-c)}+\dfrac{c(a+b)}{-(c-a)(b-c)}$

$=\dfrac{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)}{-(a-b)(b-c)(c-a)}$

$=\dfrac{(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)}=-1.$ [Vide formula 13]

21. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2-ca}{(b-a)(b-c)}+\dfrac{c^2-ab}{(c-a)(c-b)}$ [C. U. 1924]

$$=\frac{a^2-bc}{-(a-b)(c-a)}+\frac{b^2-ca}{-(a-b)(b-c)}+\frac{c^2-ab}{-(c-a)(b-c)}$$

$$=\frac{(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}-\{bc(b-c)+ca(c-a)+ab(a-b)\}}{-(a-b)(b-c)(c-a)}$$

$$=\frac{\{-(a-b)(b-c)(c-a)\}-\{-(a-b)(b-c)(c-a)\}}{-(a-b)(b-c)(c-a)}$$

$$=\frac{0}{-(a-b)(b-c)(c-a)}=0.$$

22. Simplify :—

$$\frac{(b-c)^2}{(a-b)(a-c)}+\frac{(c-a)^2}{(b-a)(b-c)}+\frac{(a-b)^2}{(c-a)(c-b)}+3.$$ [C. U. '39]

The given Exp.

$$=\frac{(b-c)^2}{-(a-b)(c-a)}+\frac{(c-a)^2}{-(a-b)(b-c)}+\frac{(a-b)^2}{-(c-a)(b-c)}+3$$

$$=\frac{(b-c)^3+(c-a)^3+(a-b)^3}{-(a-b)(b-c)(c-a)}+3=\frac{3(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)}+3$$

$=-3+3=0.$ [$\because$ $(b-c)+(c-a)+(a-b)=0.$]

23. $\dfrac{3a-b-c}{(a-b)(a-c)}+\dfrac{3b-c-a}{(b-c)(b-a)}+\dfrac{3c-a-b}{(c-a)(c-b)}$ [D. B. '33]

$$=\frac{3a-(b+c)}{-(a-b)(c-a)}+\frac{3b-(c+a)}{-(b-c)(a-b)}+\frac{3c-(a+b)}{-(c-a)(b-c)}$$

$$=\frac{\{3a-(b+c)\}(b-c)+\{3b-(c+a)\}(c-a)+\{3c-(a+b)\}(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{3a(b-c)+3b(c-a)+3c(a-b)-(b^2-c^2)-(c^2-a^2)-(a^2-b^2)}{-(a-b)(b-c)(c-a)}$$

[Here in multiplying $\{3a-(b+c)\}$ by $(b-c)$, multiply $3a$ by $(b-c)$ and $-(b+c)$ by $(b-c)$ and keep the products separately.]

$$=\frac{3\{a(b-c)+b(c-a)+c(a-b)\}-(b^2-c^2+c^2-a^2+a^2-b^2)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{3\times 0-0}{-(a-b)(b-c)(c-a)}=\frac{0}{-(a-b)(b-c)(c-a)}=0.$$

24. Simplify : $\dfrac{ax^2+bx+c}{(x-y)(x-z)}+\dfrac{ay^2+by+c}{(y-z)(y-x)}+\dfrac{az^2+bz+c}{(z-x)(z-y)}.$

Here the first term

$$=\frac{ax^2}{-(x-y)(z-x)}+\frac{bx}{-(x-y)(z-x)}+\frac{c}{-(x-y)(z-x)}$$

The second term

$$=\frac{ay^2}{-(y-z)(x-y)}+\frac{by}{-(y-z)(x-y)}+\frac{c}{-(y-z)(x-y)}$$

The third term

$$=\frac{az^2}{-(z-x)(y-z)}+\frac{bz}{-(z-x)(y-z)}+\frac{c}{-(z-x)(y-z)}$$

$\therefore$ The given Exp.

$$=a\left\{\frac{x^2}{-(x-y)(z-x)}+\frac{y^2}{-(y-z)(x-y)}+\frac{z^2}{-(z-x)(y-z)}\right\}$$

$$+b\left\{\frac{x}{-(x-y)(z-x)}+\frac{y}{-(y-z)(x-y)}+\frac{z}{-(z-x)(y-z)}\right\}$$

$$+c\left\{\frac{1}{-(x-y)(z-x)}+\frac{1}{-(y-z)(x-y)}+\frac{1}{-(z-x)(y-z)}\right\}$$

$$=a\times1+b\times0+c\times0=a.$$

25. $\dfrac{(x-c)(x-b)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-a)(b-c)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$ [C.U. '41]

$$=\frac{x^2-x(b+c)+bc}{-(a-b)(c-a)}+\frac{x^2-x(c+a)+ca}{-(a-b)(b-c)}+\frac{x^2-x(a+b)+ab}{-\;c-a)(b-c)}$$

$$=\frac{\{x^2-x(b+c)+bc\}(b-c)+\{x^2-x(c+a)+ca\}(c-a)}{-(a-b)(b-c)(c-a)}$$

$$+\frac{\{x^2-x(a+b)+ab\}(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{x^2(b-c)-x(b^2-c^2)+bc(b-c)+x^2(c-a)-x(c^2-a^2)}{-(a-b)(b-c)(c-a)}$$

$$+\frac{ca(c-a)+x^2(a-b)-x(a^2-b^2)+ab(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{x^2(b-c+c-a+a-b)-x(b^2-c^2+c^2-a^2+a^2-b^2)}{-(a-b)(b-c)(c-a)}$$

$$+\frac{bc(b-c)+ca(c-a)+ab(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{x^2\times0-x\times0-(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)}=\frac{-(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)}=1.$$

26. Simplify :—

$$\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-c)(b-a)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}.$$

Suppose, $x+a=p$, $x+b=q$ and $x+c=r$.

Now, $p-q=x+a-x-b=a-b$, $q-r=x+b-x-c=b-c$,

and $r-p=x+c-x-a=c-a$.

$\therefore$ the given Exp. $=\dfrac{1}{-(a-b)(c-a)(x+a)}$

$$+\frac{1}{-(b-c)(a-b)(x+b)}+\frac{1}{-(c-a)(b-c)(x+c)}$$

$$=\frac{1}{-(p-q)(r-p)p}+\frac{1}{-(p-q)(q-r)q}+\frac{1}{-(r-p)(q-r)r}$$

$$=\frac{qr(q-r)+rp(r-p)+pq(p-q)}{-pqr(p-q)(q-r)(r-p)}$$

$$=\frac{-(p-q)(q-r)(r-p)}{-pqr(p-q)(q-r)(r-p)}=\frac{1}{pqr}=\frac{1}{(x+a)(x+b)(x+c)}.$$

27. $$\frac{\dfrac{a-x^2+y^2}{x+y}+\dfrac{b-y^2+z^2}{y+z}+\dfrac{c-z^2+x^2}{z+x}}{\dfrac{a}{x+y}+\dfrac{b}{y+z}+\dfrac{c}{z+x}}.$$

Numerator $=\dfrac{a-(x^2-y^2)}{x+y}+\dfrac{b-(y^2-z^2)}{y+z}+\dfrac{c-(z^2-x^2)}{z+x}$

$$=\frac{a}{x+y}-(x-y)+\frac{b}{y+z}-(y-z)+\frac{c}{z+x}-(z-x)$$

$$=\frac{a}{x+y}+\frac{b}{y+z}+\frac{c}{z+x}-(x-y+y-z+z-x)$$

$$=\frac{a}{x+y}+\frac{b}{y+z}+\frac{c}{z+x}.$$

$\therefore$ The given fraction $=\dfrac{\dfrac{a}{x+y}+\dfrac{b}{y+z}+\dfrac{c}{z+x}}{\dfrac{a}{x+y}+\dfrac{b}{y+z}+\dfrac{c}{z+x}}=1.$

28. $\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$

$=\frac{a^2+ax+a^2-ax}{a^2-x^2}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$

[Adding the first two terms]

$=\frac{2a^2}{a^2-x^2}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$

$=\frac{2a^4+2a^2x^2+2a^4-2a^2x^2}{a^4-x^4}+\frac{4a^4}{a^4+x^4}=\frac{4a^4}{a^4-x^4}+\frac{4a^4}{a^4+x^4}$

$=\frac{4a^8+4a^4x^4+4a^8-4a^4x^4}{a^8-x^8}=\frac{8a^8}{a^8-x^8}$.

Exercise 5

Simplify :—

1. $\frac{x}{x-y}+\frac{y}{x+y}+\frac{2xy}{y^2-x^2}$ [C.U. 1916]

2. $\left(x+\frac{a-x}{1+ax}\right)\times\frac{x}{a}\div\left\{1-\frac{x(a-x)}{1+ax}\right\}$ [A.U. 1932]

3. $\frac{2}{x^2-1}+\frac{3}{x^2+x-2}+\frac{2}{x^2+3x+2}$ [C.U. 1930]

4. $\frac{a-2x}{a+2x}-\frac{a+2x}{a-2x}+\frac{8ax}{a^2+4x^2}$ [C.U. 1933]

5. $\frac{(a-c)^2-b^2}{a^2-(b+c)^2}+\frac{(b-a)^2-c^2}{b^2-(c+a)^2}+\frac{(c-b)^2-a^2}{c^2-(a+b)^2}$ [C.U. 1935]

6. $\frac{9x^2-(y-z)^2}{(3x+z)^2-y^2}+\frac{y^2-(z-3x)^2}{(3x+y)^2-z^2}+\frac{z^2-(3x-y)^2}{(y+z)^2-9x^2}$ [B.U. '24]

7. $\frac{bc}{(b-a)(c-a)}+\frac{ca}{(c-b)(a-b)}+\frac{ab}{(a-c)(b-c)}$

[C. U. 1912 ; D. B. 1936]

8. $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$

[C.U. 1914, '47 ; D.B. 1926 ; P.U. 1931]

9. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}$ [C.U. '19]

10. $\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-3)(x-1)}+\dfrac{1}{(x-1)(x-2)}$ [C.U. '29]

11. $\dfrac{b^2+bc+c^2}{(a-b)(a-c)}+\dfrac{c^2+ca+a^2}{(b-c)(b-a)}+\dfrac{a^2+ab+b^2}{(c-a)(c-b)}$ [C.U. '40]

12. $\dfrac{a+x}{(a-b)(a-c)}+\dfrac{b+x}{(b-c)(b-a)}+\dfrac{c+x}{(c-a)(c-b)}$ [D.B. '30]

13. $\dfrac{b+c}{2bc}(b+c-a)+\dfrac{c+a}{2ca}(c+a-b)+\dfrac{a+b}{2ab}(a+b-c).$

14. $\dfrac{\dfrac{a}{a-b}+\dfrac{b}{b-c}+\dfrac{c}{c-a}}{\dfrac{a+b}{a-b}+\dfrac{b+c}{b-c}+\dfrac{c+a}{c-a}+3}$ [Here 3 in the denominator $=1+1+1$

Add each 1 separately to each term.]

15. $\left(\dfrac{1}{xy}+\dfrac{1}{xz}+\dfrac{1}{yz}\right)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)\div\left(\dfrac{y+z}{x}+\dfrac{z+x}{y}+\dfrac{x+y}{z}+3\right)$

16. $\dfrac{(a+1)^2}{(a-b)(a-c)}+\dfrac{(b+1)^2}{(b-c)(b-a)}+\dfrac{(c+1)^2}{(c-a)(c-b)}$

17. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$

18. $\dfrac{b+c-k}{(a-b)(a-c)}+\dfrac{c+a-k}{(b-c)(b-a)}+\dfrac{a+b-k}{(c-a)(c-b)}$ [C.U. '46]

19. $\dfrac{a^2(b-c)}{(b+a)(a+c)}+\dfrac{b^2(c-a)}{(c+b)(b+a)}+\dfrac{c^2(a-b)}{(a+c)(c+b)}$ [C.U. '47]

20. $\dfrac{a^2(b+c)}{(a-c)(a-b)}+\dfrac{b^2(c+a)}{(b-c)(b-a)}+\dfrac{c^2(a+b)}{(c-a)(c-b)}$ [C.U. '48]

21. $\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}+\dfrac{8}{1+x^8}-\dfrac{16}{1-x^{16}}$

[Add a suitable fraction $\dfrac{1}{1-x}$ to the given expression at the beginning and subtract $\dfrac{1}{1-x}$ from it at the end. Then combine the first two terms, then the result thus obtained with the third term and so on.]

22. $\dfrac{1}{x-1}+\dfrac{1}{x+1}+\dfrac{2x}{x^2+1}+\dfrac{4x^3}{x^4+1}+\dfrac{8x^7}{x^8+1}$ [C.U. '50]

23. Find the value of $\dfrac{x}{y}+\dfrac{x-1}{y+1}$, when $x=\dfrac{a}{a+b}$, $y=\dfrac{b}{a-b}$.

[C. U. 1947 Sup.]

EQUATION (Simple)

N. B. The process of solving an equation is primarily based upon the following :

(1) If the same quantity be added to or subtracted from both sides of an equation, the sides remain equal.

Again, if both sides of an equation be multiplied or divided by the same quantity, they remain equal.

(2) If each side of an equation consists of a single fraction the two products obtained by cross multiplication, *i.e.*, multiplying the numerator of one side by the denominator of the other side, will be equal. Hence, if $\frac{x}{x+1}=\frac{x-1}{x+2}$, then $x(x+2)=(x+1)(x-1)$

or, $x^2+2x=x^2-1$.

(3) **Transposition**—From (1) an important principle is deduced that any term may, if necessary, be transposed from one side of an equation to the other with its sign changed. Thus $3x$ transposed from the right side to the left will be $-3x$ and -4 transposed from the left side will be $+4$ on the right.

(4) If the same quantity with the same sign ($+$ or $-$) be on both sides of an equation, they may be cancelled from both sides.

(5) By transposition place all the terms involving the unknown quantity (x) to the left-hand side of the equation and the remaining terms to the right-hand side.

Examples [6]

1. Solve $\frac{x}{2}-2=\frac{x}{4}+\frac{x}{5}-1$. [C. U. '28 ; D. B. '37]

[In such sums multiply both sides by the L. C. M. of all the denominators.]

Thus multiplying by 20 we get $10x-40=5x+4x-20$,

or, $10x-5x-4x=40-20$, or, $x=20$ (Ans.)

2. Solve $\frac{4-x}{4}-\frac{5-x}{5}+\frac{6-x}{6}=1$. [C. U. '23]

Multiplying both sides by 60 we get

$$15(4-x)-12(5-x)+10(6-x)=1\times 60$$

or, $60-15x-60+12x+60-10x=60$

or, $12x-25x=0$, or, $-13x=0$, $\therefore$ $x=\frac{0}{-13}=0$.

3. Solve $\frac{5x+6}{12}+\frac{3x-4}{5}=2(x-9)$. [C. U. '15]

Multiplying both sides by 60 we get

$5(5x+6)+12(3x-4)=120(x-9)$,

or, $25x+30+36x-48=120x-1080$,

or, $25x+36x-120x=-30+48-1080$,

or, $-59x=-1062$, $\therefore$ $x=\frac{-1062}{-59}=18$.

4. Solve $\frac{3x+2}{x-1}+\frac{2(x-2)}{x+2}-5$. [C. U. '16. '20 ; D. B. '29]

$\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5$, or, $\frac{3x+2}{x-1}-3+\frac{2x-4}{x+2}-2=0$,

[Here 5 is transposed to the left-hand side as -5. It is written in two places as -3 and -2]

or, $\frac{3x+2-3x+3}{x-1}+\frac{2x-4-2x-4}{x+2}=0$,

or, $\frac{5}{x-1}+\frac{-8}{x+2}=0$, or, $\frac{5}{x-1}=\frac{8}{x-2}$

$\left[\because +\frac{-8}{x+2}=-\frac{8}{x+2}\right]$

or, $8x-8=5x+10$ (by cross multiplication),

or, $3x=18$, $\therefore$ $x=6$.

[N. B. Second Method] Divide the numerator by the denominator of the first term $\frac{3x+2}{x-1}$.

$$\begin{array}{l} x-1\;)\;3x+2\;(\;3 \\ \qquad\;\; \underline{3x-3} \\ \qquad\quad +5 \end{array} \qquad \therefore \frac{3x+2}{x-1}=3+\frac{5}{x-1}$$

Again, in the case of the second term

$$\begin{array}{l} x+2\;)\;2x-4\;(\;2, \\ \qquad\;\; \underline{2x+4} \\ \qquad\quad -8 \end{array} \qquad \therefore \frac{2x-4}{x+2}=2-\frac{8}{x+2}$$

Now, $3+\frac{5}{x-1}+2-\frac{8}{x+2}=5$, or, $\frac{5}{x-1}=\frac{8}{x+2}$.

[The rest is to be done as before]

5. Solve $\dfrac{x^2-x-2}{x-2}+\dfrac{2x^2-x-1}{x-1}=\dfrac{4x^2+x-3}{x+1}$,

[Vide the second method above]

or, $x+1+2x+1=4x-3$, [Dividing each numerator by each denominator]

or, $x+2x-4x=-1-1-3$, or, $-x=-5$, $\therefore$ $x=5$.

6. Solve $\dfrac{x-3}{x-4}+\dfrac{x-6}{x-7}=\dfrac{x-4}{x-5}+\dfrac{x-5}{x-6}$. [C. U. '39]

$$\frac{x-3}{x-4}+\frac{x-6}{x-7}=\frac{x-4}{x-5}+\frac{x-5}{x-6},$$

or, $1+\dfrac{1}{x-4}+1+\dfrac{1}{x-7}=1+\dfrac{1}{x-5}+1+\dfrac{1}{x-6}$

[Dividing the numerator by the denominator of each term]

or, $\dfrac{1}{x-4}+\dfrac{1}{x-7}=\dfrac{1}{x-5}+\dfrac{1}{x-6}$, or, $\dfrac{1}{x-4}-\dfrac{1}{x-5}=\dfrac{1}{x-6}-\dfrac{1}{x-7}$

or, $\dfrac{x-5-x+4}{(x-4)(x-5)}=\dfrac{x-7-x+6}{(x-6)(x-7)}$

or, $\dfrac{-1}{x^2-9x+20}=\dfrac{-1}{x^2-13x+42}$

or, $x^2-9x+20=x^2-13x+42$,

[Of two equal factions if the numerators are equal, their denominators are also equal.]

or, $-9x+13x=42-20$, or, $4x=22$, $\therefore$ $x=\frac{22}{4}=5\frac{1}{2}$.

7. Solve $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}$. [C. U. '21 ; D. B. '35]

$\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}$, or, $\dfrac{6x+1}{15}-\dfrac{2x-1}{5}=\dfrac{2x-4}{7x-16}$,

[Terms having denominators free from x are brought to one side.]

or, $\dfrac{6x+1-6x+3}{15}=\dfrac{2x-4}{7x-16}$, or, $\dfrac{4}{15}=\dfrac{2x-4}{7x-16}$,

or, $30x-60=28x-64$, or. $2x=-4$. $\therefore$ $x=-2$.

8. Solve $\frac{x-a}{b}+\frac{x-b}{a}+\frac{x-3a-3b}{a+b}=0$ [C. U. '11]

or, $\frac{x-a}{b}-1+\frac{x-b}{a}-1+\frac{x-3a-3b}{a+b}+2=0$

or, $\frac{x-a-b}{b}+\frac{x-b-a}{a}+\frac{x-3a-3b+2a+2b}{a+b}=0$

or, $\frac{x-a-b}{b}+\frac{x-a-b}{a}+\frac{x-a-b}{a+b}=0$

or, $(x-a-b)\left(\frac{1}{b}+\frac{1}{a}+\frac{1}{a+b}\right)=0.$

[Here $x-a-b$ is taken as a common factor.]

$\because$ No relation of the values of a, b is known,

$\therefore$ $\left(\frac{1}{b}+\frac{1}{a}+\frac{1}{a+b}\right)$ cannot be taken as zero.

$\therefore$ $x-a-b=0$, $\therefore$ $x=a+b$.

9. Solve $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3.$ [C. U. '46 ; D. B. '43]

$\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3$, or $\frac{x-a}{b+c}-1+\frac{x-b}{c+a}-1+\frac{x-c}{a+b}-1=0$

or, $\frac{x-a-b-c}{b+c}+\frac{x-b-c-a}{c+a}+\frac{x-c-a-b}{a+b}=0$

or, $(x-a-b-c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0$

$\therefore$ $x-a-b-c=0$ ($\because$ nothing is known of the values of a, b, c the other factor cannot be 0), $\therefore$ $x=a+b+c$.

[Give this argument in such places in similar cases.]

10. Solve $\frac{x-bc}{b+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}=a+b+c$

or, $\frac{x-bc}{b+c}-a+\frac{x-ca}{c+a}-b+\frac{x-ab}{a+b}-c=0$

or, $\frac{x-bc-ab-ac}{b+c}+\frac{x-ca-bc-ab}{c+a}+\frac{x-ab-ac-bc}{a+b}=0$

or, $(x-ab-bc-ac)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0$

$\therefore$ $x-ab-bc-ac=0$ [Give the previous argument.]

$\therefore$ $x=ab+bc+ca.$

11. Solve $\frac{x-a^3}{b^2-bc+c^2}+\frac{x-b^3}{c^2-ca+a^2}+\frac{x-c^3}{a^2-ab+b^2}=2(a+b+c).$

$$\frac{x-a^3}{b^2-bc+c^2}+\frac{x-b^3}{c^2-ca+a^2}+\frac{x-c^3}{a^2-ab+b^2}=(a+b)+(b+c)+(c+a)$$

or, $\frac{x-a^3}{b^2-bc+c^2}-(b+c)+\frac{x-b^3}{c^2-ca+a^2}-(c+a)$

$+\frac{x-c^3}{a^2-ab+b^2}-(a+b)=0$

or, $\frac{x-a^3-(b^3+c^3)}{b^2-bc+c^2}+\frac{x-b^3-(c^3+a^3)}{c^2-ca+a^2}+\frac{x-c^3-(a^3+b^3)}{a^2-ab+b^2}=0$

or, $\frac{x-a^3-b^3-c^3}{b^2-bc+c^2}+\frac{x-a^3-b^3-c^3}{c^2-ca+a^2}+\frac{x-a^3-b^3-c^3}{a^2-ab+b^2}=0$

or, $(x-a^3-b^3-c^3)\left(\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}+\frac{1}{a^2-ab+b^2}\right)=0$

$\therefore \quad x-a^3-b^3-c^3=0$ [Vide previous argument]

$\therefore \quad x=a^3+b^3+c^3.$

12. Solve $\frac{ax+a^2}{b+c}+\frac{bx+b^2}{c+a}+\frac{cx+c^2}{a+b}+a+b+c=0$ [C.U. '42]

or, $\frac{ax+a^2}{b+c}+a+\frac{bx+b^2}{c+a}+b+\frac{cx+c^2}{a+b}+c=0$

or, $\frac{ax+a^2+ab+ac}{b+c}+\frac{bx+b^2+bc+ab}{c+a}+\frac{cx+c^2+ac+bc}{a+b}=0$

or, $\frac{a(x+a+b+c)}{b+c}+\frac{b(x+b+c+a)}{c+a}+\frac{c(x+c+a+b)}{a+b}=0$

or, $(x+a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=0$

$\therefore \quad x+a+b+c=0$ [Vide previous argument].

$\therefore \quad x=-(a+b+c).$

13. Solve $\frac{x+a^2+2c^2}{b+c}+\frac{x+b^2+2a^2}{c+a}+\frac{x+c^2+2b^2}{a+b}=0$

or, $\frac{x+a^2+2c^2}{b+c}+(b-c)+\frac{x+b^2+2a^2}{c+a}+(c-a)$

$+\frac{x+c^2+2b^2}{a+b}+(a-b)=0.$

[Here $(b-c)+(c-a)+(a-b)$ is added to the left-hand side. It being equal to 0, nothing is added to the right-hand side.]

17. Solve $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+c}+\dfrac{1}{x+b-c}$ [C. U. '40]

or, $\dfrac{1}{x+a}-\dfrac{1}{x+a+c}=\dfrac{1}{x+b-c}-\dfrac{1}{x+b}$,

or, $\dfrac{x+a+c-x-a}{(x+a)(x+a+c)}=\dfrac{x+b-x-b+c}{(x+b-c)(x+b)}$,

or, $\dfrac{c}{x^2+2ax+a^2+cx+ac}=\dfrac{c}{x^2+2bx+b^2-cx-bc}$,

or, $x^2+2ax+a^2+cx+ac=x^2+2bx+b^2-cx-bc$,

or, $2ax+cx-2bx+cx=b^2-bc-a^2-ac$,

or, $2ax+2cx-2bx=b^2-a^2-bc-ac$,

or, $2(a+c-b)x=(b+a)(b-a)-c(b+a)=(b+a)(b-a-c)$,

$\therefore\quad x=\dfrac{(b-a-c)(b+a)}{2(a+c-b)}=\dfrac{-(a+c-b)(b+a)}{2(a+c-b)}=-\dfrac{(a+b)}{2}$,

18. Solve $\dfrac{(x+2)(x+3)}{(x+1)(x+7)}=\dfrac{x+5}{x+8}$. [C. U. '44]

$\dfrac{(x+2)(x+3)}{(x+1)(x+7)}=\dfrac{x+5}{x+8}$, or, $\dfrac{x^2+5x+6}{x^2+8x+7}=\dfrac{x+5}{x+8}$,

or, $\dfrac{x^2+5x+6}{x+5}=\dfrac{x^2+8x+7}{x+8}$,

[Here the denominator of the left-hand side is interchanged with the numerator of the right-hand side. It is called alternendo.]

or, $x+\dfrac{6}{x+5}=x+\dfrac{7}{x+8}$, or, $\dfrac{6}{x+5}=\dfrac{7}{x+8}$,

or, $7x+35=6x+48$, $\therefore\quad x=48-35=13$.

19. Solve $\left(\dfrac{2x-10}{2x-5}\right)^2=\dfrac{x-10}{x-5}$. [C. U. '41]

$\left(\dfrac{2x-10}{2x-5}\right)^2=\dfrac{x-10}{x-5}$, or, $\dfrac{4x^2-40x+100}{4x^2-20x+25}=\dfrac{x-10}{x-5}$,

or, $\dfrac{4x^2-40x+100}{x-10}=\dfrac{4x^2-20x+25}{x-5}$

or, $4x+\dfrac{100}{x-10}=4x+\dfrac{25}{x-5}$, or, $\dfrac{100}{x-10}=\dfrac{25}{x-5}$

or, $\dfrac{4}{x-10}=\dfrac{1}{x-5}$, or, $4x-20=x-10$,

or, $3x=20-10=10$, $\therefore\quad x=\frac{10}{3}=3\frac{1}{3}$.

20. Solve $\left(\frac{x-5}{x-6}\right)^3=\frac{x-4}{x-7}$.

or, $\left(\frac{x-5}{x-6}\right)^3\times\frac{x-6}{x-5}=\frac{x-4}{x-7}\times\frac{x-6}{x-5}$,

[Multiply both sides by $\frac{x-6}{x-5}$ which is the reciprocal of $\frac{x-5}{x-6}$.]

or, $\frac{(x-5)^2}{(x-6)^2}=\frac{x^2-10x+24}{x^2-12x+35}$,

or, $\frac{x^2-10x+25}{x^2-12x+36}=\frac{x^2-10x+24}{x^2-12x+35}$,

or, $\frac{x^2-10x+25}{x^2-10x+24}=\frac{x^2-12x+36}{x^2-12x+35}$,

or, $1+\frac{1}{x^2-10x+24}=1+\frac{1}{x^2-12x+35}$

or, $\frac{1}{x^2-10x+24}=\frac{1}{x^2-12x+35}$

or, $x^2-10x+24=x^2-12x+35$,

or, $12x-10x=35-24$, or, $2x=11$, $x=\frac{11}{2}=5\frac{1}{2}$.

*21. Solve $\left(\frac{x+a}{x+b}\right)^2=\frac{x+2a-b}{x+2b-a}$.

$$\left(\frac{x+a}{x+b}\right)^3\times\frac{x+b}{x+a}=\frac{x+2a-b}{x+2b-a}\times\frac{x+b}{x+a},$$

or, $\frac{(x+a)^2}{(x+b)^2}=\frac{x^2-b^2+2ax+2ab}{x^2-a^2+2bx+2ab}$,

or, $\frac{x^2+2ax+a^2}{x^2+2bx+b^2}=\frac{x^2+2ax-b^2+2ab}{x^2+2bx-a^2+2ab}$,

$\therefore$ Each $=\frac{(x^2+2ax+a^2)-(x^2+2ax-b^2+2ab)}{(x^2+2bx+b^2)-(x^2+2bx-a^2+2ab)}$

$=\frac{a^2+b^2-2ab}{a^2+b^2-2ab}=1.$

$\therefore$ $\frac{x^2+2ax+a^2}{x^2+2bx+b^2}=1$, or, $x^2+2ax+a^2=x^2+2bx+b^2$,

or, $2ax-2bx=b^2-a^2$, or. $2(a-b)x=-(a^2-b^2)$,

$\therefore$ $x=-\frac{a^2-b^2}{2(a-b)}=-\frac{a+b}{2}$.

*22. Solve $\dfrac{x-a^2}{b+c}+\dfrac{x-b^2}{c+a}+\dfrac{x-c^2}{a+b}=4(a+b+c).$

[The equation may be written in the following way]

$$\frac{x-a^2}{b+c}-a+\frac{x-b^2}{c+a}-b+\frac{x-c^2}{a+b}-c=3(a+b+c)$$

or, $$\frac{x-a^2-ab-ac}{b+c}+\frac{x-b^2-bc-ab}{c+a}+\frac{x-c^2-ac-bc}{a+b}=3(a+b+c),$$

or, $$\frac{x-a^2-ab-ac}{b+c}-(a+b+c)+\frac{x-b^2-bc-ab}{c+a}-(a+b+c)+\frac{x-c^2-ab-bc}{a+b}-(a+b+c)=0,$$

or, $$\frac{x-a(a+b+c)-(a+b+c)(b+c)}{b+c}+\frac{x-b(a+b+c)-(a+b+c)(c+a)}{c+a}+\frac{x-c(a+b+c)-(a+b+c)(a+b)}{a+b}=0,$$

or, $$\frac{x-(a+b+c)(a+b+c)}{b+c}+\frac{x-(a+b+c)(a+b+c)}{c+a}+\frac{x-(a+b+c)(a+b+c)}{a+b}=0,$$

or, $$\frac{x-(a+b+c)^2}{b+c}+\frac{x-(a+b+c)^2}{c+a}+\frac{x-(a+b+c)^2}{a+b}=0,$$

or, $$\{x-(a+b+c)^2\}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0,$$

$\therefore\ x-(a+b+c)^2=0, \quad \therefore\ x=(a+b+c)^2.$

*23. Solve $\dfrac{x+19}{x+10}=\left(\dfrac{2x+33}{2x+24}\right)^2$

or, $\dfrac{x+19}{x+10}=\left(1+\dfrac{9}{2x+24}\right)^2$, or, $\dfrac{x+19}{x+10}=\left\{1+\dfrac{9}{3(x+12)}\right\}^2$

or, $1+\dfrac{9}{x+10}=1+\dfrac{81}{4(x+12)^2}+2\times\dfrac{9}{2(x+12)}$

or, $\dfrac{9}{x+10}=\dfrac{81}{4(x+12)^2}+\dfrac{9}{x+12}$, or, $\dfrac{9}{x+10}-\dfrac{9}{x+12}=\dfrac{81}{4(x+12)^2}$,

or, $9\times\frac{x+12-x-10}{(x+10)(x+12)}=\frac{1}{4(x+12)^2}$,

or, $\frac{9\times 2}{(x+10)(x+12)}=\frac{81}{4(x+12)^2}$,

or, $\frac{2}{x+10}=\frac{9}{4(x+12)}$ $\left(\text{dividing both sides by } \frac{9}{x+12}\right)$,

or, $9x+90=8x+96$, $\therefore$ $x=96-90=6$.

[N. B. It can be easily solved as the sum No. 11.]

24. Solve $\frac{x}{\cdot 5}-\frac{1}{\cdot 05}+\frac{x}{\cdot 005}-\frac{1}{\cdot 0005}=0$.

or, $\frac{x}{\frac{5}{10}}-\frac{1}{\frac{5}{100}}+\frac{x}{\frac{5}{1000}}-\frac{1}{\frac{5}{10000}}=0$,

or, $\frac{x}{\frac{1}{2}}-\frac{1}{\frac{1}{20}}+\frac{x}{\frac{1}{200}}-\frac{1}{\frac{1}{2000}}=0$, or, $2x-20+200x-2000=0$,

or, $202x=2020$, $\therefore$ $x=\frac{2020}{202}=10$.

25. Solve $16\left(\frac{a-x}{a+x}\right)^3=\frac{a+x}{a-x}$.

Multipiying both sides by $\frac{a-x}{a+x}$ we get

$16\left(\frac{a-x}{a+x}\right)^4=\frac{a+x}{a-x}\times\frac{a-x}{a+x}$, or, $16\left(\frac{a-x}{a+x}\right)^4=1$,

or, $\left(\frac{a-x}{a+x}\right)^4=\frac{1}{16}=\left(\frac{1}{2}\right)^4$, or, $\frac{a-x}{a+x}=\frac{1}{2}$*.

or, $a+x=2a-2x$, or, $x+2x=2a-a$, or, $3x=a$. $\therefore$ $x=\frac{a}{3}$.

[* $\frac{a-x}{a+x}=-\frac{1}{2}$ may also be taken, then $x=3a$ may be another answer.]

14. (i) $\frac{x}{2x-a}+\frac{x}{2x-b}=1.$ [D. B. 1902]
(ii) $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2.$ [W. B. S. F. '52]

15. $\frac{1}{x-a}-\frac{1}{x-a-c}=\frac{1}{x-b-c}-\frac{1}{x-b}.$

16. $\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}+\frac{x-4}{x-5}.$ [C. U. 1887]

17. $\frac{x-a}{b-a}+\frac{x-c}{b-c}=2.$ [D. B. '26]

18. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{x+a+b}+\frac{1}{x}.$

19. $\frac{x-a^2}{b+c}+\frac{x-ab}{c+a}+\frac{x-ac}{a+b}=3a$

20. $\frac{x+a^2+2bc}{b-c}+\frac{x+b^2+2ca}{c-a}+\frac{x+c^2+2ab}{a-b}=0.$

21. $\frac{a(bcx-a)}{b^2+c^2}+\frac{b(cax-b)}{c^2+a^2}+\frac{c(abx-c)}{a^2+b^2}=3.$

22. $\frac{x^2-2\frac{1}{3}}{4}-\frac{x-3\frac{1}{2}}{5}=\frac{2x^2-3}{8}-\frac{x-5\frac{1}{2}}{3}.$ [C. U. 1883]

23. $\frac{a-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx}.$ [C. U. 1886]

24. $\frac{x^2+x-1}{x-1}+\frac{x^2+3x-1}{x+2}=2x+3.$

25. $\left(\frac{x-2}{x-3}\right)^2=\frac{x-4}{x-6}.$

25. (a) $\frac{(x+2)(x+6)}{(x+4)(x+5)}=\frac{x+8}{x+9}.$ [C. U. 1949]

26. $\frac{(x+1)(x+9)}{(x+2)(x+4)}=\frac{(x+6)(x+10)}{(x+5)(x+7)}.$ [M. U. 1889]

27. $\left(\frac{3x-28}{3x-26}\right)^3=\frac{x-10}{x-8}.$

28. $\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5.$ [C. U. 1947]

29. $5+\cfrac{1}{3-\cfrac{1}{4-x}}=\frac{27}{5}.$

29. (a) $\frac{1}{x}+\frac{1}{x+a}=\frac{2}{x+b}.$ [C. U. '50]

30. $(x-a)(x-b)=(x-c)(x-d).$ [C. U. 1897]

31. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a+b}{x-a-b}.$ [C. U. 1948]

[The right-hand side $=\frac{a}{x-a-b}+\frac{b}{x-a-b}$, Now transpose]

32. $\frac{1}{x+b}-\frac{1}{x+a}=\frac{a-b}{x^2+2ab}.$ [E.B. S. B.' 55]

SIMULTANEOUS EQUATIONS

Examples [7]

1. Solve $\left.\begin{array}{l}7x-3y=31\\9x-5y=41\end{array}\right\}$ [D. B. 1934]

$$7x-3y=31 \ldots\ldots\ldots\ldots(1)$$
$$9x-5y=41 \ldots\ldots\ldots\ldots(2)$$

Multiplying (1) by 5 and (2) by 3 we get

$$\begin{array}{r}35x-15y=155\\ \underline{27x-15y=123}\end{array}$$

(Subtracting) $8x=32$. $\therefore$ $x=4$.

Now, from (1) putting the value of x, we get $28-3y=31$,

or $-3y=3$, $\therefore$ $y=-1$. $\therefore$ the solution is $x=4$, $y=-1$.

2. Solve $x+2y=3=4x-y$. [C. U. 1917]

Here, $x+2y=3 \ldots\ldots\ldots\ldots(1)$

$4x-\ \ y=3 \ldots\ldots\ldots\ldots(2)$

Multiplying (2) by 2 we get $8x-2y=6$

Again, $x+2y=3 \ldots\ldots(1)$

$\therefore$ (Adding) $9x=9$, $\therefore$ $x=1$.

Now, from (1), $1+2y=3$. or, $2y=2$, $\therefore$ $y=1$.

$\therefore$ $x=1$, $y=1$.

3. Solve $\left.\begin{array}{l}x+5y=36\\ \dfrac{x+y}{x-y}=\dfrac{5}{3}\end{array}\right\}$ [C. U. 1912]

$x+5y=36 \ldots\ldots\ldots(1)$ and $\dfrac{x+y}{x-y}=\dfrac{5}{3} \ldots\ldots\ldots(2)$

From (2) $5x-5y=3x+3y$, or, $2x-8y=0$, or, $x-4y=0 \ldots(3)$

Now $x+5y=36$

and $x-4y=0$

$\therefore$ (Subtracting) $9y=36$, $\therefore$ $y=4$.

Now, from (1), $x+20=36$, $\therefore$ $x=36-20=16$.

$\therefore$ $x=16$, $y=4$.

4. Solve $\left.\begin{array}{l}\frac{2}{x}+\frac{3}{y}=2\ldots\ldots\ldots(1)\\ \frac{5}{x}+\frac{10}{y}=5\frac{5}{6}\ldots\ldots(2)\end{array}\right\}$

Multiplying (1) by 5 and (2) by 2 we get

$$\frac{10}{x}+\frac{15}{y}=10$$

$$\frac{10}{x}+\frac{20}{y}=\frac{35}{3}$$

(Subtracting) $-\frac{5}{y}=-\frac{5}{3}$ or, $\frac{5}{y}=\frac{5}{3}$, $\therefore$ $y=3$.

Now, from (1) $\frac{2}{x}+\frac{3}{3}=2$, or, $\frac{2}{x}=2-1=1$, $\therefore$ $x=2$.

$\therefore$ $x=2$, $y=3$.

5. Solve $\frac{x+y}{xy}=2$, $\frac{x-y}{xy}=1$. [D. B. '31]

$$\frac{x+y}{xy}=2\ldots\ldots(1),\quad \frac{x-y}{xy}=1\ldots\ldots(2).$$

From (1) $\frac{x}{xy}+\frac{y}{xy}=2$, or, $\frac{1}{y}+\frac{1}{x}=2\ldots\ldots(3)$.

From (2) $\frac{x}{xy}-\frac{y}{xy}=1$, or, $\frac{1}{y}-\frac{1}{x}=1\ldots\ldots(4)$.

Adding (3)+(4) we have $\frac{2}{y}=3$, or, $3y=2$, $\therefore$ $y=\frac{2}{3}$.

Again, from (3) − (4) we get, $\frac{2}{x}=1$, $\therefore$ $x=2$.

$\therefore$ $x=2$, $y=\frac{2}{3}$.

6. Solve $23x+17y=63\ldots\ldots(1)$
$17x+23y=57\ldots\ldots(2)$

Adding (1)+(2) we get $40x+40y=120$,

or, $x+y=3\ldots\ldots(3)$ [dividing both sides by 40]

Again, from (1) − (2), $6x-6y=6$, or, $x-y=1\ldots(4)$.

Now, from (3)+(4), we have $2x=4$,
and from (3) − (4), " " $2y=2$, $\therefore$ $\left.\begin{array}{l}x=2\\ y=1\end{array}\right\}$ Ans.

7. Solve $\left.\begin{array}{l}\dfrac{x+2}{7}+\dfrac{y-x}{4}=2x-8\ldots\ldots(1)\\ \dfrac{2y-3x}{3}+2y=3x+4\ldots\ldots(2)\end{array}\right\}$ [P. U. 1892]

Multiplying (1) by 28 we get

$4x+8+7y-7x=56x-224$, or, $-59x+7y=-232\ldots\ldots(3)$

Multiplying (2) by 3 we get $2y-3x+6y=9x+12$,

or, $8y-12x=12$, or, $-3x+2y=3\ldots\ldots(4)$.

Now multiplying (3) by 2 and (4) by 7 we get

$$\begin{array}{rl} -118x+14y= & -464\\ -\ \ 21x+14y= & \ \ \ 21\\ \hline \end{array}$$

(Subtracting) $-97x \qquad =-485$, $\therefore\ x=\dfrac{-485}{-97}=5$.

Now, from (4) we get $-15+2y=3$, or, $2y=18$, $\therefore\ y=9$.

$\therefore\ x=5,\ y=9$.

8. Solve $\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{y}{b}=a+b\ldots\ldots(1).\\ \dfrac{x}{a^2}+\dfrac{y}{b^2}=2\cdots\cdots\cdots(2).\end{array}\right\}$

Multiplying (1) by a and (2) by a^2 we get

$$x+\frac{a}{b}\,y=a^2+ab\ldots\ldots(3)$$

$$\text{and}\quad x+\frac{a^2}{b^2}\,y=2a^2\ldots\ldots(4)$$

$$\therefore\quad \frac{ab-a^2}{b^2}y=ab-a^2 \quad \text{(by subtraction)}$$

$$\therefore\quad y=\frac{(ab-a^2)\times b^2}{(ab-a^2)}=b^2.$$

Now, from (3), $x+\dfrac{a}{b}\times b^2=a^2+ab$ [putting the value of y.]

or, $x+ab=a^2+ab$, $\therefore\ x=a^2$. $\therefore\ x=a^2,\ y=b^2$.

9. Solve $\left.\begin{array}{l}x+2y+3z=20\cdots\cdots\cdots(1)\\ 2x+3y-5z=-7\ldots\ldots\ldots(2)\\ 4x-5y+7z=21\ldots\ldots\ldots(4)\end{array}\right\}$ [C. U. 1988]

Multiplying (1) by 2 we get $2x+4y+6z=40$

and $2x+3y-5z=-7\ldots\ldots(2)$

$\therefore$ (by subtracting) $y+11z=47\ldots\ldots(4)$

Again by multiplying (1) by 4 we get

$$4x+8y+12z=80$$

$$\text{and } 4x-5y+7z=21\ldots(3)$$

$$\therefore\quad 13y+5z=59\ldots(5)\quad \text{(by subtraction)}$$

Multiplying (4) by 13 we get

$$13y+143z=611$$

$$\text{and } 13y+5z=59\ldots(5)$$

$$\therefore\ \text{(subtracting)}\ 138z=552,\quad \therefore\quad z=\frac{552}{138}=4.$$

Now, from (4) we get $y+44=47$, $\therefore\quad y=47-44=3.$

Again from (1) we get $x+6+12=20$, $\therefore\quad x=20-18=2.$

$$\therefore\quad x=2,\ y=3,\ z=4.$$

10. Solve $\left.\begin{array}{l} x+y=5 \\ y+z=7 \\ z+x=6 \end{array}\right\}$

$$x+y=5\ldots\ldots(1)$$

$$y+z=7\ldots\ldots(2)$$

$$z+x=6\ \ldots\ldots(3)$$

$$\therefore\quad \text{(Adding) } 2(x+y+z)=18,$$

$$\therefore\quad x+y+z=9\ldots\ldots(4)$$

Now, from (4) – (1), we get $z=9-5=4$,

from (4) – (2), ,, ,, $x=9-7=2$,

and from (4) – (3), ,, ,, $y=9-6=3$, $\therefore\ x=2,\ y=3,\ z=4.$

11. Solve : $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=6 \quad \ldots\ldots\ldots(1) \\ \dfrac{1}{y}+\dfrac{1}{z}=7 \quad \ldots\ldots\ldots(2) \\ \dfrac{1}{z}+\dfrac{1}{x}=5 \quad \ldots\ldots\ldots(3) \end{array}\right.$

$$\therefore\quad \text{(Adding) } 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=18,\quad \therefore\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9\ldots\ldots(4).$$

Now, from (4) – (1) we get $\dfrac{1}{z}=3$, or, $3z=1$, $\therefore\quad z=\dfrac{1}{3}$

,, (4) – (2) ,, ,, $\dfrac{1}{x}=2$, or, $2x=1$, $\therefore\quad x=\dfrac{1}{2}$

and ,, (4) – (3) ,, ,, $\dfrac{1}{y}=4$, or, $4y=1$, $\therefore\quad y=\dfrac{1}{4}$

$$\therefore\quad x=\frac{1}{2},\quad y=\frac{1}{4},\quad z=\frac{1}{3}.$$

12. Solve $\frac{x+y}{xy}=\frac{3}{2}$...(1), $\frac{y+z}{yz}=\frac{5}{6}$...(2), $\frac{z+x}{zx}=1\frac{1}{3}$...(3)

From (1) we get $\frac{x}{xy}+\frac{y}{xy}=\frac{3}{2}$, or, $\frac{1}{y}+\frac{1}{x}=\frac{3}{2}$......(4).

" (2) " " $\frac{y}{yz}+\frac{z}{yz}=\frac{5}{6}$, or, $\frac{1}{z}+\frac{1}{y}=\frac{5}{6}$......(5).

" (3) " " $\frac{z}{zx}+\frac{x}{zx}=\frac{4}{3}$, or, $\frac{1}{x}+\frac{1}{z}=\frac{4}{3}$......(6).

(Adding) $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{11}{3}$,

$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{6}$...(7)

Now, from (7) – (4) we get $\frac{1}{z}=\frac{11}{6}-\frac{3}{2}=\frac{1}{3}$, $\therefore \ z=3$

" (7) – (5) " " $\frac{1}{x}=\frac{11}{6}-\frac{5}{6}=1$, $\therefore \ x=1$

and " (7) – (6) " " $\frac{1}{y}=\frac{11}{6}-\frac{4}{3}=\frac{1}{2}$, $\therefore \ y=2$

$\therefore \quad x=1, \ y=2, \ z=3.$

*13. Solve
$$\left.\begin{aligned} 3x+4y&=5xy......(1)\\ 3y+5z&=6yz......(2)\\ 2z+3x&=2zx......(3)\end{aligned}\right\}$$

Dividing (1) by xy we get $\frac{3x}{xy}+\frac{4y}{xy}=5$, or, $\frac{3}{y}+\frac{4}{x}=5$......(4)

Thus dividing (2) by yz we get $\frac{3}{z}+\frac{5}{y}=6$......(5)

Dividing (3) by zx we get $\frac{2}{x}+\frac{3}{z}=2$......(6)

Now, from (5) – (6) we get $\frac{5}{y}-\frac{2}{x}=4$......(7)

Again, from (4)×1 and (7)×2 we get

$$\frac{3}{y}+\frac{4}{x}=5$$

and $$\frac{10}{y}-\frac{4}{x}=8$$

(Adding) $\frac{13}{y}=13$, or, $13y=13$, $\therefore \quad y=1.$

Now, from (4), $3+\frac{4}{x}=5$, or, $\frac{4}{x}=2$. or. $2x=4$, $\therefore$ $x=2$.

and from (5), $\frac{3}{z}+5=6$, or, $\frac{3}{z}=1$ $\therefore$ $z=3$.

$$\therefore \quad x=2, \quad y=1, \quad z=3.$$

14. Solve $\frac{xy}{x+y}=\frac{6}{5}$, $\frac{yz}{y+z}=\frac{12}{7}$ and $\frac{zx}{z+x}=\frac{4}{3}$.

[Here the equations are inverted and written as

$$\frac{x+y}{xy}=\frac{5}{6}, \quad \frac{y+z}{yz}=\frac{7}{12} \text{ and } \frac{z+x}{zx}=\frac{3}{4}.$$

Now, solve as in example 12. Ans. $x=2, y=3, z=4$.]

15. Solve $\frac{x+y}{xy}=\frac{y+z}{yz}=\frac{z+x}{zx}=\frac{2}{5}$.

$$\frac{x+y}{xy}=\frac{2}{5}, \quad \text{or,} \quad \frac{1}{y}+\frac{1}{x}=\frac{2}{5} \ldots\ldots(1)$$

$$\frac{y+z}{yz}=\frac{2}{5}, \quad \text{or,} \quad \frac{1}{z}+\frac{1}{y}=\frac{2}{5} \ldots\ldots(2)$$

$$\frac{z+x}{zx}=\frac{2}{5}, \quad \text{or,} \quad \frac{1}{x}+\frac{1}{z}=\frac{2}{5} \ldots\ldots(3)$$

(Adding) $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{6}{5}$, $\therefore$ $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{5} \ldots\ldots(4)$

Now, from (4) – (1) we get $\frac{1}{z}=\frac{1}{5}$, $\therefore$ $z=5$

" (4) – (2) " " $\frac{1}{x}=\frac{1}{5}$, $\therefore$ $x=5$

" (4) – (3) " " $\frac{1}{y}=\frac{1}{5}$, $\therefore$ $y=5$

$$\therefore \quad x=y=z=5.$$

16. Solve $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, and $a^2x+b^2y+c^2z=a^3+b^3+c^3$.

Here, $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (suppose)

$\therefore$ $x=ak$, $y=bk$, $z=ck \ldots\ldots(\text{A})$

$\therefore$ from the last equation we get

$a^3k+b^3k+c^3k=a^3+b^3+c^3$, or, $(a^3+b^3+c^3)k=a^3+b^3+c^3$,

$$\therefore \quad k=\frac{a^3+b^3+c^3}{a^3+b^3+c^3}=1.$$

$\therefore$ from (A) we get $x=a$, $y=b$, $z=c$.

17. Solve $a(x+y)=b(x-y)=2ab$. [C. U. 1930]

Here, $a(x+y)=2ab$......(1) and $b(x-y)=2ab$......(2)

From (1), $x+y=2b$ (dividing both sides by a)......(3)

" (2), $x-y=2a$ (" " " " b)

(Adding) $2x=2(a+b)$, $\therefore$ $x=a+b$.

Now, from (3), $a+b+y=2b$, $\therefore$ $y=2b-b-a=b-a$.

$$\therefore \quad x=a+b, \; y=b-a.$$

18. Solve $xy=12$......(1), $yz=20$......(2) and $zx=15$......(3).

Multiplying the three equations we get

$$x^2y^2z^2=12\times 20\times 15=3600.$$

$$\therefore \quad xyz=\sqrt{3600}=\pm 60......(4).$$

Now, from (4)$\div$(1) we have $\dfrac{xyz}{xy}=\pm\dfrac{60}{12}$, or, $z=\pm 5$.

Thus from (4)$\div$(2), $x=\pm 3$ and from (4)$\div$(3), $y=\pm 4$.

$$\therefore \quad x=\pm 3, \; y=\pm 4, \; z=\pm 5.$$

Method of Cross-multiplication :

If of the three equations the right-hand side of two equations be 0, it is easier to solve by this method.

Take those two equations and write in the following way :—

$$\frac{x}{(\text{coefficient of } y\times\text{coefficient of } z)-(\text{coefficient of } z\times\text{coefficient of } y)}$$

$$=\frac{y}{(\text{coefficient of } z\times\text{coefficient of } x)-(\text{coefficient of } x\times\text{coefficient of } z)}$$

$$=\frac{z}{(\text{coefficient of } x\times\text{coefficient of } y)-(\text{coefficient of } y\times\text{coefficient of } x)}$$

But it should be remembered here that the coefficient of a particular letter that is mentioned first within each bracket must be always taken from the upper equation and the coefficient of a particular letter that is mentioned after the sign $\times$ must be always

taken from the lower equation. The following examples are intended for illustration.

19. Solve
$$\left.\begin{aligned} x-2y+z&=0\ldots\ldots\ldots(1)\\ 3x+6y-5z&=0\ldots\ldots\ldots(2)\\ 2x+3y+4z&=20\ldots\ldots\ldots(3)\end{aligned}\right\}$$

From (1) and (2) by the method of cross-multiplication we get
$$\frac{x}{(-2\times-5)-(1\times6)}=\frac{y}{(1\times3)-(1\times-5)}=\frac{z}{(1\times6)-(-2\times3)}$$

or, $\dfrac{x}{10-6}=\dfrac{y}{3+5}=\dfrac{z}{6+6}$ or, $\dfrac{x}{4}=\dfrac{y}{8}=\dfrac{z}{12}$,

or, $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}=k$ (suppose),

$\therefore$ $x=k$, $y=2k$, $z=3k$......(A)

Now, from (3) we get $2k+6k+12k=20$, or, $20k=20$,

$\therefore$ $k=1$. $\therefore$ Putting the value of k in (A) we get
$x=1$, $y=2$, $z=3$.

20. Solve
$$\left.\begin{aligned} x+y+z&=0\ldots\ldots(1)\\ ax+by+cz&=0\ldots\ldots(2)\\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}&=3\ldots\ldots(3)\end{aligned}\right\}$$

From (1) and (2) by cross-multiplication we get

$\dfrac{x}{c-b}=\dfrac{y}{a-c}=\dfrac{z}{b-a}$, or, $\dfrac{x}{-(b-c)}=\dfrac{y}{-(c-a)}=\dfrac{z}{-(a-b)}$

or, $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}=k$ (suppose)

$\therefore$ $x=k(b-c)$, $y=k(c-a)$, $z=k(a-b)$......(4),

Now, from (3) we get $\dfrac{k(b-c)}{b-c}+\dfrac{k(c-a)}{c-a}+\dfrac{k(a-b)}{a-b}=3$,

or, $k+k+k=3$, or, $3k=3$, $\therefore$ $k=1$.

$\therefore$ from (4), $x=b-c$, $y=c-a$, $z=a-b$.

21. Solve
$$\left.\begin{aligned} x+y+z&=0\ldots\ldots(1)\\ bcx+cay+abz&=0\ldots\ldots(2)\\ ax+by+cz+(b-c)(c-a)(a-b)&=0\ldots(3)\end{aligned}\right\}$$

Here (3) may be written as

$$ax+by+cz=-(b-c)(c-a)(a-b)\ldots(4)$$

From (1) and (2), by cross-multiplication we get

$$\frac{x}{ab-ca}=\frac{y}{bc-ab}=\frac{z}{ca-bc},$$

or, $\dfrac{x}{a(b-c)}=\dfrac{y}{b(c-a)}=\dfrac{z}{c(a-b)}=k$ (suppose)

$\therefore\quad x=a(b-c)k,\ y=b(c-a)k.\ z=c(a-b)k......$(A)

Now, putting the values of x, y, z in (4) we get

$$a^2(b-c)k+b^2(c-a)k+c^2(a-b)k=-(b-c)(c-a)(a-b)$$

or, $k\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}=-(b-c)(c-a)(a-b)$

or, $-(a-b)(b-c)(c-a)k=-(b-c)(c-a)(a-b)$, $\therefore\quad k=1.$

$\therefore$ from (A) we get $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

22. Solve $\left.\begin{array}{l}24x=15y=40z......(1)\\ \text{and } 3x+y+2z=29\ ......(2)\end{array}\right\}$

Dividing each term of (1) by 120, which is the L. C. M. of 24, 15 and 40, we get $\frac{x}{5}=\frac{y}{8}=\frac{z}{3}=k$ (suppose)

$\therefore\quad x=5k,\quad y=8k,\quad z=3k......$(A)

Now, from (2) we get $15k+8k+6k=29$, or, $29k=29$.

$\therefore\quad k=1,\qquad \therefore$ from (A) we get $x=5$, $y=8$, $z=3$,

*23. Solve $\left.\begin{array}{l}x+y+z=a+b+c......(1)\\ ax+by+cz=a^2+b^2+c^2......(2)\\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3......(3).\end{array}\right\}$

From (1) we have $(x-a)+(y-b)+(z-c)=0......$(4)

From (2) ,, ,, $(ax-a^2)+(by-b^2)+(cz-c^2)=0,$

or, $a(x-a)+b(y-b)+c(z-c)=0......$(5)

From (4) and (5) by cross-multiplication we get

$$\frac{x-a}{c-b}=\frac{y-b}{a-c}=\frac{z-c}{b-a}=k \text{ (suppose)}$$

$\therefore\quad x-a=k(c-b),\ y-b=k(a-c),\ z-c=k(b-a)......$(A)

Now, from (3) by transposition we get

$$\frac{x}{a}-1+\frac{y}{b}-1+\frac{z}{c}-1=0,\quad \text{or,}\quad \frac{x-a}{a}+\frac{y-b}{b}+\frac{z-c}{c}=0,$$

or, $\frac{k(c-b)}{a}+\frac{k(a-c)}{b}+\frac{k(b-a)}{c}=0$ [Putting the values of $x-a$, $y-b$, $z-c$)

or, $k\left(\frac{c-b}{a}+\frac{a-c}{b}+\frac{b-a}{c}\right)=0$, $\therefore$ $k=0$,

$\therefore$ From (A) we get $\left.\begin{array}{ll} x-a=0, & \therefore\ x=a \\ y-b=0, & \therefore\ y=b \\ z-c=0, & \therefore\ z=c \end{array}\right\}$ (Ans)

24. Solve $\left.\begin{array}{l} \frac{2}{x}+\frac{5}{y}-\frac{3}{z}=0 \ldots\ldots(1) \\ \frac{3}{x}-\frac{12}{y}+\frac{2}{z}=0\ldots\ldots(2) \\ 4x+y+3z=4\ldots\ldots(3) \end{array}\right\}$

(1) $2.\frac{1}{x}+5.\frac{1}{y}-3.\frac{1}{z}=0$, (2) $3.\frac{1}{x}-12.\frac{1}{y}+2.\frac{1}{z}=0$.

From them. by the method of cross-multiplication we get

$$\frac{\frac{1}{x}}{(5\times 2)-(-3\times -12)}=\frac{\frac{1}{y}}{(-3\times 3)-(2\times 2)}=\frac{\frac{1}{z}}{(2\times -12)-(5\times 3)}$$

or, $\frac{\frac{1}{x}}{-26}=\frac{\frac{1}{y}}{-13}=\frac{\frac{1}{z}}{-39}$, or, $\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{1}=\frac{\frac{1}{z}}{3}=k$ (suppose)

$\therefore$ $\frac{1}{x}=2k$, $\frac{1}{y}=k$, $\frac{1}{z}=3k$, or, $x=\frac{1}{2k}$, $y=\frac{1}{k}$, $z=\frac{1}{3k}\ldots$(A)

Now, from (3), $4.\frac{1}{2k}+\frac{1}{k}+3.\frac{1}{3k}=4$, or, $\frac{2}{k}+\frac{1}{k}+\frac{1}{k}=4$,

or, $\frac{4}{k}=4$. $\therefore$ $k=1$. $\therefore$ From (A), $x=\frac{1}{2}$, $y=1$, $z=\frac{1}{3}$.

25. Solve $x(x+y+z)=6\ldots\ldots(1)$, $y(x+y+z)=12\cdots\ldots(2)$
$z(x+y+z)=18\ldots\ldots(3)$.

Adding the three equations we get $(x+y+z)(x+y+z)=36$,

or, $(x+y+z)^2=36$, $\therefore$ $x+y+z=\pm 6\ldots\ldots(4)$

Now, dividing (1) by (4) we get $x=\pm 1$
" (2) by (4) " " $y=\pm 2$
" (3) by (4) " " $z=\pm 3$ (Ans.)

26. Solve $\left.\begin{array}{l}\dfrac{1}{x-1}+\dfrac{1}{y-2}=3\ldots\ldots(1)\\ \dfrac{2}{x-1}+\dfrac{3}{y-2}=5\ldots\ldots(2)\end{array}\right\}$ [A. U. '42]

From (1)×2 we get

$$\frac{2}{x-1}+\frac{2}{y-2}=6$$

and (2) is $\dfrac{2}{x-1}+\dfrac{3}{y-2}=5$

(Subtracting) $-\dfrac{1}{y-2}=1$, or, $y-2=-1$, $\therefore\ y=2-1=1.$

Now, from (1) we get $\dfrac{1}{x-1}+\dfrac{1}{1-2}=3$, or, $\dfrac{1}{x-1}-1=3$,

or, $\dfrac{1}{x-1}=4$, or, $4x-4=1$, or, $4x=5$, $\therefore\ x=\frac{5}{4}=1\frac{1}{4}.$

$$\therefore\ x=1\tfrac{1}{4},\quad y=1.$$

27. In the cyclic quadrilateral ABCD, $\angle A=(2x+13)$ degrees, $\angle B=(2y-18)$ degrees, $\angle C=(y+31)$ degrees, $\angle D=(3x-29)$ degrees. Find the values of x and y. [P. U. 1932]

The sum of the opposite angles of a cyclic quadrilateral $=2$ right angles$=180°$. $\therefore\ \angle A+\angle C=180°$

and $B+\angle D=180°$.

$\therefore\ 2x+13+y+31=180\ldots\ldots(1)$

and $3x-29+2y-18=180\ldots\ldots(2)$

From (1) we have $2x+y=136\ldots\ldots(3)$

and from (2) ,, ,, $3x+2y=227\ldots\ldots(4).$

Solving (3) and (4) we get $x=45$, $y=46$. $\therefore\ x=45°$, $y=46°$.

28. Eliminate t from the equations

$$x=t+\frac{1}{t},\quad y^2=t^2+\frac{1}{t^2}.$$ [D. B. '32]

Here, $x^2=\left(t+\dfrac{1}{t}\right)^2=t^2+\dfrac{1}{t^2}+2.\ t.\dfrac{1}{t}=t^2+\dfrac{1}{t^2}+2=y^2+2.$

$\therefore$ eliminating t the equation is $x^2-y^2=2.$

Exercise 7

Solve :—

1. $\left.\begin{array}{r}3x+5y=69\\ x-2y=1\end{array}\right\}$ [C. U. '19]

2. $\left.\begin{array}{r}6x-7y=16\\ 9x-5y=35\end{array}\right\}$ [C. U. '22]

3. $\left.\begin{array}{l}\frac{x}{3}-\frac{2}{y}=1\\ \frac{x}{4}+\frac{3}{y}=3\end{array}\right\}$ [A. U. '23]

4. $\left.\begin{array}{r}x+y+z=1\\ 2x+3y+z=4\\ 4x+9y+z=16\end{array}\right\}$ [C.U. '11]

5. (a) $\left.\begin{array}{l}\frac{4}{x}+\frac{10}{y}=2\\ \frac{3}{x}+\frac{2}{y}=\frac{19}{20}\end{array}\right\}$

5. (b) $\left.\begin{array}{r}3x+4y=11\\ 5x-2y=1\end{array}\right\}$ [W. B. S. F. '53]

6. $\frac{2x+2y-3}{5}=\frac{3x-7y+4}{6}=\frac{8y-x+2}{7}$ [C. U. '14]

7. $\left.\begin{array}{r}y+z=6\\ z+x=4\\ x+y=2\end{array}\right\}$ [C. U. '18]

8. $\left.\begin{array}{l}6y-x=1\\ \frac{x+y}{x-y}=\frac{3}{2}\end{array}\right\}$ [C. U. '31]

9. $\frac{x+y}{xy}=5$, $\frac{x-y}{xy}=9$ [C. U. '32]

10. $\left.\begin{array}{l}\frac{5}{x}+3y=8\\ \frac{4}{x}-10y=56\end{array}\right\}$ [D. B. '39]

11. $\left.\begin{array}{r}23x-24y=21\\ 25x-16y=43\end{array}\right\}$ [D. B. '36]

12. $\frac{x+y}{xy}=\frac{y+z}{yz}=\frac{z+x}{zx}=\frac{2}{3}$.

13. $\left.\begin{array}{r}x-2y+z=0\\ 9x-8y+3z=0\\ 2x+3y+5z=36\end{array}\right\}$

14. $\left.\begin{array}{l}6x=8y=10z\\ 3x-4y+z=12\end{array}\right\}$

15. $xy=24$, $yz=42$, $zx=28$.

16. $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=6a\\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}=10a\\ \frac{1}{z}+\frac{1}{x}-\frac{1}{y}=-2a\end{array}\right\}$

16. (a) $\left.\begin{array}{r}2x-5y-3z=0\\ 3x+3y-z=0\\ -x+2y+5z=11\end{array}\right\}$

17. $\frac{7x+y}{5y-z}=\frac{11y-z}{6x-z}=\frac{5x-y}{z+1}=2$.

18. $\left.\begin{array}{l}\frac{x}{a}=\frac{y}{b}=\frac{z}{c},\\ ax+by+cz=a^2+b^2+c^2\end{array}\right\}$

19. $\left.\begin{array}{l} ax+by+cz=0 \\ a^2x+b^2y+c^2z=0 \\ x+y+z+(a-b)(b-c)(c-a)=0. \end{array}\right\}$

20. $\left.\begin{array}{l} \frac{x}{4}=\frac{y}{3}=\frac{z}{5} \\ 4x+3y+z=90 \end{array}\right\}$

21. $\left.\begin{array}{l} x+y=3xy \\ y+z=5yz \\ z+x=4zx \end{array}\right\}$

22. $\left.\begin{array}{l} \frac{3}{x}+\frac{2}{y}-\frac{3}{z}=0 \\ \frac{5}{x}-\frac{6}{y}+\frac{2}{z}=0 \\ 2x+3y+4z=6 \end{array}\right\}$ [D. B. '33]

*23. $\frac{x+ca}{b(c+a)}=\frac{y+bc}{a(b+c)}=\frac{z+ab}{c(a+b)}=\frac{x+y+z}{ab+bc+ca}$

[Hints : From the first three, by addition, we have

$$\text{Each}=\frac{x+y+z+ab+ac+bc}{2(ab+bc+ca)}=\frac{x+y+z}{ab+bc+ca}$$

$$\therefore \quad \text{Each}=\frac{x+y+z+ab+ac+bc-x-y-z}{2(ab+bc+ca)-(ab+bc+ca)}=1.$$

Now solve taking each $=1$.]

24. $x(y+z)=5,\ y(z+x)=8,\ z(x+y)=9.$

25. $ax+by=1, \quad bx+ay=\frac{(a+b)^2}{a^2+b^2}-1.$ [D. B. '51]

[Hints : Adding the two equations we have $x+y=\frac{a+b}{a^2+b^2}$ and, by subtraction, $x-y=\frac{a-b}{a^2+b^2}$; again, by addition and subtraction of these two......]

26. $\frac{1}{x}+\frac{1}{y}=3,\ y+z=5yz,\ z+x=4xz.$ [A.U. '43]

27. $\left.\begin{array}{l} ax+by=c \\ bx+ay=1+c \end{array}\right\}$ [E.B.S.B. '52]

28. $\frac{x}{a}+\frac{y}{b}=1, \quad \frac{y}{b}+\frac{z}{c}=1, \quad \frac{z}{c}+\frac{x}{a}=1$ [U.U. '48]

[Hints : Adding the three equations we have $2\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=3,$

$\therefore \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{3}{2}.$]

Ratio and Proporiion

N. B. :

(1) If a, b, c are in proportion, then $\frac{a}{b}=\frac{b}{c}$.

(2) If a, b, c, d are in proportion, then $\frac{a}{b}=\frac{c}{d}$.

(3) If a, b, c are in continued proportion, then $\frac{a}{b}=\frac{b}{c}$.

(4) If a, b, c, d are in continued proportion, then $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$.

(5) If $a+b$ is a mean proportional between $b+c$ and $c+a$ then $\frac{b+c}{a+b}=\frac{a+b}{c+a}$.

(6) If $a : b :: c : d$, then $a : c=b : d$ (Alternendo).

(7) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$ (Invertendo).

(8) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo).

(9) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo).

(10) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Comp. Div.)

(11) If $\frac{a}{b}=\frac{c}{d}$, then each $\frac{a+c}{b+d}$ (Addendo).

(12) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, then each $=\frac{a+c+e+g}{b+d+f+h}$.

(13) If $\frac{a}{b}=\frac{c}{d}$, then each $\frac{a-c}{b-d}$.

Examples [8]

1. If $x : y=3 : 4$, find the ratio $3y-x : 2x+y$. [P.U. '20]

[First method] $\because \frac{x}{y}=\frac{3}{4}$, $\therefore x=\frac{3}{4}y$.

Now, $\frac{3y-x}{2x+y}=\frac{3y-\frac{3}{4}y}{2\times\frac{3}{4}y+y}=\frac{\frac{9}{4}y}{\frac{5}{2}y}=\frac{9}{4}\times\frac{2}{5}=\frac{9}{10}=9 : 10.$

[The second method] $\because \frac{x}{y}=\frac{3}{4}$, $\therefore \frac{x}{3}=\frac{y}{4}=k$ (suppose)

$\therefore \quad x=3k,\ y=4k.$

Now, $\frac{3y-x}{2x+y}=\frac{12k-3k}{6k+4k}=\frac{9k}{10k}=\frac{9}{10}=9:10.$

2. If $3(x+y)=11(x-y)$, find the ratio of $x:y$.

Here, $3(x+y)=11(x-y)$, or, $3x+3y=11x-11y$, or, $-8x=-14y$,

or, $4x=7y$, or, $\frac{x}{y}=\frac{7}{4}$, $\therefore\ x:y=7:4.$

3. Find the fourth proportional to 4, 6, 10.

Let the fourth proportional be x.

Then 4, 6, 10, x are in proportion,

$\therefore \quad \frac{4}{6}=\frac{10}{x}$ or, $4x=60$. $\therefore\ x=15.$

$\therefore$ the reqd. fourth proportional$=15$.

4. If $a:b=c:d$, prove that $a^2+b^2:a^2-b^2=ac+bd:ac-bd$. [D.B. '27]

Here $\frac{a}{b}=\frac{c}{d}=k$ (suppose), $\therefore\ a=bk,\ c=dk.$

Now, $\frac{a^2+b^2}{a^2-b^2}$ (left-hand side)$=\frac{b^2k^2+b^2}{b^2k^2-b^2}=\frac{b^2(k^2+1)}{b^2(k^2-1)}=\frac{k^2+1}{k^2-1}$,

and $\frac{ac+bd}{ac-bd}$ (right-hand side)$=\frac{bk.dk+bd}{bk.dk-bd}=\frac{bdk^2+bd}{bdk^2-bd}$

$=\frac{bd(k^2+1)}{bd(k^2-1)}=\frac{k^2+1}{k^2-1}$. $\therefore\ \frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

5. If $x:a=y:b$,

show that $\frac{x^3}{a^2}+\frac{y^3}{b^2}=\frac{(x+y)^3}{(a+b)^2}$. [P.U. 1928]

Here $\frac{x}{a}=\frac{y}{b}=k$ (suppose), $\therefore\ x=ak,\ y=bk.$

Now, the left-hand side$=\frac{a^3k^3}{a^2}+\frac{b^3k^3}{b^2}=ak^3+bk^3=(a+b)k^3$,

and the right-hand side$=\frac{(ak+bk)^3}{(a+b)^2}=\frac{k^3(a+b)^3}{(a+b)^2}=(a+b)k^3$.

$$\therefore\ \frac{x^3}{a^2}+\frac{y^3}{b^2}=\frac{(x+y)^3}{(a+b)^2}.$$

6. If $\frac{x}{a}=\frac{y}{b}$, prove that $(x^2+y^2)(a^2+b^2)=(ax+by)^2$.

[C. U. 1910 ; 1928]

Here, $\frac{x}{a}=\frac{y}{b}=k$ (suppose), $\therefore$ $x=ak$, $y=bk$.

Now, $(x^2+y^2)(a^2+b^2)=(a^2k^2+b^2k^2)(a^2+b^2)$
$$=k^2(a^2+b^2)(a^2+b^2)=k^2(a^2+b^2)^2$$
and $(ax+by)^2=(a^2k+b^2k)^2=\{k(a^2+b^2)\}^2=k^2(a^2+b^2)^2$.
$$\therefore \quad (x^2+y^2)(a^2+b^2)=(ax+by)^2.$$

7. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a}{b}=\left(\frac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}$. [C.U. '30]

Here, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ (suppose), $\therefore$ $a=bk$, $c=dk$, $e=fk$.

Now, $\frac{a}{b}=\frac{bk}{b}=k$ and $\left(\frac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}=\left(\frac{b^2k^2+d^2k^2+f^2k^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}$
$$=\left\{\frac{k^2(b^2+d^2+f^2)}{(b^2+d^2+f^2)}\right\}^{\frac{1}{2}}=(k^2)^{\frac{1}{2}}=k^{2\times\frac{1}{2}}=k.$$
$$\therefore \quad \frac{a}{b}=\left(\frac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}.$$

8 If a, b, c be in continued proportion,
then $a : c=a^2+b^2 : b^2+c^2$. [C. U. '17]

Here, $\frac{a}{b}=\frac{b}{c}=k$ (suppose), $\therefore$ $a=bk=ck^2$ and $b=ck$.

Now, $\frac{a}{c}=\frac{ck^2}{c}=k^2$,

and $\frac{a^2+b^2}{b^2+c^2}=\frac{c^2k^4+c^2k^2}{c^2k^2+c^2}=\frac{c^2k^2(k^2+1)}{c^2(k^2+1)}=k^2$.
$$\therefore \quad \frac{a}{c}=\frac{a^2+b^2}{b^2+c^2}.$$

9. If $a : b=b : c$, show that $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

[C.U. 1912 ; D. B. '34, '37]

Here, $\frac{a}{b}=\frac{b}{c}=k$ (suppose), $\therefore$ $a=bk=ck^2$, $b=ck$.

Now, the left-hand side $=(ck^2+ck+c)(ck^2-ck+c)$
$$=c(k^2+k+1)c(k^2-k+1)=c^2(k^4+k^2+1).$$
The right-hand side $=c^2k^4+c^2k^2+c^2=c^2(k^4+k^2+1)$.
$$\therefore \quad (a+b+c)(a-b+c)=a^2+b^2+c^2.$$

10. If $\frac{p}{q}=\frac{r}{s}$, show that $\frac{pq}{p^2+q^2}=\frac{rs}{r^2+s^2}$. [C. U. 1940]

Here, $\frac{p}{q}=\frac{r}{s}=k$ (suppose), $\therefore$ $p=qk, r=sk$.

Now, the left-hand side$=\frac{q^2k}{q^2k^2+q^2}=\frac{q^2k}{q^2(k^2+1)}=\frac{k}{k^2+1}$.

The right-hand side$=\frac{s^2k}{s^2k^2+s^2}=\frac{s^2k}{s^2(k^2+1)}=\frac{k}{k^2+1}$,

$\therefore$ $\frac{pq}{p^2+q^2}=\frac{rs}{r^2+s^2}$.

11. If $a:b=c:d$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2. [D. B. 1928]

Here we have to prove that, $\frac{a^2+c^2}{ab+cd}=\frac{ab+cd}{b^2+d^2}$.

$\frac{a}{b}=\frac{c}{d}=k$ (suppose), $\therefore$ $a=bk, c=dk$.

Now, $\frac{a^2+c^2}{ab+cd}=\frac{b^2k^2+d^2k^2}{b^2k+d^2k}=\frac{k^2(b^2+d^2)}{k(b^2+d^2)}=k$,

and $\frac{ab+cd}{b^2+d^2}=\frac{b^2k+d^2k}{b^2+d^2}=k$. $\therefore$ $\frac{a^2+c^2}{ab+cd}=\frac{ab+cd}{b^2+d^2}$.

12. If $a:b=c:d$, then $a+c:b+d=\sqrt{a^2-c^2}:\sqrt{b^2-d^2}$.

Here, $\frac{a}{b}=\frac{c}{d}=k$ (suppose). $\therefore$ $a=bk, c=dk$.

Now, the left-hand side$=\frac{a+c}{b+d}=\frac{bk+dk}{b+d}=\frac{k(b+d)}{b+d}=k$,

and the right-hand side$=\frac{\sqrt{a^2-c^2}}{\sqrt{b^2-d^2}}=\frac{(b^2k^2-d^2k^2)^{\frac{1}{2}}}{(b^2-d^2)^{\frac{1}{2}}}$

$$=\frac{\{k^2(b^2-d^2)\}^{\frac{1}{2}}}{(b^2-d^2)^{\frac{1}{2}}}=\frac{k(b^2-d^2)^{\frac{1}{2}}}{(b^2-d^2)^{\frac{1}{2}}}=k.$$

$\therefore$ both sides are equal. [Here it is better to write both sides]

13. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that $\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=3\left(\frac{x+y+z}{a+b+c}\right)^3$. [P. U. 1926]

Here, $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (suppose), $\therefore$ $x=ak, y=bk, z=ck$.

Now, the left-hand side$=\frac{a^3k^3}{a^3}+\frac{b^3k^3}{b^3}+\frac{c^3k^3}{c^3}$

$=k^3+k^3+k^3=3k^3,$

and the right-hand side$=3\left(\frac{ak+bk+ck}{a+b+c}\right)^3=3\left\{\frac{k(a+b+c)}{(a+b+c)}\right\}^3$

$=3(k)^3=3k^3.$

$\therefore$ Both sides are equal. [It is better not to write in this way, you should write both sides here.]

14. If $(a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$, prove that $a:b=c:d$.

[C. U. 1929]

From the given condition we have $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$.

$\therefore \quad \frac{2(a+b)}{2(c+d)}=\frac{2(a-b)}{2(c-d)}$, [Comp. & Div.]

or, $\frac{a+b}{c+d}=\frac{a-b}{c-d}$, $\therefore \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}$ [Alternendo]

$\therefore \quad \frac{2a}{2b}=\frac{2c}{2d}$ [Comp. & Div.] $\therefore \quad \frac{b}{a}=\frac{c}{d}$.

15. If $\frac{x}{y}=\frac{y}{z}$, find the simplest value of $\frac{xyz(x+y+z)^3}{(xy+yz+zx)^3}$.

[D. B. '28]

$\because \quad \frac{x}{y}=\frac{y}{z}$, $\therefore \quad y^2=xz$. $\therefore$ the given fraction$=\frac{y.xz(x+y+z)^3}{(xy+yz+zx)^3}$

$=\frac{y^3(x+y+z)^3}{(xy+yz+y^2)^3}$ [substituting y^2 for xz]$=\frac{y^3(x+y+z)^3}{y^3(x+y+z)^3}=1$.

16. If $a:b=b:c=c:d$, prove that $(a^2-b^2)(c^2-d^2)$ $=(b^2-c^2)^2$.

[C. U. 1943]

$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$ (suppose), $\therefore$ $a=bk=ck^2=dk^3$,
$b=ck=dk^2$,
$c=dk$.

Now, $(a^2-b^2)(c^2-d^2)=(d^2k^6-d^2k^4)(d^2k^2-d^2)$

$=d^2k^4(k^2-1).\ d^2(k^2-1)=d^4k^4(k^2-1)^2,$

and $\quad (b^2-c^2)^2=(d^2k^4-d^2k^2)^2=\{d^2k^2(k^2-1)\}^2=d^4k^4(k^2-1)^2.$

$\therefore \quad (a^2-b^2)(c^2-d^2)=(b^2-c^2)^2.$

17. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each of the ratios$=\left(\frac{la^n+mc^n+ne^n}{lb^n+md^n+nf^n}\right)^{\frac{1}{n}}$.

Here, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ (suppose), $\therefore\quad a=bk,\ c=dk=,\ e=fk.$

Now, $\left(\frac{la^n+mc^n+ne^n}{lb^n+md^n+nf^n}\right)^{\frac{1}{n}}=\left(\frac{lb^nk^n+md^nk^n+nf^nk^n}{lb^n+md^n+nf^n}\right)^{\frac{1}{n}}$

$=\left\{\frac{k^n(lb^n+md^n+nf^n)}{lb^n+md^n+nf^n}\right\}^{\frac{1}{n}}=(k^n)^{\frac{1}{n}}=k^{n\times\frac{1}{n}}=k.$

$\because$ each given ratio$=k$,

$\therefore$ each ratio$=\left(\frac{la^n+mc^n+ne^n}{lb^n+md^n+nf^n}\right)^{\frac{1}{n}}$.

18. If $a:b=b:c=c:d$, prove that

$(b-c)^2+(c-a)+(b-d)^2=(a-d)^2.$ [D. B. 1933]

$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$ (suppose), $\therefore\ a=bk=ck^2=dk^3,\ b=ck=dk^2,\ c=dk.$

Now, the left-hand side$=(dk^2-dk)^2+(dk-dk^3)^2+(dk^2-d)^2$

$=d^2k^4+d^2k^2-2d^2k^3+d^2k^2+d^2k^6-2d^2k^4+d^2k^4$

$-2d^2k^2+d^2=d^2k^6-2d^2k^3+d^2.$

The right-hand side$=(dk^3-d)^2=d^2k^6-2d^2k^3+d^2.$

$\therefore$ both sides are equal.

19. If $a:b=c;d=e;f$, prove that

$\frac{a^2ce}{b^2df}=\sqrt[4]{\frac{a^5c^8e^8}{b^5d^8f^8}}.$ [C. U. 1933]

Here, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ (suppose), $\therefore\ a=bk,\ c=dk,\ e=fk.$

Now, the left-hand side$=\frac{b^2k^2.dk.fk}{b^2df}=\frac{b^2dfk^4}{b^2df}=k^4$,

and the right-hand side$=\sqrt[4]{\frac{b^5k^5.d^8k^8.f^8k^8}{b^5d^8f^8}}=\left(\frac{b^5d^8f^8k^{16}}{b^5d^8f^8}\right)^{\frac{1}{4}}$

$=(k^{16})^{\frac{1}{4}}=k^{16\times\frac{1}{4}}=k^4.\quad\therefore$ both sides are equal.

20. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, show that $(a^2+b^2+c^2)(b^2+c^2+d^2)$

$=(ab+bc+cd)^2.$ [C. U. 1944]

Here $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$ (suppose), $\therefore\quad a=bk,\ b=ck,\ c=dk.$

Now, $(a^2+b^2+c^2)(b^2+c^2+d^2)=(b^2k^2+c^2k^2+d^2k^2)\times(b^2+c^2+d^2)=k^2(b^2+c^2+d^2)(b^2+c^2+d^2)=k^2(b^2+c^2+d^2)^2$.

Again, $(ab+bc+cd)^2=(bk.b+ck.c+dk.d)^2$
$=(b^2k+c^2k+d^2k)^2=k^2(b^2+c^2+d^2)^2$. $\therefore$ both sides are equal.

21. If $\dfrac{x}{lm-n^2}=\dfrac{y}{mn-l^2}=\dfrac{z}{nl-m^2}$,

show that $lx+my+nz=0$. [C. U. 1934]

Here, $\dfrac{x}{lm-n^2}=\dfrac{y}{mn-l^2}=\dfrac{z}{nl-m^2}=k$ (suppose)

$\therefore$ $x=k(lm-n^2)$, $y=k(mn-l^2)$, $z=k(nl-m^2)$.

Now, $lx+my+nz=kl(lm-n^2)+km(mn-l^2)+kn(nl-m^2)$
$=k(l^2m-n^2l+m^2n-l^2m+n^2l-m^2n)=k\times 0=0$.

23. If $\dfrac{x}{a+b-c}=\dfrac{y}{b+c-a}=\dfrac{z}{c+a-b}$,

show that each fraction $=\dfrac{x+y+z}{a+b+c}$. [C. U. '11 ; D. B. '36]

$\because$ $\dfrac{x}{a+b-c}=\dfrac{y}{b+c-a}=\dfrac{z}{c+a-b}$,

$\therefore$ each of them $=\dfrac{(x+y+z)}{(a+b-c)+(b+c-a)+(c+a-b)}$

$=\dfrac{x+y+z}{a+b+c}$ (proved) [By Addendo]

23. If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$.

show that $(b-c)x+(c-a)y+(a-b)z=0$.

Here, $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}=k$ (suppose)

$\therefore$ $x=(b+c)k$, $y=(c+a)k$, $z=(a+b)k$.

Now, $(b-c)x+(c-a)y+(a-b)z=(b-c)(b+c)k+(c-a)(c+a)k$
$+(a-b)(a+b)k=k(b^2-c^2)+k(c^2-a^2)+k(a^2-b^2)$
$=k(b^2-c^2+c^2-a^2+a^2-b^2)=k\times 0=0$.

24. If $\dfrac{a}{a+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$, prove that each ratio is equal to $\frac{1}{2}$ or -1.

$\because$ $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$,

$\therefore$ Each ratio$=\dfrac{\text{sum of the numerators}}{\text{sum of the denominators}}=\dfrac{a+b+c}{2(a+b+c)}=\dfrac{1}{2}$.

Again, $\because$ $\dfrac{a}{b+c}=\dfrac{b}{c+a}$,

$\therefore$ each ratio$=\dfrac{\text{difference of the two numerators}}{\text{difference of the two denominators}}$

$=\dfrac{a-b}{b+c-c-a}=\dfrac{a-b}{b-a}=\dfrac{(a-b)}{-(a-b)}=-1$. $\therefore$ each ratio$=\frac{1}{2}$ or -1.

25. If $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$ and $a+b+c\neq 0$,

shew that $a=b=c$. [The sign $\neq$ denotes 'is not equal to.']

$\because$ $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$, $\therefore$ $\dfrac{a}{b+c}+1=\dfrac{b}{c+a}+1=\dfrac{c}{a+b}+1$

or, $\dfrac{a+b+c}{b+c}=\dfrac{a+b+c}{c+a}=\dfrac{a+b+c}{a+b}$, $\therefore$ $\dfrac{1}{b+a}=\dfrac{1}{c+a}=\dfrac{1}{a+b}$

[Dividing by $a+b+c$ which is not zero]

$\therefore$ $b+c=c+a=a+b$. Now $\because$ $b+c=c+a$, $\therefore$ $a=b$.

Again, $\because$ $c+a=a+b$, $\therefore$ $c=b$. $\therefore$ $a=b=c$.

26. If $\dfrac{a}{ax+by+cz}=\dfrac{b}{bx+cy+az}=\dfrac{c}{cx+ay+bz}$,

and $a+b+c\neq 0$, prove that each of the ratios$=\dfrac{1}{x+y+z}$.

Each of the given ratio$=\dfrac{\text{sum of the numerators}}{\text{sum of the denominators}}$

$=\dfrac{a+b+c}{ax+by+cz+bx+cy+az+cx+ay+bz}$

$=\dfrac{a+b+c}{a(x+y+z)+b(x+y+z)+c(x+y+z)}$

$=\dfrac{a+b+c}{(x+y+z)(a+b+c)}=\dfrac{1}{x+y+z}$.

27. If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$,

then $\dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{w}=\dfrac{1}{x}$. [C. U. 1905]

$(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w=k$ (suppose)

Now, $\because$ $(a+b+c)x=k$, $\therefore$ $x=\dfrac{k}{a+b+c}$, $\therefore$ $\dfrac{1}{x}=\dfrac{a+b+c}{k}$.

Again $\because$ $(b+c-a)y=k$, $\therefore$ $y=\frac{k}{b+c-a}$, $\therefore$ $\frac{1}{y}=\frac{b+c-a}{k}$

Similarly, $z=\frac{k}{c+a-b}$, $\therefore$ $\frac{1}{z}=\frac{c+a-b}{k}$,

and $w=\frac{k}{a+b-c}$, $\therefore$ $\frac{1}{w}=\frac{a+b-c}{k}$.

Now, $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{b+c-a}{k}+\frac{c+a-b}{k}+\frac{a+b-c}{k}$

$=\frac{b+c-a+c+a-b+a+b-c}{k}=\frac{a+b+c}{k}=\frac{1}{x}$.

28. If $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c+a}{b}=1$ and $a-b+c\neq 0$,

prove that $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$.

[C. U. 1875]

Here from the given condition $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c+a}{b}-1=0$

or, $\frac{a-b}{c}+1+\frac{b-c}{a}-1+\frac{c+a}{b}-1=0$,

or, $\frac{a-b+c}{c}+\frac{b-c-a}{a}+\frac{c+a-b}{b}=0$,

or, $\frac{a-b+c}{c}+\frac{-(a-b+c)}{a}+\frac{a-b+c}{b}=0$,

or, $(a-b+c)\left(\frac{1}{c}-\frac{1}{a}+\frac{1}{b}\right)=0$,

$\therefore$ $\frac{1}{c}-\frac{1}{a}+\frac{1}{b}=0$ [$\because$ $a-b+c\neq 0$], $\therefore$ $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

*29. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, then $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Multiplying the numerator and the denominator of the first raction by a, those of the second fraction by b and those of he third fraction by c we have

$$\frac{acy-bcx}{c^2}=\frac{bcx-abz}{b^2}=\frac{abz-acy}{a^2}$$

$\therefore$ each ratio $=\frac{\text{sum of the numerators}}{\text{sum of the denominators}}$

$=\frac{acy-bcx+bcx-abz+abz-acy}{c^2+b^2+a^2}=\frac{0}{a^2+b^2+c^2}=0.$

Now $\because \dfrac{acy-bcx}{c^2}=0, \qquad \therefore \; acy-bcx=0,$

or, $acy=bcx,$ or, $ay=bx. \qquad \therefore \; \dfrac{y}{b}=\dfrac{x}{a}.$

Again, $\because \dfrac{bcx-abz}{b^2}=0, \qquad \therefore \; bcx-abz=0,$

or, $bcx=abz,$ or, $cx=az, \qquad \therefore \; \dfrac{x}{a}=\dfrac{z}{c}, \quad \therefore \; \dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}.$

*30. If $\dfrac{bz+cy}{b-c}=\dfrac{cx+az}{c-a}=\dfrac{ay+bx}{a-b}$,

prove that $(a+b+c)(x+y+z)=ax+by+cz.$ [C. U. '35 Addl.]

Here, $\dfrac{bz+cy}{b-c}=\dfrac{cx+az}{c-a}=\dfrac{ay+bx}{a-b}=k$ (suppose)

$\therefore \; bz+cy=k(b-c),\; cx+az=k(c-a),\; ay+bx=k(a-b).$

Now taking the sum of the above we have

$bz+cy+cx+az+ay+bx=k(b-c+c-a+a-b)=0.$

Adding $ax+by+cz$ to both sides we have

$bz+bx+by+cy+cx+cz+az+ay+ax=ax+by+cz,$

or, $b(z+x+y)+c(y+x+z)+a(z+y+x)=ax+by+cz,$

$\therefore \; (x+y+z)(a+b+c)=ax+by+cz.$

31. If $\dfrac{a}{b}=\dfrac{c}{d}$, show that $bd\left(\dfrac{a+b}{b}+\dfrac{c+d}{d}\right)^2=4(a+b)(c+d).$

Here, $\dfrac{a}{b}=\dfrac{c}{d}=k$ (suppose), $\therefore \; a=bk,\; c=dk.$

Now, the left-hand side $=bd\left(\dfrac{bk+b}{b}+\dfrac{dk+d}{d}\right)^2=bd(k+1+k+1)^2$

$=bd(2k+2)^2=bd.4(k+1)^2=4bd(k+1)^2,$

The right-hand side

$=4(bk+b)(dk+d)=4b(k+1).d(k+1)=4bd(k+1)^2.$

$\therefore \; bd\left(\dfrac{a+b}{b}+\dfrac{c+d}{d}\right)^2=4(a+b)(c+d).$

32. If $\dfrac{a+b}{b+c}=\dfrac{c+d}{d+a}$, show that either $c=a$, or $a+b+c+d=0.$

[C. U, 1891]

$\because \dfrac{a+b}{b+c}=\dfrac{c+d}{d+a}, \; \therefore \dfrac{a+b}{c+d}=\dfrac{b+c}{d+a}$ [Alternendo]

$\therefore \quad \frac{a+d+c+b}{c+d}=\frac{b+c+d+a}{d+a}$ (by comp.)

or, $\frac{a+b+c+d}{c+d}-\frac{a+b+c+d}{a+d}=0$

or, $(a+b+c+d)\left(\frac{1}{c+d}-\frac{1}{a+d}\right)=0.$

$\therefore$ either $a+b+c+d=0$, (proved) ; or, $\frac{1}{c+d}-\frac{1}{a+d}=0$,

or, $\frac{1}{c+d}=\frac{1}{a+d}$, or, $c+d=a+d$, or, $a=c$ (proved).

Exercise 8

If $a : b=c : d$, *prove the following* :—

1. $a^2+ab+b^2 : a^2-ab+b^2=c^2+cd+d^2 : c^2-cd+d^2$. [C. U. 1894, 1945]
2. $(a^2+c^2)(b^2+d^2)=(ab+cd)^2$. [C. U. 1946 ; A. U. 1890]
3. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2 : (c^{\frac{1}{2}}+d^{\frac{1}{2}})^2=a-b : c-d$. [C. U. 1895]
4. $\sqrt{a^2+c^2} : \sqrt{b^2+d^2}=ma+nc : mb+nd$. [C. U. 1880]
5. $\frac{a^2+c^2}{b^2+d^2}=\frac{(a+c)c}{(b+d)d}+$ [C. U. 1937]
6. $\frac{a^2+b^2}{a^2-b^2}=\frac{c^2+d^2}{c^2-d^2}$. [C. U. 1939 Suppl.]
7. If $5(x-y)=3(x+y)$, find $x : y$. [C. U. '13 ; Ans.$=4 : 1$]
8. If $\frac{a}{b}=\frac{p}{q}$, prove that $(a+b)(a^2+b^2)p^3=(p+g)(p^2+q^2)a^3$. [C. U. 1936]
9. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, show that $\frac{x^3+y^3+z^3}{a^3+b^3+c^3}=\frac{xyz}{abc}$. [D. B. '32]
10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $(a-b)^3 : (b-c)^3=a : d$. [C. U. '38 ; G. U. '48 ; P. U. '22]

If a, b, c, d be *in continued proportion, prove that* :—

11. $a+b : c+d=a^2+b^2+c^2 : b^2+c^2+d^2$. [C. U. 1239]
12. $a : d=a^3+b^3+c^3 : b^3+c^3+d^3$. [C. U. '34 ; D. B. '35]
13. $(ab+cd) : (ab-cd)=(b^2+d^2) : (b^2-d^2)$. [P. U. '13]

If a, b, c be in continued proportion, prove the following :—

14. $(a+b+c)^2 : (a^2+b^2+c^2)=(a+b+c) : (a-b+c)$. [P. U. 1934]

15. $\dfrac{abc(a+b+c)^3}{(ab+bc+ca)^3}=1$. 16. $\left(\dfrac{a+b}{b+c}\right)^2=\dfrac{a^2+b^2}{b^2+c^2}$. [B. U. '34]

17. $a^2b^2c^2\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right)=a^3+b^3+c^3$.

If $a : b=c : d=e : f$, show that :—

18. $\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}=\dfrac{ce}{df}$. [C. U. 1941

19. $\left(\dfrac{2a+3c+5e}{2b+3d+5f}\right)^3=\dfrac{ace}{bdf}$. [C. U. 1942]

20. Each$=\sqrt[3]{a^3+c^3+e^3} : \sqrt[3]{b^3+d^3+f^3}$.

21. $(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2$. [P. U. '31 ; W. B. S. F. '52]

22. If $x : a=y : b=z : c$, prove that

$\dfrac{x^3}{a^2}+\dfrac{y^3}{b^2}+\dfrac{z^3}{c^2}=\dfrac{(x+y+z)^3}{(a+b+c)^2}$. [D. B. 1930]

23. If $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$,

and $a+b+c\neq 0$, prove that $a=b=c$. [C. U. 1873]

24. If $\dfrac{x}{(b-c)(b+c-2a)}=\dfrac{y}{(c-a)(c+a-2b)}=\dfrac{z}{(a-b)(a+b-2c)}$,

find the value of $x+y+z$. [C. U. 1889. Ans.=0]

*25. If $x : ax+by+cz=y : bx+cy+az=z : cx+ay+bz$, and $x+y+z\neq 0$, then show that each of the ratios is equal to $\dfrac{1}{a+b+c}$. [M. U. 1902]

26. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, find the value of

$(b-c)x+(c-a)y+(a-b)z$. [C. U. '48 ; Ans.=0]

27. If $a : b=x : y$, prove that $a^2+b^2 : \dfrac{a^3}{a+b} :: x^2+y^2 : \dfrac{x^3}{x+y}$. [C. U.]

28. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, prove that

$$\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$ [D. B. '49]

29. If $\frac{x}{y}=\frac{a+2}{a-2}$, find the value of $\frac{x^2-y^2}{x^2+y^2}$. [D. B. '51]

[Hints : $\frac{x^2}{y^2}=\frac{a^2+4a+4}{a^2-4a+4}$, $\therefore\ \frac{x^2+y^2}{x^2-y^2}=\frac{2(a^2+4)}{8a}$ (by comp. & div.)

$\therefore\ \frac{x^2-y^2}{x^2+y^2}=\frac{4a}{a^2+4}$]

IDENTITIES AND DETERMINATION OF VALUES

Examples [9]

1. Show that $(a+b+c)^2+a^2+b^2+c^2$

$$=(a+b)^2+(b+c)^2+(c+a)^2.$$

The left-hand side$=2a^2+2b^2+2c^2+2ab+2bc+2ca$

$=(a^2+b^2+2ab)+(b^2+c^2+2bc)+(c^2+a^2+2ca)$

$=(a+b)^2+(b+c)^2+(c+a)^2.$

2. Prove that $(1+ab)^2-(1-a^2)(1-b^2)=(a+b)^2.$

The left-hand side$=1+a^2b^2+2ab-1-a^2b^2+a^2+b^2$

$=a^2+b^2+2ab=(a+b)^2.$

3. Show that

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a).$$

The left-hand side$=\{(a+b)+c\}^3=(a+b)^3+3(a+b)^2c$

$+3(a+b)c^2+c^3$

$=a^3+b^3+3ab(a+b)+3(a+b)^2c+3(a+b)c^2+c^3$

$=a^3+b^3+c^3+3(a+b)(ab+ac+bc+c^2)$

$=a^3+b^3+c^3+3(a+b)(b+c)(c+a).$

4. Show that $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$

$$=3(2a+b+c)(2b+c+a)(2c+a+b).$$

$8(a+b+c)^3=\{2(a+b+c)\}^3=(2a+2b+2c)^3$

$=\{(a+b)+(b+c)+(c+a)\}^3=(a+b)^3+(b+c)^3+(c+a)^3+$

$3(a+b+b+c)(b+c+c+a)(c+a+a+b)$, [vide formula No. 10]

$\therefore$ the whole left side $=(a+b)^3+(b+c)^3+(c+a)^3$
$+3(a+2b+c)(a+b+2c)(2a+b+c)-(a+b)^3-(b+c)^3-(c+a)^3$
$=3(2a+b+c)(2b+c+a)(2c+a+b)$.

5. Prove that $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.
[C. U. 1926]

The left side $=a^2c^2+b^2d^2+b^2c^2+a^2d^2=(a^2c^2+b^2d^2+2abcd)$
$+(a^2d^2-2abcd+b^2c^2)=(ac+bd)^2+(ad-bc)^2$.

6. Show that $x+(1-x)y+(1-x)(1-y)z$
$=1-(1-x)(1-y)(1-z)$.

Adding 1 to and deducting 1 from the left side we have the left side $=1-1+x+(1-x)y+(1-x)(1-y)z=1-(1-x)+(1-x)y$
$+(1-x)(1-y)z=1-(1-x)(1-y)+(1-x)(1-y)z$
$=1-(1-x)(1-y)(1-z)$.

7. If $ab+a+b=a^2$, show that $1+a^2=(1+a)(1+b)$.
$1+a^2=1+a+b+ab$ [substituting the value of a^2.]
$=(1+a)+b(1+a)=(1+a)(1+b)$.

8. If $xy+xz+yz=1$, prove that $1+x^2=(x+y)(x+z)$, $1+y^2=(y+z)(y+x)$ and $1+z^2=(z+x)(z+y)$.

Now, $1+x^2=xy+xz+yz+x^2$ [$\because$ $1=xy+xz+yz$.]
$=(x+y)(x+z)$
Similarly, $1+y^2=xy+xz+yz+y^2=(y+z)(y+x)$
and $1+z^2=xy+xz+yz+z^2=(z+x)(z+y)$.

9. Express $2(a^2+b^2+c^2-ab-ac-bc)$ as the sum of 3 squares.

The given expression $=2a^2+2b^2+2c^2-2ac-2ab-2bc$
$=(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)$
$=(a-b)^2+(b-c)^2+(c-a)^2$.

10. Express $3(a-b)(b-c)(c-a)$ as the sum of 3 cubes.
Suppose, $x=a-b$, $y=b-c$, $z=c-a$
Then $x+y+z=a-b+b-c+c-a=0$,
$\therefore$ $x^3+y^3+z^3=3xyz$ (vide Example 12 below)
$\therefore$ $3(a-b)(b-c)(c-a)=(a-b)^3+(b-c)^3+(c-a)^3$.

11. If $a+b+c=0$, then $a^2+b^2+c^2=-2(ab+bc+ca)$.
$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$
$=(0)^2-2(ab+bc+ca)=-2(ab+bc+ca)$.

12. If $a+b+c=0$, show that $a^3+b^3+c^3=3abc$.

$\because\ a+b+c=0, \qquad \therefore\ a+b=-c,$

$\therefore\ (a+b)^3=(-c)^3,$ or, $a^3+b^3+3ab(a+b)=-c^3,$

or, $a^3+b^3+3ab\times -c=-c^3$, or, $a^3+b^3-3abc=-c^3,$

$\therefore\ a^3+b^3+c^3=3abc.$

Alternative method : $a^3+b^3+c^3=a^3+b^3+c^3-3abc+3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)+3abc$

$=0\times(a^2+b^2+c^2-ab-ac-bc)+3abc=3abc.$

13. If $a+b+c=0$, prove that $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$.

[C. U. 1943]

$\because\ a+b+c=0, \qquad \therefore\ a+b=-c,$

$\therefore\ a^2+b^2+2ab=c^2,$ or, $a^2+b^2-c^2=-2ab,$

$\therefore\ (a^2+b^2-c^2)^2=(-2ab)^2,$

or, $a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=4a^2b^2$

$\therefore\ a^4+b^4+c^4=4a^2b^2-2a^2b^2+2a^2c^2+2b^2c^2$

$=2a^2b^2+2a^2c^2+2b^2c^2=2(a^2b^2+b^2c^2+c^2a^2).$

14. If $a+b+c=0$, then $2(a^4+b^4+c^4)=(a^2+b^2+c^2)^2$.

As in example 13, here $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

Now adding $a^4+b^4+c^4$ to both sides we have

$2(a^4+b^4+c^4)=a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2$

$=(a^2+b^2+c^2)^2.$

15. If $a+b+c=0$, show that $a^2-bc=b^2-ca=c^2-ab$.

[C. U. '23 ; D. B. '22, '27, '29]

$\because\ a+b+c=0, \quad \therefore\ a=-b-c$. Multiplying both sides by a,

$a^2=-ab-ac$, Similarly $b^2=-ab-bc$ and $c^2=-ac-bc$.

Now $a^2-bc=-ab-ac-bc=-(ab+ac+bc)$,

$b^2-ca=-ab-bc-ac=-(ab+ac+bc)$,

and $c^2-ab=-ac-bc-ab=-(ab+bc+ac)$.

$\therefore\ a^2-bc=b^2-ca=c^2-ab.$

16. If $a+b+c=0$, prove that

$a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2.$ [C. U. 1926]

$\because\ a+b+c=0, \quad \therefore\ a=-b-c, \quad \therefore\ a^2=-ab-ac.$

Similarly $b^2=-ab-bc$ and $c^2=-ac-bc.$

Now, $a^2+ab+b^2=-ab-ac+ab-ab-bc=-(ab+bc+ac)$,

$b^2+bc+c^2=-ab-bc+bc-ac-bc=-(ab+bc+ac)$,

$c^2+ca+a^2=-ac-bc+ca-ab-ac=-(ab+bc+ac)$

$\therefore\quad a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.

17. If $a+b+c=0$, show that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0. \qquad \text{[C. U. 1938]}$$

$\because\quad a+b+c=0, \qquad \therefore\quad a+b=-c$,

$\therefore\quad a^2+b^2+2ab=c^2$ [squaring both sides.]

or, $a^2+b^2-c^2=-2ab$. Similarly $b^2+c^2-a^2=-2bc$ and $c^2+a^2-b^2=-2ca$.

Now, $\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}+\dfrac{1}{a^2+b^2-c^2}$

$$=\frac{1}{-2bc}+\frac{1}{-2ca}+\frac{1}{-2ab}=\frac{a+b+c}{-2abc}=\frac{0}{-2abc}=0.$$

18. If $a+b+c=0$, prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}=0,$$

$\because\quad a+b+c=0 \qquad \therefore\quad a=-b-c$

$\therefore\quad a^2=-ab-ac$ (Multiplying both sides by a)

Similarly, $b^2=-ab-bc$ and $c^2=-ac-bc$.

Now, $2a^2+bc=a^2+a^2+bc=a^2-ab-ac+bc=(a-b)(a-c)$,

$2b^2+ca=b^2+b^2+ca=b^2-ab-bc+ca=(b-c)(b-a)$,

and $2c^2+ab=c^2+c^2+ab=c^2-ac-bc+ab=(c-a)(c-b)$,

$$\therefore\quad \frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}$$

$$=\frac{1}{-(a-b)(c-a)}+\frac{1}{-(a-b)(b-c)}+\frac{1}{-(c-a)(b-c)}$$

$$=\frac{b-c+c-a+a-b}{-(a-b)(b-c)(c-a)}=\frac{0}{-(a-b)(b-c)(c-a)}=0.$$

19. If $a+b+c=0$, show that

$$\frac{2a^2}{b^2+c^2-a^2}+\frac{2b^2}{c^2+a^2-b^2}+\frac{2\ ^2}{a^2+b^2-c^2}=-3.$$

$\because\quad a+b+c=0, \quad \therefore\quad a+b=-c$, or, $a^2+b^2+2ab=c^2$,

or, $a^2+b^2-c^2=-2ab$. Similarly $b^2+c^2-a^2=-2bc$ and $c^2+a^2-b^2=-2ca$.

Now, the left-hand side$=\frac{2a^2}{-2bc}+\frac{2b^2}{-2ca}+\frac{2c^2}{-2ab}$

$$=\frac{a^2}{-bc}+\frac{b^2}{-ca}+\frac{c^2}{-ab}=\frac{a^3+b^3+c^3}{-abc}=\frac{3abc}{-abc}=-3.$$

[$\because$ $a+b+c=0$. $\therefore$ $a^3+b^3+c^3=3abc$]

20. If $a+b+c=0$, then

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{(a+b)^3}+\frac{1}{(b+c)^3}+\frac{1}{(c+a)^3}=0.$$

$\because$ $a+b+c=0$, $\therefore$ $a+b=-c$, $b+c=-a$, $c+a=-b$

Now, the left-hand side

$$=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{(-c)^3}+\frac{1}{(-a)^3}+\frac{1}{(-b)^3}$$

$$=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}-\frac{1}{c^3}-\frac{1}{a^3}-\frac{1}{b^3}=0.$$

21. If $bc+ca+ab=0$, prove that

$$\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{c^2-ab}=0,$$

[C. U. '27, '45, '51 ; D. B. '37]

$\because$ $bc+ca+ab=0$, $\therefore$ $bc+ca=-ab$, $ca+ab=-bc$, and $bc+ab=-ca$.

Now, $\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{c^2-ab}$

$$=\frac{1}{a^2+ca+ab}+\frac{1}{b^2+bc+ab}+\frac{1}{c^2+bc+ca}$$

[substituting the values of $-ab$, $-bc$, $-ca$]

$$=\frac{1}{a(a+b+c)}+\frac{1}{b(a+b+c)}+\frac{1}{c(a+b+c)}$$

$$=\frac{bc+ca+ab}{abc(a+b+c)}=\frac{0}{abc(a+b+c)}=0.$$

22. If $s=a+b+c$, then $(as+bc)(bs+ca)(cs+ab)$ $=(b+c)^2(c+a)^2(a+b)^2$.

Putting the value of s we have $(as+bc)(bs+ca)(cs+ab)$

$$=(a^2+ab+ac+bc)(b^2+ab+bc+ca)(ac+bc+c^2+ab)$$
$$=(a+b)(a+c)(b+a)(b+c)(a+c)(b+c)$$
$$=(b+c)^2(c+a)^2(a+b)^2.$$

23. If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, prove that
$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc)$. [D. B. '30]

$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$
[vide formula No. 9]

$$=\tfrac{1}{2}(b+c-a+c+a-b+a+b-c)\{(b+c-a-c-a+b)^2$$
$$+(c+a-b-a-b+c)^2+(a+b-c-b-c+a)^2\}$$
$$=\tfrac{1}{2}(a+b+c)\{(2b-2a)^2+(2c-2b)^2+(2a-2c)^2\}$$
$$=\tfrac{1}{2}(a+b+c)\{4(b-a)^2+4(c-b)^2+4(a-c)^2\}$$
$$=4\times\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=4(a^3+b^3+c^3-3abc).\quad [\because\ (b-a)^2=(a-b)^2\ldots]$$

24. If $x=a+b+c$, then $(x+a)^3+(x+b)^3+(x+c)^3$
$-3(x+a)(x+b)(x+c)=4(a^3+b^3+c^3-3abc)$.

The left-hand side $=\frac{1}{2}(x+a+x+b+x+c)$

$$\times\{(x+a-x-b)^2+(x+b-x-c)^2+(x+c-x-a)^2\}$$
$$=\tfrac{1}{2}(3x+a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=\tfrac{1}{2}(4a+4b+4c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=4\times\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=4(a^3+b^3+c^3-3abc).$$

25. If $2s=a+b+c$, show that
$s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$. [C. U. '37, '41]

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2$$
$$=s^2+s^2-2as+a^2+s^2-2bs+b^2+s^2-2cs+c^2$$
$$=4s^2-2s(a+b+c)+a^2+b^2+c^2=4s^2-2s\times2s+a^2+b^2+c^2$$
$$=4s^2-4s^2+a^2+b^2+c^2=a^2+b^2+c^2.$$

26. If $2s=a+b+c$, prove that
$s^2+(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=ab+ac+bc$.
[C. U. 1944]

The left-hand side

$$=s^2+s^2-as-bs+ab+s^2-bs-cs+bc+s^2-as-cs+ac$$
$$=4s^2-2as-2bs-2cs+ab+ac+bc$$

The left-hand side of the identity

$$=\frac{(s-a)^3+(s-b)^3+(s-c)^3}{(s-a)(s-b)(s-c)}=\frac{3(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)}=3.$$

[Here $\because$ $s-a+s-b+s-c=0$,

$\therefore$ $(s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c)$]

42. $x+y+z=xyz$, prove that

$$\frac{1+x^2}{(x+y)(x+z)}+\frac{1+y^2}{(y+z)(y+x)}+\frac{1+z^2}{(z+x)(z+y)}=1.$$

The left-hand side

$$=\frac{1+x^2}{x^2+xy+xz+yz}+\frac{1+y^2}{y^2+yx+yz+xz}+\frac{1+z^2}{z^2+zx+yz+xy}$$

$$=\frac{1+x^2}{x(x+y+z)+yz}+\frac{1+y^2}{y(y+x+z)+xz}+\frac{1+z^2}{z(z+x+y)+xy}$$

$$=\frac{1+x^2}{x^2yz+yz}+\frac{1+y^2}{xy^2z+xz}+\frac{1+z^2}{xyz^2+xy}$$ [Putting xyz for $x+y+z$]

$$=\frac{1+x^2}{yz(1+x^2)}+\frac{1+y^2}{xz(1+y^2)}+\frac{1+z^2}{xy(1+z^2)}=\frac{1}{yz}+\frac{1}{xz}+\frac{1}{xy}$$

$$=\frac{x+y+z}{xyz}=\frac{xyz}{xyz}=1.$$

43. If $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}=1$,

find the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$.

Here $\because$ $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}=1$,

$\therefore$ $\frac{a}{1-a}+1+\frac{b}{1-b}+1+\frac{c}{1-c}+1=1+3$

[Adding 3 to both sides]

or, $\frac{a+1-a}{1-a}+\frac{b+1-b}{1-b}+\frac{c+1-c}{1-c}=4$,

$\therefore$ $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=4$.

44. If $x=\frac{4ab}{a+b}$, prove that $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=2$. [D. B. '32]

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=\frac{x+2a}{x-2a}-1+\frac{x+2b}{x-2b}-1+2$$

$$=\frac{x+2a-x+2a}{x-2a}+\frac{x+2b-x+2b}{x-2b}+2$$

$$=\frac{4a}{x-2a}+\frac{4b}{x-2b}+2=\frac{4ax-8ab+4bx-8ab}{(x-2a)(x-2b)}+2$$

$$=\frac{4x(a+b)-16ab}{(x-2a)(x-2b)}+2$$

$$=\frac{4\times 4ab-16ab}{(x-2a)(x-2b)}+2 \quad [\because\ x=\frac{4ab}{a+b},\ \therefore\ x(a+b)=4ab]$$

$$=\frac{0}{(x-2a)(x-2b)}+2=2.$$

45. If $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$,

show that $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=0$.

The left-hand side $=\frac{a^2}{b+c}+a+\frac{b^2}{c+a}+b+\frac{c^2}{a+b}+c-(a+b+c)$

$$=\frac{a^2+ab+ac}{b+c}+\frac{b^2+bc+ab}{c+a}+\frac{c^2+ac+bc}{a+b}-(a+b+c)$$

$$=\frac{a(a+b+c)}{b+c}+\frac{b(a+b+c)}{c+a}+\frac{c(a+b+c)}{a+b}-(a+b+c)$$

$$=(a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)-(a+b+c)$$

$$=(a+b+c)-(a+b+c)\quad\left[\because\ \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1\right]=0.$$

46. If $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$,

prove that $\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=\frac{3xyz}{abc}$. [D.B. '24]

$\because\ x=a(b-c)$, $\therefore\ \frac{x}{a}=b-c$. Similarly, $\frac{y}{b}=c-a$ and $\frac{z}{c}=a-b$.

$$\therefore\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=b-c+c-a+a-b=0.$$

$$\therefore\ \left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=3.\frac{x}{a}.\frac{y}{b}.\frac{z}{c}=\frac{3xyz}{abc}.$$

47. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, show that

$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}=\frac{1}{(a+b+c)^3}$. [C.U. '41 ; D.B. '42]

$\because \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, $\therefore \frac{bc+ca+ab}{abc}=\frac{1}{a+b+c}$,

or, $(a+b+c)(bc+ca+ab)=abc$ (by cross multiplication)

or, $(a+b+c)(bc+ca+ab)-abc=0$,

$\therefore (a+b)(b+c)(c+a)=0$,

If the product of three quantities be 0, at least one of them must be 0.

Here suppose, $a+b=0$, $\therefore a=-b$, $\therefore a^3=-b^3$.

Now, $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{-b^3}+\frac{1}{b^3}+\frac{1}{c^3}=-\frac{1}{b^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{c^3}$.

Again, $\frac{1}{a^3+b^3+c^3}=\frac{1}{-b^3+b^3+c^3}=\frac{1}{c^3}$

and $\frac{1}{(a+b+c)^3}=\frac{1}{(-b+b+c)^3}=\frac{1}{(c)^3}=\frac{1}{c^3}$.

$\therefore \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}=\frac{1}{(a+b+c)^3}$ $\left[\because \text{ each}=\frac{1}{c^3}\right]$

48. If $\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}$, prove that either

$a+b+c+0$, or, $a=b=c$. [C.U. 1931]

$\therefore \frac{b+c}{a}=\frac{c+a}{b}$, $\therefore \frac{b+c}{a}+1=\frac{c+a}{b}+1$, or, $\frac{b+c+a}{a}=\frac{c+a+b}{b}$

or, $\frac{a+b+c}{a}-\frac{a+b+c}{b}=0$, or, $(a+b+c)\left(\frac{1}{a}-\frac{1}{b}\right)=0$.

$\therefore$ Either $a+b+c=0$, or, $\frac{1}{a}-\frac{1}{b}=0$, i.e., $\frac{1}{a}=\frac{1}{b}$, $\therefore a=b$.

Again, $\frac{c+a}{b}+1=\frac{a+b}{c}+1$, or, $\frac{a+b+c}{b}-\frac{a+b+c}{c}=0$,

or, $(a+b+c)\left(\frac{1}{b}-\frac{1}{c}\right)=0$, $\therefore$ Either $a+b+c=0$.

or, $\frac{1}{b}-\frac{1}{c}=0$, i.e., $b=c$. $\therefore$ Either $a+b+c=0$, or, $a=b=c$.

49. If $a+b+c=0$, show that

$$\frac{a^2+b^2+c^2}{a^3+b^3+c^3}+\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0. \quad \text{[M. U. 1877]}$$

$\because \; a+b+c=0, \qquad \therefore \; (a+b+c)^2=0,$

or, $a^2+b^2+c^2=-2(ab+bc+ca)$.

Again $\because \; a+b+c=0, \;\; \therefore \; a^3+b^3+c^3=3abc.$

The left side of the identity $=\dfrac{-2(ab+bc+ca)}{3abc}+\dfrac{2}{3}\left(\dfrac{bc+ca+ab}{abc}\right)$

$$=-\frac{2}{3}\left(\frac{ab+bc+ca}{abc}\right)+\frac{2}{3}\left(\frac{ab+bc+ca}{abc}\right)=0.$$

50. If $(a+b+c)^2=3(ab+bc+ca)$, prove that $\dfrac{a^2}{bc}+\dfrac{b^2}{ca}+\dfrac{c^2}{ab}=3.$

$\because \; (a+b+c)^2=3ab+3bc+3ca,$

$\therefore \; a^2+b^2+c^2+2ab+2bc+2ca=3ab+3bc+3ca,$

or, $a^2+b^2+c^2-ab-bc-ca=0,$

or, $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

[Multiplying both sides by $a+b+c$.]

or, $a^3+b^3+c^3-3abc=0$, or, $a^3+b^3+c^3=3abc,$

or, $\dfrac{a^3}{abc}+\dfrac{b^3}{abc}+\dfrac{c^3}{abc}=3$ [dividing by abc]

$$\therefore \; \frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3.$$

51. If $a^2=by+cz$, $b^2=cz+ax$, $c^2=ax+by$,

show that $\dfrac{x}{x+a}+\dfrac{y}{y+b}+\dfrac{z}{z+c}=1.$

$$\frac{x}{x+a}+\frac{y}{y+b}+\frac{z}{z+c}=\frac{ax}{ax+a^2}+\frac{by}{by+b^2}+\frac{cz}{cz+c^2}$$

[The numerator and denominator of the first term are multiplied by a, those of the second by b and those of the third by c.]

$$=\frac{ax}{ax+by+cz}+\frac{by}{by+cz+ax}+\frac{cz}{cz+ax+by}$$

[substituting the value of a^2, b^2, c^2]

$$=\frac{ax}{ax+by+cz}+\frac{by}{ax+by+cz}+\frac{cz}{ax+by+cz}=\frac{ax+by+cz}{ax+by+cz}=1.$$

52. If $abc=1$, prove that,

$$(a+b-c)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)+(a+b)(b-c)(a-c)=1. \qquad \text{[B. U. '21]}$$

The left side $=(a+b-c)\left(\dfrac{bc+ca-ab}{abc}\right)+(a+b)(b-c)(a-c)$

$=(a+b-c)(bc+ca-ab)+(a+b)(b-c)(a-c) \quad [\because abc=1]$
$=\{(a+b)-c\}\{c(a+b)-ab\}+(a+b)(b-c)(a-c)$
$=\{c(a+b)^2-c^2(a+b)-ab(a+b)\}+abc+(a+b)(b-c)(a-c)$
$=(a+b)(ac+bc-c^2-ab)+abc+(a+b)(b-c)(a-c)$
$=(a+b)(a-c)(c-b)+abc+(a+b)(b-c)(a-c)$
$=-(a+b)(a-c)(b-c)+(a+b)(b-c)(a-c)+abc=abc=1.$

53. If $a+b+c=6$ and $ab+ac+bc=11$, find the value of $bc(b+c)+ca(c+a)+ab(a+b)+3abc.$ [C. U. '39]

The given expression

$=bc(b+c)+abc+ca(c+a)+abc+ab(a+b)+abc$
$=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$
$=(a+b+c)(bc+ca+ab)=6\times 11=66.$

54. If $x+y+z=6$ and $xy+yz+zx=9$, prove that

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0. \qquad \text{[B. U. '12]}$$

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=\frac{(1-y)(1-z)+(1-x)(1-z)+(1-x)(1-y)}{(1-x)(1-y)(1-z)}$$

Now, the numerator $=1-y-z+yz+1-x-z+xz+1-x-y+xy$
$=3-2x-2y-2z+xy+yz+zx=3-2(x+y+z)+(xy+yz+xz)$
$=3-2\times 6+9=12-12=0.$

$$\therefore \quad \frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=\frac{0}{(1-x)(1-y)(1-z)}=0.$$

55. If $x^2=ab+bc+ca$,
show that $(x^2+a^2)(x^2+b^2)(x^2+c^2)$ is a perfect square.

The given expression

$=(ab+bc+ca+a^2)(ab+bc+ca+b^2)(ab+bc+ca+c^2)$, [substituting the value of x^2]
$=(a+b)(a+c)(b+c)(a+b)(a+c)(b+c)$
$=(a+b)^2(b+c)^2(a+c)^2=\{(a+b)(b+c)(c+a)\}^2$, which is a perfect square.

56. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$,

show that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

From the given condition we have

$$a^2x^2+b^2x^2+c^2x^2+a^2y^2+b^2y^2+c^2y^2+a^2z^2+b^2z^2+c^2z^2$$
$$=a^2x^2+b^2y^2+c^2z^2+2abxy+2bcyz+2acxz,$$

or, $b^2x^2+c^2x^2+a^2y^2+c^2y^2+a^2z^2+b^2z^2$
$$-2abxy-2bcyz-2acxz=0,$$

or, $(b^2x^2+a^2y^2-2abxy)+(c^2x^2+a^2z^2-2acxz)$
$$+(b^2z^2+c^2y^2-2bcyz)=0,$$

or, $(bx-ay)^2+(cx-az)^2+(bz-cy)^2=0.$

Now every term of the left side is a perfect square and therefore positive. If the sum of three positive quantities is 0, then each of them must be 0.

$\therefore$ $(bx-ay)^2=0$, or, $bx-ay=0$, or, $bx=ay$, $\therefore \frac{x}{a}=\frac{y}{b}$.

Again, $(cx-az)^2=0$, or, $cx-az=0$, or, $cx=az$, $\therefore$ $\frac{x}{a}=\frac{z}{c}$.

$$\therefore \quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

57. If $x(b-c)+y(c-a)+z(a-b)=0$,

prove that $\frac{b-c}{bz-cy}=\frac{c-a}{cx-az}=\frac{a-b}{ay-bx}$.

From the given condition, $x(b-c)+y(c-a)+z(a-b)=0$...(1)

and being an identity $a(b-c)+b(c-a)+c(a-b)=0$......(2)

From (1) and (2), by cross multiplication we have

$$\frac{b-c}{cy-bz}=\frac{c-a}{az-cx}=\frac{a-b}{bx-ay},$$

or, $$\frac{b-c}{-(bz-cy)}=\frac{c-a}{-(cx-az)}=\frac{a-b}{-(ay-bx)}$$

or, $$\frac{b-c}{bz-cy}=\frac{c-a}{cx-az}=\frac{a-b}{ay-bx}.$$

58. Prove that $\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2$

$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$. [A. U. '22]

The left-hand side$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$

$+2\left\{\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}\right\}$

$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+2\left\{\frac{c-a+a-b+b-c}{(a-b)(b-c)(c-a)}\right\}$

$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+2\times 0$

$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$.

59. If $a+b=2c$, show that $\frac{a}{a-c}+\frac{c}{b-c}=1$. [C. U. '46]

$\because \quad a+b=2c, \qquad \therefore \quad b-c=c-a.$

Now, $\frac{a}{a-c}+\frac{c}{b-c}=\frac{a}{a-c}+\frac{c}{c-a}=\frac{a}{a-c}+\frac{c}{-(a-c)}=\frac{a-c}{a-c}=1.$

60. If $ab+bc+ca=0$,

prove that $(a+b+c)^3=a^3+b^3+c^3-3abc.$ [M. U.]

$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

$=a^3+b^3+c^3+3\{(a+b+c)(ab+bc+ca)-abc\}$

$=a^3+b^3+c^3+3\{(a+b+c)\times 0-abc\}=a^3+b^3+c^3-3abc.$

61. If $x+y+z=1$, $xy+yz+zx=3$ and $xyz=5$,

find the value of $\frac{1}{x+yz}+\frac{1}{y+zx}+\frac{1}{z+xy}$.

The given exp.$=\frac{1}{x(x+y+z)+yz}+\frac{1}{y(x+y+z)+zx}$

$+\frac{1}{z(x+y+z)+xy}$ $\quad[\because x+y+z=1]$

$=\frac{1}{(x+y)(x+z)}+\frac{1}{(y+z)(y+x)}+\frac{1}{(z+y)(z+x)}$

$=\frac{2(x+y+z)}{(x+y)(y+z)(z+x)}=\frac{2\times 1}{(x+y+z)(xy+yz+zx)-xyz}$

$=\frac{2}{1.3-5}=\frac{2}{-2}=-1.$

62. If $a+b+c=0$, show that
$$a(a^2-b^2-c^2)+b(b^2-c^2-a^2)+c(c^2-b^2-a^2)=6abc.$$
[C. U. '49 Sup.]

$\because\ a+b+c=0,\ \therefore\ a+b=-c,$

$\therefore\ a^2+b^2+2ab=c^2,$ or, $2ab=c^2-a^2-b^2.$

Similarly, $2bc=a^2-b^2-c^2$, and $2ac=b^2-a^2-c^2$.

Now, the left side of the identity
$$=a(2bc)+b(2ac)+c(2ab)=2abc+2abc+2abc=6abc.$$

63. If $a+b+c=0$, show that $a^5+b^5+c^5$
$$=\tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3).$$ [C. U. '50]

$\because\ a+b+c=0,\ \therefore\ a^2+b^2+c^2+2(ab+bc+ca)=0,$

$\therefore\ a^2+b^2+c^2=-2(ab+bc+ca).$

Again, $\because\ a+b+c=0,\ \therefore\ a^3+b^3+c^3=3abc.$

$\therefore\ \tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3)=\tfrac{5}{6}\times-2(ab+bc+ca)\times3abc$
$$=-5abc(ab+bc+ca).$$

Now, $\because\ a+b+c=0,\ \therefore\ a+b=-c,\ \therefore\ (a+b)^5=-c^5,$

or, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5$

or, $a^5+b^5+c^5=-5a^4b-10a^3b^2-10a^2b^3-5ab^4$
$$=-5ab(a^3+2a^2b+2ab^2+b^3)$$
$$=-5ab\{(a+b)(a^2-ab+b^2)+2ab(a+b)\}$$
$$=-5ab(a+b)(a^2-ab+b^2+2ab)$$
$$=-5ab\times-c(a^2+b^2+ab)\qquad[\because\ a+b=-c]$$
$$=5abc\{(a+b)(a+b)-ab\}$$
$$=5abc\{(a+b)\times-c-ab\}=5abc(-ac-bc-ab)$$
$$=-5abc(ab+ac+bc).$$

$\therefore\ a^5+b^5+c^5=\tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3).$

Exercise 9

Show that :—

1. $(ax+by)^2+(bx-ay)^2=(a^2+b^2)(x^2+y^2).$
2. $27(a+b+c)^3-(2a+b)^3-(2b+c)^3-(2c+a)^3$
$$=3(2a+3b+c)(2b+3c+a)(2c+3a+b).$$
3. $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2.$ [C. U. '26]
4. $-x^3-y^3-z^3+(x+y+z)^3=3(x+y)(y+z)(z+x).$
[C. U. '33]

5. If $a+b+c=0$, show that $(a+b)(b+c)(c+a)=-abc$.

6. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that $ax+by+cz=(a+b+c)(x+y+z)$. [C. U. '27]

7. Find the value of $a^3+b^3+c^3-3abc$, when $a=y-z$, $b=z-x$, $c=x-y$. [C. U. '23]

8. If $\left(x+\frac{1}{x}\right)^2=3$, show that $x^3+\frac{1}{x^3}=0$.

9. If $x+y=2z$, then

   (i) $\frac{x}{x-z}+\frac{y}{y-z}=2$; (ii) $\frac{x}{x-z}+\frac{z}{y-z}=1$ [W.B. S.F. '53]

10. If $x+\frac{1}{z}=1$ and $y+\frac{1}{x}=1$, then $z+\frac{1}{y}=1$.

11. If $xy+yz+zx=1$, prove that $(1+x^2)(1+y^2)(1+z^2)=(x+y)^2(y+z)^2(z+x)^2$.

12. If $2s=a+b+c$, show that $s(s-c)+(s-a)(s-b)=ab$.

13. If $a^2+b^2+c^2=ab+bc+ca$, then $a^3+b^3+c^3=3abc$.

14. If $a=y+z$, $b=z+x$, $c=x+y$, prove that $a^3+b^3+c^3-3abc=2(x^3+y^3+z^3-3xyz)$.

15. If $x^2+y^2=1=z^2+p^2$, then $(xz-yp)^2+(xp+yz)^2=1$.

16. If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$, prove that $a^3+b^3+c^3-3abc=(x^3+y^3+z^3-3xyz)^2$. [C.U. '44]

17. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, show that $\frac{1}{x^2-yz}+\frac{1}{y^2-zx}+\frac{1}{z^2-xy}=0$.

18. If $x^2+y^2+z^2=xy+yz+zx$, then $x^3+y^3+z^3=3xyz$.

19. If $a+b+c=0$, prove that $a^4+b^4+c^4=2(ab+bc+ca)^2$.

20. If $\frac{a}{b}+\frac{c}{d}=\frac{b}{a}+\frac{d}{c}$, then $\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}$. [C. U. ; M. U.]

21. (a) If $x+y=7$, $y+z=9$, $z+x=8$, evaluate $x^3+y^3+z^3$.

    (b) If $a+b+c=8$, $a^2+b^2+c^2=30$, find the value of $a^3+b^3+c^3-3abc$. [C. U. '50]

22. Given $a+2b+3c=0$, find the numerical value of $\frac{2c}{a+c}-\frac{a}{b+c}$. [D. B. '28]

23. If $a^2=b+c$, $b^2=c+a$, $c^2=a+b$,
show that $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=1$. [C. U. '42]

24. If $a+b+c=0$,
show that $\frac{ab}{a^2+ab+b^2}+\frac{bc}{b^2+bc+c^2}+\frac{ca}{c^2+ca+a^2}=-1$.

25. Show that $\frac{(a-b)^2}{(b-c)(c-a)}+\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}=3$.
[C. U. '30 ; D. B. '41]

26. If $a+b+c=0$, then $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}=1$.

27. If $a+b+c=1$,
show that $\frac{a+bc}{(a+b)(c+a)}+\frac{b+ca}{(b+c)(a+b)}+\frac{c+ab}{(c+a)(b+c)}=3$.

28. If $x=\frac{2ab}{a+b}$, show that $\frac{x+a}{x-a}+\frac{x+b}{x-b}=2$. [C. U. '20]

29. If $2s=a+b+c$,
prove that $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.

30. If $a+b+c=6$, $a^2+b^2+c^2=14$ and $abc=6$,
find the value of $a^3+b^3+c^3$.

31. If a, b, c, d are all real and $(a+b)^2+(b+c)^2+(c+d)^2$
$=4(ab+bc+cd)$, show that $a=b=c=d$.

32. If $a+b+c=0$, show that $a^2-bc=-(ab+bc+ca)$.
[C. U. '46]

33. Prove that,
$$\frac{2}{b-c}+\frac{2}{c-a}+\frac{2}{a-b}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}=0.$$
[P. U. 1888]

34. If $ap=bq=cr$,
show that $\frac{p^2}{qr}+\frac{q^2}{pr}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}$. [P. U. '29]

35. If $2s=a+b+c$, show that $2(s-a)(s-b)(s-c)$
$+a(s-b)(s-c)+b(s-a)(s-c)+c(s-a)(s-b)=abc$. [P.U.]

36. Prove that $\{(b-c)^2+(c-a)^2+(a-b)^2\}^2$
$=2\{(b-c)^4+(c-a)^4+(a-b)^4\}$.

*37. $a^2-b^2=b^2-c^2=c^2-a^2$,

show that $\dfrac{ab-c^2}{a-b}+\dfrac{bc-a^2}{b-c}+\dfrac{ca-b^2}{c-a}=0.$ [B. U.]

[*Hints* : From the given condition we have

$$b^2=\frac{c^2+a^2}{2},\ c^2=\frac{a^2+b^2}{2},\ a^2=\frac{b^2+c^2}{2}.$$

$$\therefore\ \frac{ab-c^2}{a-b}=\frac{ab-\frac{a^2+b^2}{2}}{a-b}=\frac{\frac{2ab-a^2-b^2}{2}}{a-b}=\frac{\frac{-(a-b)^2}{2}}{a-b}=-\frac{a-b}{2};$$

thus $\dfrac{bc-a^2}{b-c}=-\dfrac{b-c}{2}$, and $\dfrac{ca-b^2}{c-a}=-\dfrac{c-a}{2}$]

38. If $\dfrac{1}{b+c}+\dfrac{1}{c+a}=\dfrac{2}{a+b}$, prove that $a^2+b^2=2c^2$. [C.U. '48]

[The right side $\dfrac{2}{a+b}=\dfrac{1}{a+b}+\dfrac{1}{a+b}$, now transpose.]

39. If $x+y+z=xyz$, prove that

$$(1+x)(1+y)(1+z)-(1-x)(1-y)(1-z)=4xyz.$$

[C. U. '48 Sup.]

40. Given $ax+by=m$, $bx-ay=n$ and $a^2+b^2=1$, show that $x^2+y^2=m^2+n^2$. [C. U. '39, Sup.]

[*Hints* : Squaring $(ax+by)=m$, $(bx-ay)=n$, add the results...]

41. If $a^2+b^2=1=c^2+d^2$, then $(ac-bd)^2+(ad+bc)^2=1$.

42. If $a=b+c$, show that $a^3-b^3-c^3=3abc$.

43. If $x=p^2+2pq-q^2$, $y=q^2+2pq-p^2$ and $z=p^2+q^2$, prove that $x^2+y^2=2z^2$. [Oxford]

[*Hints* . Here, $x+y=4pq$; $xy=\{2pq+(p^2-q^2)\}\{2pq-(p^2-q^2)\}$ $=6p^2q^2-p^4-q^4$. Now $x^2+y^2=(x+y)^2-2xy$......]

44. If $(a+b+c)(ab+bc+ca)=abc$,

then $(a+b+c)^3=a^3+b^3+c^3$.

[From the given condition $(a+b+c)(ab+bc+ca)-abc=0$

$\therefore\ (a+b)(b+c)(c+a)=0.$

Now, $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$
$=a^3+b^3+c^3.$]

45. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$,

show that $(b+c)x+(c+a)y+(a+b)z=a+b+c$.

[**Hints :** Substituting the values of x, y and z

the left side $=\dfrac{b+c}{2bc}(b^2+c^2-a^2)+\dfrac{c+a}{2ca}(c^2+a^2-b^2)$

$+\dfrac{a+b}{2ab}(a^2+b^2-c^2)=$...... [See Ex. 37 (page 38)]

46. If $a+b+c=0$, prove that $a(b-c)^3+b(c-a)^3+c(a-b)^3=0$ [W. B. S. F. '52]

[**Hints :** $a\{b^3-c^3-3bc(b-c)\}+b\{c^3-a^3-3ca(c-a)\}$
$+c\{a^3-b^3-3ab(a-b)\}=-\{b^3(c-a)+c^3(a-b)+a^3(b-c)\}$
$-3abc(b-c+c-a+a-b)$......]

47. If $\dfrac{y+z-x}{b+c-a}=\dfrac{z+x-y}{c+a-b}=\dfrac{x+y-z}{a+b-c}$, show that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$. [G. U. '55]

47. (a) $\dfrac{1}{b^3(a-c)}+\dfrac{1}{a^3(b-c)}=\dfrac{1}{ab(a-c)(b-c)}$,

then either $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$, or, $a^2+b^2=ab$.

48. If $1+x+x^2=0$, show that $(a+bx+cx^2)+(ax+bx^2+c)$ $+(ax^2+b+cx)=0$. [G. U. '51]

49. If $x=by+cz$, $y=cz+ax$, $z=ax+by$, show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1. \qquad \text{[E. B. S. B. '55]}$$

[**Hints :** Multiply the numerator and the denominator of the first, the second and the third terms of the left side of the identity by x, y and z respectively.]

50. If $4(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$,
prove that $a=b=c=d$. [W. B. S. B. '52 Addl.]

[**Hints :** From the given condition
$4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2$,
or, $(a-b)^2+(a-c)^2+(b-c)^2+(b-d)^2+(a-d)^2+(c-d)^2=0$.
Each term of the left side is a perfect square and is therefore positive. The sum of the positive terms is 0, $\therefore$ each term is 0.
$\therefore$ $(a-b)^2=0$, $\therefore$ $a=b$; similarly $b=c$, $c=d$, $\therefore$ $a=b=c=d$.]

PROBLEMS ON EQUATIONS

[N. B. Many miscellaneous problems can be expressed symbolically in the form of an equation and easily solved. Represent the quantity required to be found out in the problem by x. If there be more than one unknown quantity, they should be represented by x, y, z etc. The following examples will serve as illustrations.]

Examples [10]

Ex. 1. The half of a certain integer exceeds the third of the next greater integer by 2. Find the number. [C. U. '17]

Let x be the number.

Now, from the condition of the problem we have

$$\frac{x}{2}-\frac{x+1}{3}=2, \quad \text{or,} \quad 3x-2x-2=12, \quad \text{or,} \quad x=12+2=14.$$

$\therefore$ the required number $=14$.

[N. B. Here it is stated that the half of a certain integer is greater than $\frac{1}{3}$ of the next greater integer by 2. The required number is represnted by x. Half of $x=\frac{x}{2}$. The integer next to x is $x+1$ and $\frac{1}{3}$ of it $=\frac{x+1}{3}$. It is given that $\frac{x}{2}$ is greater than $\frac{x+1}{3}$ by 2. $\therefore$ if $\frac{x+1}{3}$ is subtracted from $\frac{x}{2}$, the remainder is 2. So the equation is written as $\frac{x}{2}-\frac{x+1}{3}=2$. Now the value of x which is found on solving the equation is the required number. In every problem first form the equation in terms of the symbol x of the unknown quantity. On solving the equation you will get the answer. Two equations are to be formed, if there be two unknown quantities.]

2. The present age of a father is twice that of his son. Eight years hence their ages would be as 7 : 4. Find the son's present age. [C. U. 1932]

Let the present age of the son be x years.

$\therefore$ the present age of the father $=2x$ years.

Now, by the condition of the problem we get

$$\frac{2x+8}{x+8}=\frac{7}{4}, \quad \text{or,} \quad 8x+32=7x+56, \quad \text{or,} \quad x=56-32=24.$$

$\therefore$ the present age of the son $=24$ years.

3. A man pays Rs. 200 more than one-third of his debt and still owes Rs. 210 more than what he has paid. What was the original debt ? [C. U. 1913]

Let the original debt be x rupees.

He first pays Rs. 200 more than one-third of his debt. *i.e.* $\left(\frac{x}{3}+200\right)$ rupees. He still owes Rs. 210 more than what he has paid ; *i.e.*, he has still to pay $(\frac{1}{3}x+200+210)$ rupees. The total debt is equal to the sum of the debts paid and unpaid.

$\therefore$ by the question, we have $x=\frac{x}{3}+200+\frac{x}{3}+200+210$

or, $x-\frac{2x}{3}=610$, or, $\frac{x}{3}=610$, $\therefore$ $x=610\times 3=1830$.

$\therefore$ The original debt was Rs. 1830.

4. A monarch who came to the throne at the age of 30 reigned for $\frac{5}{11}$ of his life. How long did he reign ? [C. U. '30]

Suppose the monarch reigned for x years after he had come to the throne at the age of 30.

$\therefore$ he lived for $(30+x)$ years.

$\therefore$ by the question, $x=\frac{5}{11}(30+x)$, or, $11x=150+5x$,

or, $6x=150$, or, $x=25$. $\therefore$ he reigned for 25 years.

5. Twenty years ago a father was 4 times as old as his son and 4 years hence he will be twice as old as his son. What are their present ages ? [C. U. 1940]

Let the present ages of the father and the son be x and y yrs. From the two given conditions we have,

$$x-20=4(y-20)\ldots\ldots(1) \text{ and } x+4=2(y+4)\ldots\ldots(2)$$

From (1), $x-4y=-60\ldots\ldots(3)$ and from(2), $x-2y=4\ldots\ldots(4)$.

From (3) $-$ (4) we have $-2y=-64$, $\therefore$ $y=32$;

and from (4) we have $x=4+64=68$.

$\therefore$ the age of the father is 68 years and that of the son is 32 years.

6. A father's age is four times that of his elder son and five times that of his younger son; when the elder son has lived to three times his present age the father's age will exceed twice that of his younger son by 4 years. Find their present ages.

[W. B. S. F. '53]

Let x be the present age of the father. $\therefore$ from the given condition the ages of the elder and the younger sons are $\frac{x}{4}$ and $\frac{x}{5}$ years respectively. The age of the elder son will be $\frac{x}{4}\times 3$ yrs. after $\left(\frac{3x}{5}-\frac{x}{4}\right)$ yrs. or $\frac{x}{2}$ years. Then the ages of the father and the younger son will be $\left(x+\frac{x}{2}\right)$ and $\left(\frac{x}{5}+\frac{x}{2}\right)$ years respectively.

$\therefore$ From the given condition, $x+\frac{x}{2}=2\left(\frac{x}{5}+\frac{x}{2}\right)+4$,

or, $x+\frac{x}{2}=\frac{2x}{5}+x+4$, or, $\frac{x}{10}=4$, $\therefore$ $x=40$.

$\therefore$ the ages of the father, the elder son and the younger son are 40 yrs., 10 yrs. and 8 yrs. respectively.

7. Add 1 to the numerator and the denominator of a certain fraction and it reduces to $\frac{4}{5}$; subtract 5 from each and it reduces to $\frac{1}{2}$; required the fraction.

[C. U. 1916]

Let $\frac{x}{y}$ be the given fraction. From the given condition we have

$$\frac{x+1}{y+1}=\frac{4}{5}\ldots\ldots(1) \quad \text{and} \quad \frac{x-5}{y-5}=\frac{1}{2}\ldots\ldots(2)$$

From (1), we have $5x+5=4y+4$, or, $5x-4y=-1\ldots(3)$
and from (2), we have $2x-10=y-5$. or, $2x-y=5 \cdots(4)$
Now from (3)$\times$1 and (4)$\times$4 we have

$$\begin{aligned} 5x-4y&=-1\\ 8x-4y&=20 \end{aligned}$$

Subtracting $\quad -3x \quad =-21$, $\therefore$ $x=7$,

and from (4) we have $14-y=5$, or, $-y=-9$, $\therefore$ $y=9$,

$\therefore$ the reqd. fraction$=\frac{7}{9}$.

8. Find the fraction which reduces to $\frac{1}{2}$ when 1 is added to its denominator, and $\frac{1}{3}$ when 2 is subtracted from its numerator. [D. B. 1932]

Let $\frac{x}{y}$ be the given fraction. $\therefore$ from the given condition

we have $\frac{x}{y+1}=\frac{1}{2}$......(1) and $\frac{x-2}{y}=\frac{1}{3}$......(2).

From (1), $2x-y=1$......(3) and from (2), $3x-y=6$......(4).

Solving the equations (3) and (4) we have $x=5$, $y=9$.

$\therefore$ the required fraction$=\frac{5}{9}$.

9. The denominator of a fraction exceeds the numerator by 3, and if the numerator be increased by 7, the fraction is increased by unity. Find the fraction. [C. U. 1933]

Let x be the numerator, then the denominator$=x+3$,

i. e. the fraction$=\frac{x}{x+3}$. Now from the given condition we have $\frac{x+7}{x+3}=\frac{x}{x+3}+1$, or, $\frac{x+7}{x+3}=\frac{2x+3}{x+3}$, or, $2x+3=x+7$,

or, $x=4$, $\therefore$ the required fraction$=\frac{4}{7}$.

10. A market woman bought a certain number of eggs at two a penny and as many at 3 a penny and sold them at the rate of 5 for two pence losing $4d.$ by the bargain. What number of eggs did she buy ? [D.B. 1927]

Let x be the number of eggs bought of each kind.

Their total cost$=\left(\frac{x}{2}+\frac{x}{3}\right)d.$ and the total selling price

$=\left(\frac{2}{5}\times 2x\right)d.=\frac{4x}{5}d.$

$\therefore$ from the given condition we have $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+4$.

We get on solution $x=120$.

$\therefore$ the required number of eggs$=120\times 2=240$.

11. A says to B "I am twice as old as you were when I was as old as you are." The sum of their present ages in 63. Find their ages. [A. U. '31]

Let x be the present age of A and y be the present age of B in years. From the given conditions $x+y=63$...(1) and

$x=2\{y-(x-y)\}$*...(2).

From (2) we have $x=4y-2x$. or, $3x-4y=0$...(3)

Solving (1) and (3) we have $y=27$, $\therefore$ $x=63-27=36$.

$\therefore$ the age of A$=36$ yrs. and the age of B$=27$ yrs.

*[B is younger than A by $(x-y)$ years, $\therefore$ when the age of A was y years, the age of B was $y-(x-y)$ years.]

12. Nine chairs and 5 tables cost Rs. 90, while 5 chairs and 4 tables cost Rs. 61. Find the price of 6 chairs and 3 tables. [P. U. '30]

Let x rupees be the price of one chair and y rupees be the price of one table. From the given conditions we have, $9x+5y=90$...(1) and $5x+4y=61$......(2). Solving the equations $x=5$ and $y=9$.
$\therefore$ the price of 6 chairs and 3 tables$=$Rs.$5\times6+$Rs. $9\times3=$Rs.57.

13. The area of a floor is 192 sq. ft. Had each of the sides been 2 ft. longer, the area would have been increased by 60 sq. ft. Find the sides of the floor. [C. U. 1924]

Let x feet be the length and y feet be the breadth.

From the given conditions we have $xy=192$......(1)

and $(x+2)\times(y+2)=192+60$...(2). From(2), $xy+2x+2y=248$,

or, $192+2(x+y)=248$, or, $2(x+y)=56$, or, $x+y=28$...(3),

Now, $(x-y)^2=(x+y)^2-4xy=28^2-4\times192=16$. $\therefore x-y=4$...(4)

Solving (3) and (4) we have $x=16$, $y=12$.

$\therefore$ the length$=16$ ft., and the breadth$=12$ ft.

14. The perimeter of a rectangular courtyard is 60 ft. If the length is increased by 3 ft. and the width be decreased by 3 ft., the area is decreased by 21 sq. ft. Find the dimensions of the courtyard. [C. U. 1927]

Let x feet be the length and y feet be the breadth. From the first condition, $2(x+y)=60$, or, $x+y=30$...(1). From the second

condition, $(x+3)(y-3)=xy-21$, or, $xy-3x+3y-9=xy-21$, or, $x-y=4$...(2).

Now, solving (1) and (2) we have $x=17$, $y=13$.

$\therefore$ the required length $=17$ ft. and the breadth $=13$ ft.

15. A certain number consisting of two digits is equal to eight times the sum of its digits ; if 45 be subtracted from the number, the digits interchange their places. Find the number. [C. U. 1919]

Let x be the digit in the units' place and y be the digit in the tens' place. Then the number $=10y+x$.

From the given conditions we have, $10y+x=8(x+y)$...(1)
and $10y+x-45=10x+y$...(2). From (1) we have
$2y-7x=0$...(3) and from (2), $9y-9x=45$, or, $y-x=5$...(4).
Solving (3) and (4) we have $x=2$ and $y=7$.
$\therefore$ the required number $=10\times 7+2=72$.

[**N. B.** x and y denote the digits only. But the number $=10y+x$. If there is z in the hundreds' place, the number will be $100z+10y+x$.]

16. The sum of the digits of a number less than 100 is 6 ; if the digits be reversed the resulting number will be less by 18 than the original number. Find the number. [C. U. '25]

The number being less than 100, it is of two digits.

Let x be the digit in the units' place and y be the digit in the tens' place. $\therefore$ the number $=10y+x$.

From the given conditions $=x+y=6$...(1)
and $10x+y=10y+x-18$...(2).
From (2) we have $9x-9y=-18$, or, $x-y=-2$...(3)
Solving (1) and (3) we have $x=2$, and $y=4$.
$\therefore$ the required number $=4\times 10+2=42$.

17. A number consists of two digits ; the sum of the digits is 11, and if the left digit be increased by 2, it (the digit) will be equal to $\frac{1}{8}$th of the number. Find the number. [C. U. '36]

Let the number be $10y+x$. From the given conditions $x+y=11$...(1) and $y+2=\frac{1}{8}(10y+x)$...(2).

From (2) we have $8y+16=10y+x$, or, $2y+x=16\cdots(3)$

Solving (1) and (3) we have $x=6$, $y=5$.

$\therefore$ the reqd. number$=5\times10+6=56$.

18. The sum of a number consisting of two digits and the number formed by interchanging the digits is 110 and the difference between the digits is 6. Find the number. [A.U. '28]

Let the number be $10y+x$.

$$\therefore\quad (10y+x)+(10x+y)=110\ldots(1) \text{ and } \left.\begin{array}{l} x-y=6 \\ \text{or } y-x=6\ldots(2). \end{array}\right\}$$

From (1) we get $11x+11y=110$, $\therefore$ $x+y=10\cdots(3)$.

Now, solving (2) and (3) we have $x=8$, $y=2$, or $x=2$, $y=8$.

$\therefore$ the number is 28 or 82.

If the digit in the units' place be greater, the number$=28$. If the digit in the tens' place be greater, the number$=32$.

19. One of the digits of a number is greater by 5 than the other. When the digits are inverted the number becomes $\frac{3}{8}$th of the original number. Find the number. [D. B. '28]

Let x be the digit in the units' place and y be the digit in the tens' place, $\therefore$ the number$=10y+x$.

[N. B. Here if the two digits are inverted the new number is less than the original number, $\therefore$ it is evident that the digit in the units' place is less than the digit in the tens' place. Again, if the new number becomes greater than the original number when the two digits are inverted, then it is evident that the digit in the units' place is greater than that in the tens' place.]

From the condition we have $y-x=5\ldots(1)$

and $10x+y=\frac{3}{8}(10y+x)\ldots(2)$.

From (2) we have $80x+8y=30y+3x$, or, $77x-22y=0$,

or, $7x-2y=0\ldots(3)$. Solving (1) and (3) we have $x=2$, $y=7$.

$\therefore$ the reqd. number$=7\times10+2=72$.

20. There is a number of two digits whose difference is 2, and if it be diminished by $\frac{3}{2}$ times the sum of the digits, the digits will be inverted ; find it. [C.U. 1899]

Let x be the digit in the units' place and y be the digit in the tens' place. $\therefore$ the number$=10y+x$. Here if the two digits

are inverted, the new number is less than the original number, $\therefore$ the digit in the units' place is less than the other.

$\therefore \quad y-x=2\ldots(1)$, and $(10y+x)-\frac{3}{2}(x+y)=10x+y\ldots(2)$.

From (2), $20y+2x-3x-3y=20x+2y$ [multiplying by 2]

or, $15y-21x=0$, or, $5y-7x=0\ldots\ldots(3)$.

Solving (1) and (3) we have $x=5$, and $y=7$.

$\therefore$ the reqd. number$=7\times10+5=75$.

21. A number consists of three digits of which the middle one is 0 and the sum 8; the number formed by interchanging the digits is greater than the number itself by 198. Find the number. [C. U. 1922]

Let x be the digit in the units' place, 0 be the digit in the tens' place and y be the digit in the hundreds' place.

Then the number$=100y+x$.

From the given conditions we have $x+y=8\ldots\ldots(1)$

and $100x+y=100y+x+198\ldots\ldots(2)$

From (2) we have $99x-99y=198$; or, $x-y=2\ldots\ldots(3)$

Solving (1) and (3) we have $x=5$, $y=3$,

$\therefore$ the reqd. number$=3\times100+5=305$.

22. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two; and the number will be increased by 99 if the first and third digits be interchanged. Find the number. [C. U. 1923]

[N. B. Suppose 325 is a number. Its first digit is 3 (and not 5), because while writing 325 we first write 3, then 2 and then 5]

Suppose the digits in the hundreds', tens' and units' places are x, y and z respectively. $\therefore$ the number$=100z+10y+x$.

From the given conditions $x+y+z=10\ldots(1)$, $x+z=y\ldots(2)$,

and $100x+10y+z=100z+10y+x+99\ldots(3)$.

From (1) and (2) we have $x+(x+z)+z=10$, or, $2(x+z)=10$, or, $x+z=5\ldots(4)$. From (3), $99x-99z=99$, or, $x-z=1\ldots(5)$.

Solving (4) and (5) we get $x=3$, $z=2$, and $\therefore y=5$,

$\therefore$ the reqd. number$=2\times100+5\times10+3=253$.

23. A number consists of 3 digits, each greater by 2 than that which precedes it ; if 16 be subtracted from the number, the remainder will be less than 20 times the sum of the digits by 10. Find the number.

Let x be the digit in the hundreds' place. Then from the given condition the digit in the tens' place is $x+2$ and that in the units' place is $x+4$.

$\therefore$ the number $=100x+10(x+2)+(x+4)=111x+24$.

From the given second condition we get $111x+24-16$

$$=20(x+x+2+x+4)-10,$$

or, $111x+8=60x+110$, or, $51x=102$, $\therefore$ $x=2$.

$\therefore$ the required number $=111\times 2+24=246$.

24. A number consists of three consecutive digits. If the digits are reversed, the difference of the numbers is 33 times the greatest digit involved. Find the number. [C. U. 1939]

Let x be the digit in the hundreds' place, $x+1$ that in the tens' place and $x+2$ that in the units' place,

$\therefore$ the number $=100x+10(x+1)+(x+2)=111x+12$.

From the given second condition,

$$\{100(x+2)+10(x+1)+x\}-(111x+12)=33(x+2),$$

or, $111x+210-111x-12=33x+66$, or, $33x=132$, $\therefore$ $x=4$.

$\therefore$ the required number $=4\times 111+12=456$.

25. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours ; it also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the rate of the stream and of the boat.

[D. B. 1949]

Suppose that the boat goes x miles per hour if there be no current, and that the current flows at the rate of y miles per hour.

Then it is clear that *with the current* the boat goes $x+y$ miles per hour, and *against the current*, $x-y$ miles per hour.

Hence, the time taken to go 30 miles up stream $=\dfrac{30}{x-y}$ hours,

and the time taken to go 44 miles down stream $=\dfrac{44}{x+y}$ hours.

Now, from the given conditions of the problem we have

$$\frac{30}{x-y}+\frac{44}{x+y}=10\ldots(1) \text{ and } \frac{40}{x-y}+\frac{55}{x+y}=13\ldots(2).$$

From (1)×4 and (2)×3, $\frac{120}{x-y}+\frac{176}{x+y}=40$

and $\frac{120}{x-y}+\frac{165}{x+y}=39$

$\therefore$ (Subtracting) $\frac{11}{x+y}=1$, $\therefore$ $x+y=11\ldots(3)$.

Now substituting the value of $x+y$ in (1) we get $x-y=5\ldots(4)$.
Solving (3) and (4) we get $x=8$, $y=3$.

Thus, the rates of the stream and the boat are 3 miles and 8 miles per hour respectively.

26. A man rowing at the rate of 5 miles an hour in still water takes thrice as much time in going 40 miles up a river as in going 40 miles down; find the rate at which the river flows. [C.U. 1935]

Suppose the current flows at the rate of x miles per hour. $\therefore$ the boat goes $5-x$ miles per hour up-stream and $5+x$ miles per hour down stream. $\therefore$ the boat takes $\frac{40}{5-x}$ and $\frac{40}{5+x}$ hours to go 40 miles against and with the current respectively.

Now, from the given condition, $\frac{40}{5-x}=3\times\frac{40}{5+x}$, or, $\frac{1}{5-x}=\frac{3}{5+x}$

or, $5+x=15-3x$, or, $4x=10$, $\therefore$ $x=2\frac{1}{2}$.

$\therefore$ the river flows at the rate of $2\frac{1}{2}$ miles per hour.

27. A person rowed down a river, a distance of 70 miles in 10 hours with the stream and rowed back again in 70 hours. Find the rate of the flow of the river per hour. [C.U. 1941]

Suppose the rates of the stream and of the boat are x and y miles per hour respectively.

From the given conditions we have

$$10(x+y)=70\ldots(1) \text{ and } 70(y-x)=70\ldots(2).$$

From (1), $x+y=7\ldots(3)$ and from (2), $y-x=1\ldots(4)$.
Solving (3) and (4) we have $x=3$.
$\therefore$ the rate of the stream is 3 miles per hour.

Solution of problems relating to clocks

(1). The dial of the clock is divided into 60 equal parts, each part being called a 'minute-division' or 'minute space'.

(2) While the minute-hand makes a complete revolution round the dial, that is, moves through 60 minute-divisions, the hour-hand moves through only 5 minute-divisions. Therefore in a given time the hour-hand moves $\frac{5}{60}$ or $\frac{1}{12}$ of the space moved by the minute-hand.

(3). (*a*) When the hands are coincident, the number of minute-divisions between them is zero. (*b*) When the hands are opposite to each other, the number of minute-divisions between them is 30. (*c*) The hands are said to be in the same straight line (i) when they are either coincident or in the same straight line being opposite to each other in direction. (*d*) When the hands are at right angles, the minute-hand will be either ahead of or behind the hour-hand by 15 minute-divisions.

28. Find at what time between 7 and 8 o'clock the hands of a clock are coincident.

[C. U. '33]

Let the time be x minutes past 7, when the hands of the clock are coincident.

At 7 o'clock the minute-hand is 35 minute divisions behind the hour-hand. $\therefore$ the two hands to be coincident between 7 and 8, the minute-hand has to gain 35 divisions over the hour-hand.

Now, in x minutes the minute-hand moves x minute-divisions and the hour-hand moves $\frac{x}{12}$ minute-divisions.

$$\therefore \quad x-\frac{x}{12}=35, \quad \text{or,} \quad \frac{11x}{12}=35, \quad \text{or,} \quad x=\tfrac{12}{11}\times 35=\tfrac{420}{11}=38\tfrac{2}{11}.$$

$\therefore$ The two hands will be coincident at $38\frac{2}{11}$ mins. past 7.

29. When between 4 and 5 o'clock will the hands of a clock be opposite to each other?

Suppose it is x minutes past 4 when the hands of the clock are opposite to each other. At 4 o'clock the minute hand is 20

minute divisions behind the hour-hand. $\therefore$ the minute hand has to gain 20 divisions over the hour-hand to coincide with the hour-hand and then again it has to move 30 divisions more than the hour-hand, *i.e.*, 50 divisions more in all so that the two hands may be coincident.

$$\therefore \quad x-\frac{x}{12}=50, \text{ or, } \frac{11x}{12}=50, \text{ or, } 11x=600, \therefore x=54\tfrac{6}{11}.$$

$\therefore$ the required time is $54\frac{6}{11}$ minutes past 4.

30. Find the time between 3 and 4 o'clock, when the hour and minute hands of a clock are in the same straight line.
[C. U. 1934]

Let the required time be x minutes past 3. They will be in the same straight line (1) if one hand coincides with the other and (2) if they are opposite to each other.

In the first case, $\because$ at 3 o'clock the minute-hand is 15 minute divisions behind the hour-hand,

$$\therefore \quad x-\frac{x}{12}=15, \text{ or, } \frac{11x}{12}=15, \quad \text{or, } \quad x=\tfrac{180}{11}=16\tfrac{4}{11}.$$

In the second case, $\because$ the minute-hand has to gain 15 minute divisions over the hour-hand to coincide with the hour-hand and then again it has to move 30 divisions more than the hour-hand to be opposite to each other,

$$\therefore \quad x-\frac{x}{12}=15+30, \quad \therefore \quad \frac{11x}{12}=45, \qquad \therefore \quad x=49\tfrac{1}{11}.$$

$\therefore$ the two hands are in the same straight line at $16\frac{4}{11}$ mins. past 3 and at $49\frac{1}{11}$ minutes past 3.

31. Find the time between 4 and 5 o'clock when the hands of a clock are at right angles. [C. U. 1935, 1945]

Let the required time be x minutes past 4. The two hands are at right angles when the distance between them is 15 minute-divisions. At 4 o'clock the minute-hand is 20 minute-divisions behind the hour-hand. $\therefore$ (1) if the minute-hand gains $(20-15)$ or 5 divisions over the hour-hand, the distance between them is 15 minute divisions. Again (2) the minute-hand has to gain 20 divisions over the hour-hand to coincide with the hour-hand and

then again it has to gain 15 divisions more *i.e.* (20+15) or 35 divisions in all to be 15 minute divisions apart from the hour-hand.

$$\therefore \quad x-\frac{x}{12}=5\ldots(1) \quad \text{and } x-\frac{x}{12}=35\ldots(2)$$

From (1) we have $11x=60$, $\quad \therefore \quad x=5\frac{5}{11}$.
From (2) we have $11x=420$, $\quad \therefore \quad x=38\frac{2}{11}$.

$\therefore$ At $5\frac{5}{11}$ mins. past 4 and $38\frac{2}{11}$ minutes past 4 the two hands will be at right angles to each other.

32. A clock is 12 minutes too fast at noon; it loses $2\frac{1}{2}$ minutes an hour. What is the true time when the hands are at right angles between 3 and 4 o'clock ? [C. U. 1936]

Suppose it is x minutes past 3 when the hands are at right angles.

$$\therefore \quad x-\frac{x}{12}=15+15, \quad \text{or, } \frac{11x}{12}=30, \quad \therefore \quad x=32\frac{8}{11}.$$

$\therefore$ the hands are at right angles at $32\frac{8}{11}$ minutes past 3 by that clock. But the clock was 12 mins. too fast at 12 o'clock.

$\therefore$ this clock has run (3 hrs. $32\frac{8}{11}$ min. − 12 min.)
or, 3 hrs. $20\frac{8}{11}$ min., or, $\frac{2208}{11}$ minutes.

The clock loses $2\frac{1}{2}$ minutes an hour.

$\therefore$ while that clock shows $(60-2\frac{1}{2})$ or, $57\frac{1}{2}$ minutes, the true clock will show 60 minutes,

$\therefore$ if that clock show $\frac{2208}{11}$ minutes, the true clock will show $\frac{60\times2}{115}\times\frac{2208}{11}$ min., or, $\frac{2304}{11}$ min., or, 3 hrs. $29\frac{5}{11}$ minutes.

$\therefore$ the required time is $29\frac{5}{11}$ minutes past 3.

33. When will the hands of a clock be 10 minutes apart between 5 and 6 o'clock ?

Hints : Suppose it is x minutes past 5 when the hands of the clock are 10 minutes apart.

$$\therefore \quad x-\frac{x}{12}=(25-10)\ldots(1), \quad \text{or,} \quad x-\frac{x}{12}=(25+10)\ldots(2)$$

From (1) we have $x=16\frac{4}{11}$. From (2) we have $x=38\frac{2}{11}$.

$\therefore$ the required time is $16\frac{4}{11}$ mins., or, $38\frac{2}{11}$ mins. past 5.

34. A man who went out between 4 and 5 P.M. and returned between 5 and 6 P.M. found that the hands of his watch had exactly changed places. When did he go out? [C.U. '30, Addl. '51]

Suppose it was x minutes past 4 when the man went out and at that time the hour-hand was at the point B between 4 and 5 o'clock marks and the minute-hand was at the point C between 5 and 6 o'clock marks.

Now at x minutes past 4 the minute-hand was at a distance of x minute-divisions from the 12 o'clock mark and the hour-hand was at a distance of $\frac{1}{12}x$ minute-divisions form the 4 o'clock mark. Now see the figure :—

The arc AC$=x$ minute-divisions.

The arc $AB=\left(20+\frac{x}{12}\right)$ minute-divisions.

The arc $CKA=(60-x)$ „ „

The arc $BC=x-\left(20+\frac{x}{12}\right)$ minute-divisions.

While the man was out, the minute-hand moved over the arc CKAB, *i.e.*, $\left(60-x+20+\frac{x}{12}\right)$ minute-divisions and the hour-hand moved over the arc BC, *i.e.*, $x-\left(20+\frac{x}{12}\right)$ minute-divisions.

The minute-hand moves 12 times the hour-hand,

$\therefore\;60-x+20+\frac{x}{12}=12\left\{x-\left(20+\frac{x}{12}\right)\right\}$, or, $80-\frac{11x}{12}=11x-240$

or, $\frac{143}{12}x=320$, $\therefore\;x=\frac{3840}{143}=26\frac{122}{143}$.

$\therefore$ the man went out at $26\frac{122}{143}$ minutes past 4.

43. A man who went out between 3 P.M. and 4 P.M. and returned between 4 P.M. and 5 P.M. found that the hands of the clock had exactly changed places. When did he go out? When did he return and how long did he stay outside? [cf. C.U. 1942]

[*The Second Method*] Suppose the man went out at x minutes past 3 and returned at y minutes past 4. The hour-hand goes $\frac{x}{12}$ and $\frac{y}{12}$ minute-divisions while the minute-hand goes x and

y minute divisions respectively. $\therefore$ When the man went out, the hour-hand was at a distance of $15+\frac{x}{12}$ minute spaces from the 12 o'clock mark and on his return at y minutes past 4, he found the minute-hand at that position. $\therefore$ $y=15+\frac{x}{12}$......(1).

Again, when he returned the hour-hand was at a distance of $20+\frac{y}{12}$ minute divisions from the 12 o'clock mark, where the minute-hand was when the man previously went out.

$$\therefore \quad x=20+\frac{y}{12}......(2).$$

On solving (1) and (2) we get $x=21\frac{57}{143}$ and $y=16\frac{112}{143}$.

$\therefore$ the man went out at $21\frac{57}{143}$ minutes past 3 and returned at $16\frac{112}{143}$ minutes past 4, and he stayed outside for $55\frac{5}{13}$ minutes *i.e.*, the interval between the above two times.

HOLLOW SQUARE

The number of men in a solid square having x men in the front row$=x^2$. The number of men in a hollow square three deep having x men in the front row$=x^2-(x-3\times 2)^2$. Thus the general rule is this: the number of men in a hollow square n deep having x men in the front row$=x^2-(x-2n)^2$. A hollow square 3 deep means that from whatever side you may enter it, there is vacant space within it behind every three men.

36. A general can arrange his regiment in a hollow square 3 deep; if he had 800 more men, he could have arranged them in a hollow square 4 deep with the same number of men in the front row; find the number of men in the regiment. [C.U. '42]

Let x be the number of men in the front row in both the formations.

$\therefore$ the number of men in the first square

$$=x^2-(x-3\times 2)^2=x^2-(x-6)^2...(1)$$

and that in the second square$=x^2-(x-4\times 2)^2=x^2-(x-8)^2...(2)$

But he had 800 men more in the second formation,

$\therefore$ $x^2-(x-6)^2=x^2-(x-8)^2-800$,

or, $x^2-x^2+12x-36=x^2-x^2+16x-64-800$,

or, $4x=828$, $\therefore$ $x=207$. $\therefore$ from (1) the reqd. total number of men $=207^2-(207-6)^2=207^2-201^2$

$$=(207+201)(207-201)=408\times6=2448.$$

37. A company of men is formed into a hollow square 10 deep. If the company be increased by 1600 men, the whole number may also be formed into a hollow square 10 deep so that the front in the latter formation shall contain twice the number of men contained in the front of the former. Find the original number of men. [C. U. 1905]

Let x be the number of men in the front row of the first formation. $\therefore$ its total number of men $=x^2-(x-20)^2$.

Then $2x$ is the number of men in the front row of the second formation.

$\therefore$ the total number of men in the second formation

$$=(2x)^2-(2x-20)^2.$$

Hence, $x^2-(x-20)^2=(2x)^2-(2x-20)^2-1600$,

or, $x^2-x^2+40x-400=4x^2-4x^2+80x-400-1600$,

or, $40x=1600$, $\therefore$ $x=40$.

$\therefore$ the required number of men $=40^2-(40-20)^2=1200$.

38. A general wishing to draw up his regiment in the form of a hollow square found that he had 50 men over when it was 4 deep, but that he wanted 50 men to complete it when it was 5 deep, the number of men in the front being the same in both cases. Find the number of men in the regiment. [C. U. 1903]

Let x be the number of men in the front row of the two arrangements.

$\therefore$ in the first square $x^2-(x-8)^2$ men were arranged and 50 men were over,

$\therefore$ the total number of men $=x^2-(x-8)^2+50$.

In the second square the total number of men $=x^2-(x-10)^2$ and it required 50 men more to complete it.

$\therefore$ $x^2-(x-10)^2=x^2-(x-8)^2+50+50$,

or, $x^2-x^2+20x-100=x^2-x^2+16x-64+100$,
or, $4x=136$, $\therefore$ $x=34$.
$\therefore$ the required total number of men $=34^2-(34-8)^2+50$
$=34^2-26^2+50=530$.

39. A regiment of soldiers was marching in regular column with 5 men more in depth than in front. On the enemy coming in sight the front was increased by 405 men, and by this movement the regiment was drawn up in 5 lines. Find the number of soldiers in the regiment.

[C. U. 1937]

Let x be the number of men in the front row of the first arrangement, $\therefore$ its depth or the number of rows $=x+5$.

$\therefore$ The total number of men $=x(x+5)$.

In the second arrangement the number of men in the front row $=x+405$ and depth $=5$. $\therefore$ the total number $=5(x+405)$.

$\therefore$ $x(x+5)=5(x+405)$. or. $x^2+5x=5x+2025$,
or. $x^2=2025$, $\therefore$ $x=\sqrt{2025}=45$.
$\therefore$ the reqd. number of men $=45(45+5)=2250$.

40. A man bought an equal number of two kinds of mangoes, one kind at 2*as.* each and the other at 1*a.* 3*p.* each. If he had spent his money equally in the two kinds, he would have had 9 mangoes more. How many of each kind did he buy? [C.U. '43]

Let x be the number of mangoes bought of each kind. The cost of x mangoes of the first kind $=2x$ annas and that of the second kind $=\frac{5}{4}x$ annas.

$\therefore$ the total cost $=\left(2x+\dfrac{5x}{4}\right)$ or $\dfrac{13x}{4}$ *as.* If he had spent his money equally, the cost of each kind of mangoes would be $\dfrac{13x}{4}\times\dfrac{1}{2}$ or, $\dfrac{13x}{8}$ *as.*

Then with this money the number of mangoes of the first kind would be $\dfrac{13x}{8\times 2}$ or, $\dfrac{13x}{16}$ and the number of the second kind $=\dfrac{13x}{8}\times\dfrac{4}{5}=\dfrac{13x}{10}$, $\therefore$ $\dfrac{13x}{16}+\dfrac{13x}{10}=2x+9$, or, $65x+104x=160x+720$,

$\therefore$ $x=80$, $\therefore$ he bought 80 mangoes of each kind.

41. A person bought an article and sold it at a profit of 6%. Had he bought it at 4 p. c. less and sold it at Rs. 2.6 as. more, his profit would have been 12%, For how much did he buy it ? [C. U. 1944]

[Vide Example 37 of Profit and Loss in arithmetic.]

Let the cost price of the article be x rupees. If it is sold at a profit of 6%, the selling price$=\frac{106}{100}x$ rupees.

Had he bought it at 4% less, the cost price would have been $\frac{96}{100}x$ rupees,

Again, if the article worth $\frac{96}{100}x$ rupees be sold at a profit of 12%, the selling price$=\frac{112}{100}\times\frac{96}{100}x$ rupees. This selling price is greater than the first selling price by Rs. 2. 6 as. or Rs. $\frac{19}{8}$.

$\therefore\ \frac{112\times96}{100\times100}x=\frac{106}{100}x+\frac{19}{8}$, or, $\frac{672x}{625}-\frac{53x}{50}=\frac{19}{8}$, or, $\frac{19x}{1250}=\frac{19}{8}$,

$\therefore\ x=\frac{19}{8}\times\frac{1250}{19}=156\frac{1}{4}$. $\therefore$ the cost price=Rs. 156. 4 as.

42. A man rode a certain distance at a uniform rate in $2\frac{1}{2}$ hours. If the distance had been a mile less and his rate per hour 2 miles more, he would have taken half an hour less. Find his rate. [D. B. 1930]

Suppose the man rode at the rate of x miles per hour. He rode the whole distance in $2\frac{1}{2}$ hours.

$$\therefore \text{ the distance}=x\times\frac{5}{2}\text{ miles}=\frac{5x}{2}\text{ miles.}$$

Now, from the given condition we have

$$\frac{\frac{5x}{2}-1}{x+2}=2\tfrac{1}{2}-\tfrac{1}{2}.\quad\text{or,}\quad\frac{5x-2}{2x+4}=2,\quad\therefore\quad x=10.$$

$\therefore$ the man rode at the rate of 10 miles per hour.

[The distance being 1 mile less it is $\frac{5x}{2}-1$ and the speed being 2 miles more it is $x+2$ and the time taken being $\frac{1}{2}$ hr. less it is $2\frac{1}{2}-\frac{1}{2}$].

43. The manager of a boarding house having already 50 boarders, finds that an addition of 10 more boarders increases

the gross monthly expenditure by Rs. 20, but diminishes the average cost per head by Re 1. What did the monthly expenses originally amount to ?

[D. B. '29]

Let x rupees be the original monthly expenses.

$\because$ the original number of boarders$=50$,

$\therefore$ the average expenses per head were $\frac{x}{50}$ rupees.

The increased number of boarders$=(50+10)$ or 60 and the total monthly expenses$=(x+20)$ rupees.

$\therefore$ the average of monthly expenses$=\frac{x+20}{60}$ rupees, it is less than the previous average by Re. 1. $\therefore \frac{x}{50}=\frac{x+20}{60}+1$, $\therefore x=400$.

$\therefore$ the required monthly expenses$=$Rs. 400.

44. Divide the number 77 into three parts such that the sum of the first and second multiplied by 3, the sum of the second and third diminished by 3, and the sum of the first and third increased by 3 may be all equal.

[A. U. 33]

Suppose the three parts are x, y, z respectively.

$\therefore$ from the given conditions $x+y+z=77$......(1)

and $3(x+y)=y+z-3=x+z+3$......(2).

Now, $3x+3y=y+z-3$, or, $3x+2y-z=-3$......(3)

Again, $3x+3y=x+z+3$, or, $2x+3y-z=3$.........(4)

$\therefore$ from (3)+(4), $5x+5y-2z=0$......(5)

and from (1)$\times$5, $5x+5y+5z=385$

(subtracting) $-7z=-385$, $\therefore$ $z=55$.

Now, from (1) we have $x+y=77-55=22$......(6)

From the last part of (2) we have $y-x=6$......(7)

$\therefore$ solving (6) and (7) we get $x=8$, $y=14$.

$\therefore$ the required parts$=8, 14, 55$.

45. Two mixtures contain wine and water in the ratio 2 : 3 and 5 : 4 respectively ; in what ratio must the two mixtures be mixed together so that the resulting mixture may contain equal quantities of wine and water ?

Suppose x quantity is taken from the first mixture and y quantity from the second mixture.

$\therefore$ the first mixture of x contains $\frac{2}{5}x$ of wine and $\frac{3}{5}x$ of water. The second mixture contains $\frac{5}{9}y$ of wine and $\frac{4}{9}y$ of water.

$\therefore$ In $(x+y)$ of the mixture the quantity of wine$=\frac{2}{5}x+\frac{5}{9}y$, and the quantity of water$=\frac{3}{5}x+\frac{4}{9}y$;

$\therefore \quad \frac{2}{5}x+\frac{5}{9}y=\frac{3}{5}x+\frac{4}{9}y$, or, $\frac{5}{9}y-\frac{4}{9}y=\frac{3}{5}x-\frac{2}{5}x$,

or, $\frac{y}{9}=\frac{x}{5}$, $\therefore \frac{x}{y}=\frac{5}{9}$, $\therefore$ the required ratio$=5:9$.

[N. B. Here in the last mixture if the ratio of wine and water$=3:4$ (say), then $\frac{x}{y}$ is to be ascertained from $\dfrac{\frac{2}{5}x+\frac{5}{9}y}{\frac{3}{5}x+\frac{4}{9}y}=\frac{3}{4}$.]

46. The boarders in a hostel found that the rooms were 8 too few for each to have one, but 10 rooms would remain empty if each room was occupied by two. Find the number of boarders and the rooms.

Let x be the number of boarders and $x-8$ be the number of rooms. If each room was occupied by two, $\frac{1}{2}x$ rooms were needed for x men.

$\therefore \quad x-8=\frac{1}{2}x+10$, or, $x-\frac{1}{2}x=10+8$, or, $\frac{1}{2}x=18$. $\therefore x=36$.

$\therefore$ the number of boarders$=36$,

and the number of rooms$=36-8=28$.

47. The hypotenuse of a right-angled triangle exceeds one side by six inches and the other side by a foot. Find the area of the triangle.

Suppose the hypotenuse$=x$ inches, $\therefore$ the other sides are $x-6$ and $x-12$ inches respectively.

$\therefore (x-6)^2+(x-12)^2=x^2$, or, $x^2-12x+36+x^2-24x+144=x^2$,

or, $x^2-36x+180=0$, or, $(x-30)(x-6)=0$, $\therefore x=30$, or 6.

$\therefore$ the hypotenuse$=30$ inches. Here 6 inches is not taken into consideration as it will make the length of other sides absurd. The other two sides$=24$ inches and 18 inches.

$\therefore$ the reqd. area$=\frac{1}{2}\times 24\times 18$ sq. inches$=216$ sq. inches.

48. At an election there were two candidates A and B. $\frac{5}{8}$ of the electors voted for A who was elected by a majority of

200 over B, while $\frac{1}{3}$ of the electors did not vote at all. How many electors were there altogether ? [A. U. 1934]

Let x be the number of voters. $\therefore$ A has got $\frac{2}{5}x$ votes and B $(\frac{2}{3}-\frac{2}{5})x$ votes or $\frac{4}{15}x$ votes ($\because$ $\frac{2}{3}$ of the voters only voted and A has got $\frac{3}{5}$ of the total votes).

$\therefore$ $\frac{2}{5}x-\frac{4}{15}x=200$, or, $\frac{2}{15}x=200$, $\therefore$ $x=\frac{200\times15}{2}=1500$.

$\therefore$ the total number of voters $=1500$.

49. Two boys start together to walk 33 miles. One walks 12 miles in the same time that the other takes to walk 11 miles and arrives at the end of the journey an hour beforehand. Find their rates of walking in miles per hour. [C. U. 1909]

Suppose the first boy walks x miles per hour.

$\because$ the first boy walks 12 miles, while the second walks 11 miles, $\therefore$ the second boy walks $\dfrac{11x}{12}$ miles per hour.

$\therefore$ To go 33 miles the first boy takes $\dfrac{33}{x}$ hrs. and the second boy $\dfrac{33}{\frac{11}{12}x}$ hrs. or $\dfrac{36}{x}$ hrs., $\therefore$ $\dfrac{36}{x}=\dfrac{33}{x}+1$, or, $\dfrac{3}{x}=1$. $\therefore$ $x=3$.

$\therefore$ the first boy walks 3 miles and the second boy $3\times\frac{11}{12}$ or $2\frac{3}{4}$ miles per hour.

Exercise 10

1. (a) $\frac{1}{3}$ of a post is under mud, $\frac{1}{6}$ of its length is under water and the part that stands above the water level is 2 ft. long. Find the entire length of the post. [D. B. '48]

(b) Ten years ago, a father was seven times as old as his son; two years hence twice his age will be equal to 5 times his son's; what are their present ages ? [C. U. 1920]

2. A man performed a journey of 7 miles in 1 hour 15 mins. He walked part of the way at 4 miles an hour and rode the rest of the way at 10 miles an hour. How far did he walk ? [C. U. 1921]

3. Divide 50 into two parts so that three times the greater may exceed 100 by as much as five times the less falls short of 80. [P. U. 1918]

4. The sum of two numbers is 61 and twice the first number exceeds two-thirds of the second by 10. Find the numbers. [D. B. 1942]

5. A man distributed Rs. 100 equally among his friends; if there had been five more friends, each would have received one rupee less. How many friends had he? [A. U. 1932]

6. A motorist does a journey of 80 miles in 6 hours. During the first part of the journey he travels at the rate of 10 miles and during the latter part at 18 miles an hour. How far does he travel at each rate? [C. U. '18, '29]

7. A number consists of two digits. The sum of the digits falls short of the number by 54; if the digits be reversed the number exceeds the old number by 27; find the number. [P.U.'35]

8. The product of two numbers is 18225 and the quotient when the larger number is divided by the smaller is 81. Find the numbers. [C. U. 1945]

9. I bought a certain number of articles at 7 for 6*d.*; if they had been 13 for 1*s.*, I should have spent 6*d.* more. How many did I buy? [D.B. 1939]

10. Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from denominator. [C. U. 1928]

11. A man's age is 4 times the sum of the ages of his three children and in 8 years it will be twice the sum of their ages. What is the man's age? [P.U. 1930]

12. A number consists of two digits. The digit in the tens' place is 3 times the digit in the units' place. If 54 is subtracted from the number, the digits are inverted. Find the number. [C. U. 1943]

13. A certain number between 10 and 100 is 8 times the sum of its digits, and if 45 be subtracted from it, the digits will be reversed. Find the number. [C. U. 1919]

14. Reverse the digits of a number and it will become $\frac{5}{6}$ths of what it was before; and also the difference between the two digits is 1. Find the number. [C. U. 1883, '49 Sup.]

15. At what time between 2 and 3 o'clock are the hands of a watch (1) coincident, (2) opposite to each other, (3) at right angles and (4) 12 minute-spaces apart ?

16. A man who went out for an evening walk between 5 and 6 returned between 6 and 7 and found that the hands of the clock had exactly changed places. When did he go out ? [C. U. 1944]

17. A watch is 10 minutes too fast at noon ; if it loses 2 minutes in 1 hour, find the true time when its hands are at right angles between 2 and 3 o'clock.

18. In a lake the tip of a lotus-bud is half a cubit above the water. Forced by the wind, it begins to move and just sinks at a distance of two cubits. Find the depth of the water.

[Lilavaty]

19. An officer can form his men into a hollow square 5 deep and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former ; find the number of men.

[G. U '48 ; C. U. 1887]

20. A horse was sold at a loss for Rs. 840 ; but if it had been sold for Rs. 1050, the gain would have been $\frac{3}{4}$ of the former loss. Find its real value.

[C. U. 1942, Addl.]

21. A boy spends his money in oranges. If he had received 4 more for his money, they would have averaged a half-penny each less ; if three less, a half-penny each more. How much did he spend ?

[P. U. 1921, Addl.]

22. A man rowing at the rate of 4 miles an hour takes thrice as much time in going 30 miles up a river as in going 30 miles down ; find the rate at which the river flows.

23. A mixture contains wine and water in the ratio of 5 : 3 and another in the ratio of 4 : 5. In what ratio must the two kinds of mixture be mixed to give a mixture of wine and water in the ratio of 31 : 29 ?

24. How much gold, at Rs. 20 a tola, must be mixed with 14 tolas of Rs. 15 a tola, so that the mixture may be worth Rs. 18 a tola ?

[P. U. 1921]

25. If a cyclist had gone 2 miles an hour faster, he would have taken 1 hour 40 minutes less to ride 100 miles. What time did he take ?

[P. U. 1925]

26. A number consists of two digits, the digit in the units' place being four times that in the tens' place. If the digits be inverted, the new number increased by 2 equals three times the old number. Find the number. [C. U. 1901]

27. Two men, 40 miles apart, walking in opposite directions, meet in $6\frac{2}{3}$ hours; but if one of them had doubled his pace, they would have met in $\frac{3}{4}$ths of the time. Find their respective speeds, [P. U. 1931]

28. A man rows 30 miles down a river in 6 hours and returns in 10 hours. Find the rate at which the man rows and also the rate at which the river flows. [P. U. 1933]

29. A number consists of two digits; the digit in the tens' place is twice the digit in the units' place; if 36 be subtracted from the number, the digits are inverted; find the number. [C. U. 1946]

30. The number of months in the age of a man on his birthday in the year 1875 was exactly half of the number denoting the year in which he was born. In what year was he born? [A. U.]

31. A boy buys a certain number of oranges at 3 for 2*d.*, and one-third of that at 2 for 1*d.*; at what price must he sell them to get 20% profit? If his profit be 5*s.* 4*d.*, find the number bought. [D. B. 1936]

32. Find the time between 1 and 2 o'clock when the hands of a clock are at right angles. [C. U. '48]

33. One customer buys 14lb. of tea and 10 lb. of coffee for £ 2. 3*s.*, and another buys 11 lb. of tea and 15 lb. of coffee for £ 2. 4*s.* 6*d.* Find the price of tea and coffee per lb. [D.B. '40]

34. A number consists of three digits each less by unity than that which follows it and if 3 be subtracted from the number, the remainder will be 20 times the sum of the digits. Find the number. [G. U. '48]

35. *P* and *Q* start at the same time from Howrah and Madhupur and proceed towards each other at the rate of 20 and 30 miles respectively. They meet when *Q* has proceeded 36 miles farther than *P*. Find the distance between Howrah and Madhupur. [C. U. '49]

36. If 40 men be arranged into a hollow square 2 deep, find the number in the front. [B. C. S. '50]

37. If the numerator of a certain fraction is doubled and its denominator increased by 1, its value becomes $\frac{1}{2}$; but if its denominator is doubled and its numerator increased by 1, its value becomes $\frac{1}{5}$. Find the fraction. [E. B. S. B. '55]

GRAPHS

N. B. (1) Let XOX' and YOY' be two given straight lines intersecting at the point O at right angles to each other (vide Fig. no. 1). Each of the straight lines is called an **axis** of co-ordinates, XOX' being the x-axis and YOY' being the y-axis. and the point O is called the **origin.**

(2) The distance along the straight line XOX' or the x-axis from the point O to the right is taken as positive and that to the left is taken as negative. And the distance along the straight line YOY' or y-axis from the point O above XOX' is taken as positive and that below XOX' is takin as negative. Every time when you are to draw a graph, you should draw the co-ordinate axes just as is shown in the diagram and not otherwise.

(3) The perpendicular distance of a point from the y-axis is called the **abscissa** of the point and this distance is taken along the x-axis. The perpendicular distance of a point from the x-axis is called the **ordinate** of the point and this distance is taken along the y-axis. The abscissa and the ordinate of a point are together called its **co-ordinates.**

(4) If the co-ordinates of a point be given, it can be plotted *i.e.*, its position can be found out in reference to the co-ordinate axes. To write the co-ordinates of a point first write the abscissa and then the ordinate within brackets with a comma (,) between them.

(5) Generally one small division or one side of the small square on the graph paper is taken to represent the unit of length. Each small division is generally equal to $\frac{1}{10}$th inch or ·1 inch. It may be equal to a centimetre or a millimetre as we like. Two or more small divisions may also be conveniently taken to represent the unit of length. Whenever you draw a graph, never forget to mention the unit of length chosen by you.

(6) The co-ordinates of the origin O are $(0, 0)$.

Examples [11]

1. Draw the graph of $6x-7y=42$. [C. U. 1941]

[N. B. Place y of the given equation on the left side and all other terms on the right. If there is negative sign ($-$) preceding y on the left, multiply both sides by -1. Then write $y=$the equivalent expression.

Now, corresponding to a chosen value of x, we get a corresponding value of y from the given equation. Those values will represent the co-ordinates of the points on the graph. It is preferable to have integral values of x and y. Thus tabulating the co-ordinates of at least three points, the points are plotted on the graph paper with a chosen unit. Then the straight line drawn through these points and produced both ways is obtained as the required graph.]

Here $6x-7y=42$, or, $-7y=42-6x$, or, $7y=6x-42$.

$\therefore\ y=\dfrac{6x-42}{7}$, whence

The value of x	0	7	-7
The value of y	-6	0	-12

In the graph paper two axes XOX' and YOY' intersect at right angles at the origin O. Now the points $(0, -6)$, $(7, 0)$ $(-7, -12)$ are plotted on the graph paper taking one small division on the graph paper as the unit of length. We have the straight line AB by joining those points. This line is the required graph of the equation. (See the st. line AB of the diagram No. 1)]

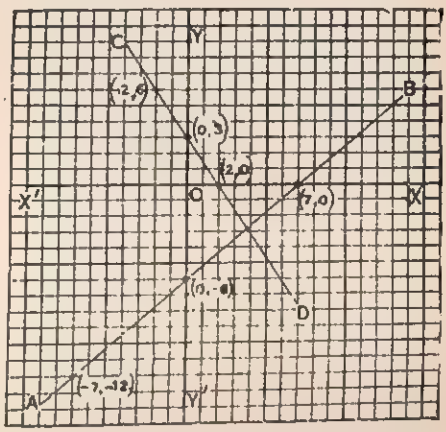


Fig. 1

[N. B. Every time you are to draw the graph of an equation you must show the two axes, tables of values of x and y and the unit of length chosen by you. The co-ordinates should be noted by the side of each point plotted. The graph should be drawn with a pencil very neatly.]

2. Draw the graph of $\frac{x}{2}+\frac{y}{3}=1$. [C. U. 1936]

Here, $\frac{x}{2}+\frac{y}{3}=1$, or, $3x+2y=6$ (multiplying both sides by 6)

or, $2y=6-3x$, $\therefore\ y=\dfrac{6-3x}{2}$, whence

value of x	0	2	-2
value of y	3	0	6

One side of the small square on the graph paper is taken as the unit of length. The straight line CD is the required graph. [See Fig. No. 1]

3. Draw the graphs of (*i*) $x=3$, (*ii*) $y=-4$, (*iii*) $x=0$, (*iv*) $y=0$.

Draw the above graphs taking one side of a small square on the graph paper as the unit of length.

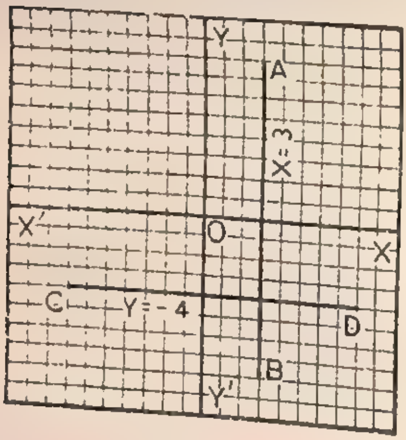


Fig. 2

(*i*) The equation is $x=3$ (constant), so it is evident that whatever be the values of y, the corresponding values of x will always be 3. ∴ the points (3, 0), (3, 4), (3, −2) etc. will be on the graph. On drawing the graph it is seen that the required graph is the straight line AB parallel to the y-axis at a distance of 3 units from the y-axis. (Vide Fig. No. 2)

[N. B. If $x=-3$, the required graph is the straight line parallel to the y-axis at a distance of 3 units from it in the negative direction of the x-axis.

(*ii*) $y=-4$. Here the points (1, −4), (2, −4), (5, −4) etc. will lie on the graph. The graph is the straight line CD parallel to x-axis at a distance of 4 units from it, but in the negative direction. [Vide CD of diagram No. 2]

(iii) The graph of the equation $x=0$ is the y-axis, *i.e.*, the straight line YOY'. Here, (0, 1), (0, 2), (0, −3), etc. are points on the graph.

(iv) The graph of the equation $y=0$ is the x-axis or the straight line XOX'. Here, (1, 0), (2, 0), (−4, 0) etc. are points on the graph.

4. Draw the graphs of $3x+4y=25$ and $4x-3y=0$ and measure the co-ordinates of their point of intersection. [C.U. '14]

$3x+4y=25\ldots$(i)

or, $4y=25-3x$,

or, $y=\dfrac{25-3x}{4}$, from which we have $\begin{array}{c|c|c|c} x & 3 & -1 & -5 \\ \hline y & 4 & 7 & 10 \end{array}$.

The straight line drawn through the points (3, 4), (−1, 7), (−5, 10) is the required graph of (i) [See Fig. No. 3]

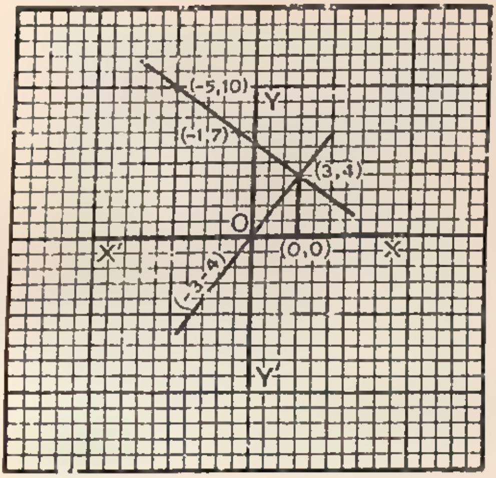


Fig. 3

$4x-3y=0\ldots$(ii) or, $-3y=-4x$, or, $3y=4x$,

$\therefore\quad y=\dfrac{4x}{3}$, whence $\begin{array}{c|c|c|c} x & 0 & 3 & -3 \\ \hline y & 0 & 4 & -4 \end{array}$ The straight line drawn through the points (0, 0), (3, 4), (−3, −4) is the required graph of (ii). [Fig. No. 3]

The co-ordinates of the point where the two graphs intersect are found to be (3, 4).

[N. B. The two graphs must be drawn with reference to the same co-ordinate axes, if the co-ordinates of their point of intersection are to be determined.]

Ex. 5. Draw the graphs of $3x-2y=6$ and $2x+3y=0$ and measure the angle of intersection. [C. U. 1932]

Here, $3x-2y=6\ldots\ldots$(i), or, $-2y=6-3x$, or, $2y=-6+3x$,

$\therefore\quad y=\dfrac{-6+3x}{2}$, whence $\begin{array}{c|c|c|c} x & 0 & 2 & 4 \\ \hline y & -3 & 0 & 3 \end{array}$.

Again, $2x+3y=0$......(ii), or, $3y=-2x$,

$\therefore \quad y=\frac{-2x}{3}$, whence

x	0	3	-3
y	0	-2	2

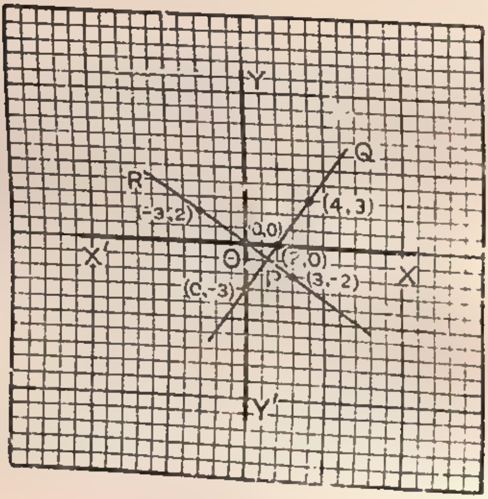


Fig. 4

One side of the small square is taken as the unit of length.

The two graphs have intersected each other at P. [Diagram No. 4]. Now we are to measure the angle at P between the two graphs.

(1) By measuring with a protractor we find each angle at P to be 90° or a right angle.

Otherwise : (2) From PR cut off PA equal to 3 units of length and from PQ cut off PB equal to 4 units. Now, by measuring we find that $AB=5$ units of length ;

$\therefore \quad AB^2=25$ and $PA^2+PB^2=3^2+4^2=25$.

$\therefore \quad AB^2=PA^2+PB^2$, $\therefore \ \angle P=1$ right angle.

[**N. B.** The two graphs must intersect at right angles, if the sum of the product of the coefficients of x of the two equations and that of the coefficients of y be 0. Here the two coefficients of x are 3 and 2 and their product is 6, and the two coefficients of y are -2 and 3 and their product is -6.

Now $6-6=0$, $\quad \therefore \quad \angle P=1$ right angle.]

6. Draw the graph of the expression $\frac{x+3}{2}$ and find its value from the graph when $x=3$ [D. B. '34] and also find the value of x for which the given expression is 0.

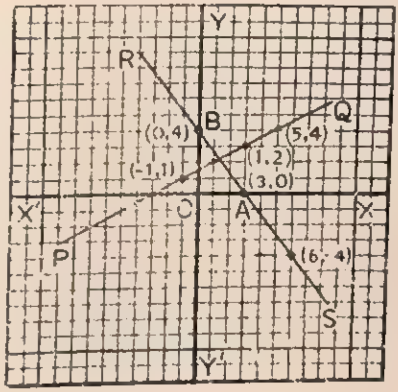


Fig. 5

The graph of $\frac{x+3}{2}$ is the same as that of the equation $y=\frac{x+3}{2}$. From the equation we have

x	1	-1	5
y	2	1	4

Now the graph PQ is drawn by taking one side of the small square on the graph paper as the unit of length. [See Fig. No. 5]

Now it is evident from the graph that if $x=3$, then $y=3$, and if $x=-3$, the given expression, *i.e.*, $y=0$.

7. Solve graphically $3x=17-2y$ and $3y=2x+6$. [A. U.'27]

$3x=17-2y \cdots$(i),

or, $2y=17-3x$,

$\therefore \quad y=\frac{17-3x}{2}$,

whence

x	-1	3	5
y	10	4	1

$3y=2x+6 \ldots$(2),

or, $y=\frac{2x+6}{3}$,

whence

x	0	-3	6
y	2	0	6

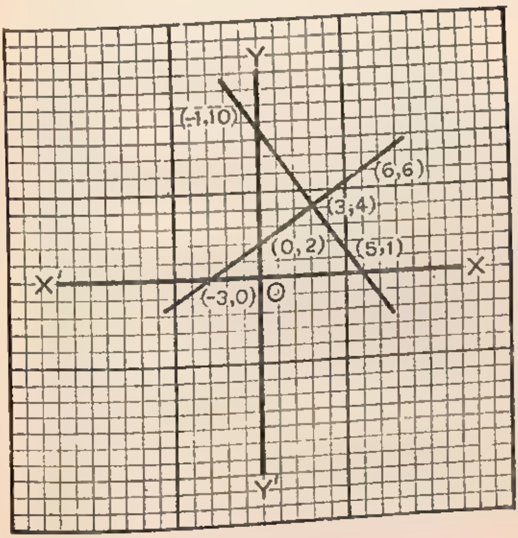


Fig. 6

The two graphs are drawn, taking one side of the small square on the graph paper as the unit of length. The co-ordinates of the point where they intersect are found to be (3, 4).

$\therefore$ the required solution is $x=3$, $y=4$.

8. Solve graphically $\frac{8-x}{2}=\frac{x-3}{3}$.

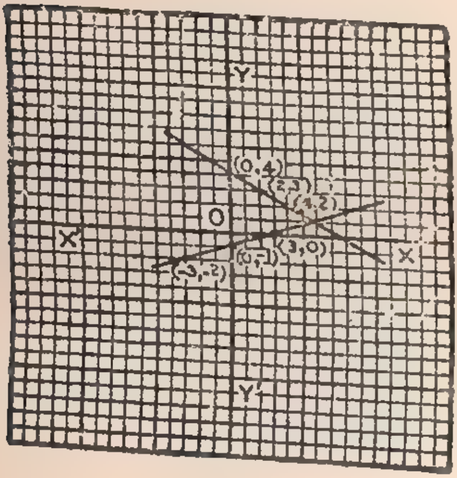


Fig. 7

Suppose $y=\frac{8-x}{2}$,

$\therefore \quad y=\frac{x-3}{3}$.

Now from $y=\frac{8-x}{2}$,

we have $\begin{array}{c|c|c|c} x & 0 & 2 & 4 \\ \hline y & 4 & 3 & 2 \end{array}$,

and from $y=\frac{x-3}{3}$,

we have $\begin{array}{c|c|c|c} x & 0 & 3 & -3 \\ \hline y & -1 & 0 & -2 \end{array}$,

Draw the graphs of the two equations taking one side of the small square on the graph paper as the unit of length. The co-ordinates of their point of intersection are found to be (6, 1) [See Fig. No. 7]

$\therefore$ the required solution is $x=6$.

9. Draw the graph of $\frac{x}{3}+\frac{y}{4}=1$ and measure the intercept between the two axes.

[D. B. '30, '33]

$\frac{x}{3}+\frac{y}{4}=1$, or, $4x+3y=12$, or, $3y=12-4x$,

$\therefore \quad y=\frac{12-4x}{3}$, whence $\begin{array}{c|c|c|c} x & 0 & 3 & 6 \\ \hline y & 4 & 0 & -4 \end{array}$.

The graph RS is drawn by taking one side of the small square on the graph paper as the unit of length. [See diagram No. 5] Let the graph cut the two axes at A and B. It is found by measurement that $AB=5$ units of length.

[Otherwise : $AB^2=AO^2+BO^2=3^2+4^2=25$,

$\therefore \quad AB=5$ units of length]

10. Draw the graph of $y-2x+4=0$ and find from it the solution of the equation $2x-4=0$. [D. B. 1929]

$y-2x+4=0$, $\therefore$ $y=2x-4 \ldots (A)$, whence $\begin{array}{c|c|c|c} x & 0 & -1 & 3 \\ \hline y & -4 & -6 & 2 \end{array}$.

The graph is drawn by taking one small division on the graph paper as the unit of length.

[Diagram No. 8]

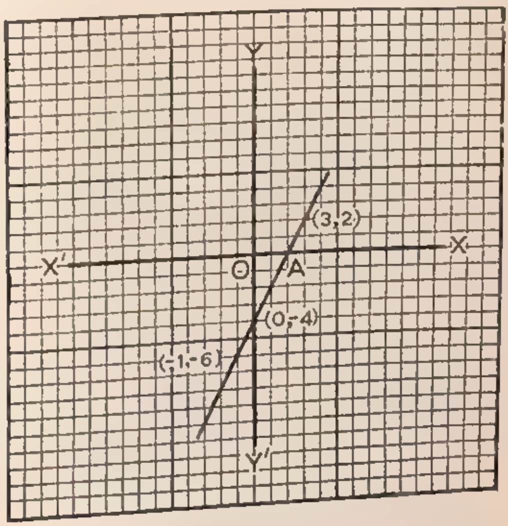


Fig. 8

Now, to solve the equation $2x-4=0$ from that graph, we have to find the abscissa of the point A where the graph intersects the x-axis, for there $y=0$ and therefore $0=2x-4$, [see the equation (A)]. Here, the abscissa of A is 2. Hence the required solution is $x=2$.

11. Find the area of the triangle formed by joining in order the points (3, 5), (– 2, 4), (5, 2) and (3, 5).

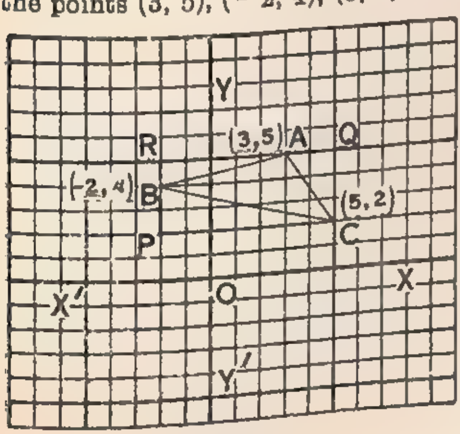


Fig. 9

Plot the points taking one side of the small square on the graph paper as the unit of length and join them in order so as to form the triangle ABC.

Now, we are to find the area of this triangle.

Draw QR and CP parallel to the x-axis through A and C respectively. And draw PR and QC parallel to the y-axis through B and C respectively. Then $PCQR$ is a rectangle.

Now $\triangle ABC$ = rectangle $PCQR - \triangle ABR - \triangle AQC - \triangle BPC$.

Of these, the rectangle $PCQR = PC \times PR = 7$ units $\times$ 3 units

$= 21$ sq. units ;

$\triangle ABR = \frac{1}{2} AR.BR = \frac{1}{2} \times 5 \times 1$ sq. units $= 2\frac{1}{2}$ sq. units ;

$\triangle AQC = \frac{1}{2} AQ \times QC = \frac{1}{2} \times 2 \times 3$ sq. units $= 3$ sq. units ;

$\triangle BPC = \frac{1}{2} BP.PC = \frac{1}{2} \times 2 \times 7$ sq. units $= 7$ sq. units ;

$\therefore \quad \triangle ABC = (21 - 2\frac{1}{2} - 3 - 7)$ sq. units $= 8\frac{1}{2}$ sq. units.

[**N. B.** If one side of the small square on the graph paper be $\frac{1}{10}$ inch, then the unit of length $= \frac{1}{10}$ inch ; $\therefore$ the square unit $= \frac{1}{10} \times \frac{1}{10}$ square inch $= \frac{1}{100}$ sq. inch. Then the answer may be given either as $8\frac{1}{2}$ sq. units or as ·085 sq. inch.]

12. Find the equation of the linear graph passing through the points (3, 2) and (– 2, 5).

The graph being linear, its equation must be of the first degree. Suppose the equation is $y = mx + c$. The straight line passes through the points (3, 2) and (– 2, 5), $\therefore$ the equation $y = mx + c$ is satisfied by their co-ordinates. For the point (3, 2), $x = 3$, $y = 2$,

$\therefore$ from $y = mx + c$ we have $2 = 3m + c$......(1)

Similarly, for the second point $5 = -2m + c$...(2)

Solving (1) and (2) we have $m = -\frac{3}{5}$, $c = \frac{19}{5}$.

Now putting the values of m and c we have the required equation $y = -\frac{3}{5}x + \frac{19}{5}$, or, $5y = -3x + 19$, or, $3x + 5y = 19$.

Ex. 1. If 1 lb of tea costs Re. 1. 8 as., find by means of a graph (*i*) the price of 5 lbs. of tea and (*ii*) the quantity of tea that can be had for Rs. 12.

Suppose x pounds of tea cost y rupees. Here since 1 lb. of tea costs Rs. $\frac{3}{2}$, $\therefore$ x lbs. of tea cost $\frac{3x}{2}$ rupees.

$\therefore$ We have $y=\frac{3x}{2}$ and this is the equation of the required graph.

Now, on the graph paper let a small division measured along OX represent one pound, and two such divisions measured along OY represent one rupee.

From the equation we have:

x	2	4	6	...
y	3	6	9	...

The graph OA of the equation $y=\frac{3}{2}x$ is drawn.

(*i*) Now if any point be taken on this graph, the abscissa of the point will represent the quantity of tea (in pound) the price of which will be represented by its ordinate (in rupees).

Now, on the graph find the point P (5, $7\frac{1}{2}$) whose abscissa (5 units) consequently represents the quantity of tea and whose ordinate ($7\frac{1}{2}$ units) represents the corresponding price of tea.

$\therefore$ 5 lbs. of tea cost Rs. $7\frac{1}{2}$.

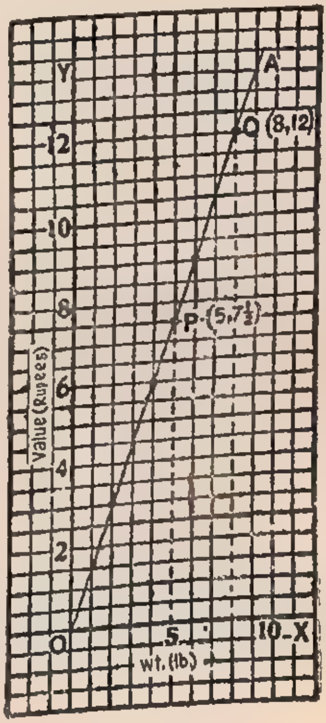


Fig. 10

(*ii*) Again, on the graph find the point Q (8,12) whose ordinate (12 units) represents the price of tea and whose abscissa (8 units) represents the corresponding quantity of tea.

$\therefore$ we can have 8 lbs. of tea for Rs. 12.

[N.B. Here the straight line OA is called the *price-graph* of tea.]

Ex. 2. A man walks at the rate of 3 miles an hour. Draw his motion-graph and find from it (*i*) how far he will walk in 2 hours 20 mins. and (*ii*) in what time he will walk 14 miles.

Suppose the man walks y miles in x hours. Here he walks at the rate of 3 miles an hour. $\therefore$ he walks $3x$ miles in x hours.

$\therefore$ y miles $=3x$ miles. $\therefore$ $y=3x$. The graph of this equation will be the *motion-graph* of the man.

Let three small divisions on the graph paper measured along OX represent one hour *i.e.*, let one division represent 20 minutes and let one small division measured along OY represent one mile.

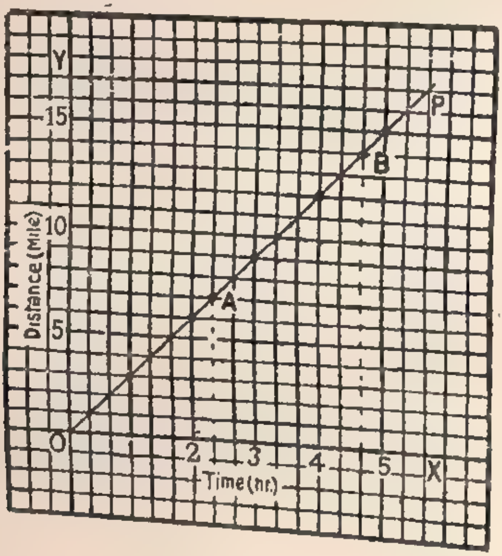


Fig. 11

Thus the graph OP of the equation $y=3x$ is drawn. The straight line OP is the required motion-graph. The abscissa and the ordinate of any point on this graph will represent the time and the distance travelled in that time.

(*i*) 2 hrs. 20 mins. $=\frac{7}{3}$ hrs. Here it is evident from the graph that the abscissa of the point $A=7$ small divisions $=\frac{7}{3}$ units, and its corresponding ordinate $=7$ small divisions $=7$ units.

$\therefore$ the man will walk 7 miles in $\frac{7}{3}$ hrs or in 2 hrs. 20 minutes.

Again (*ii*) we find that the abscissa of the pt. B, whose ordinate is 14 units, is $\frac{14}{3}$ units (14 small divisions).

$\therefore$ the man will take $\frac{14}{3}$ hrs. or 4 hrs. 40 mins. to walk 14 miles.

[N. B. The straight line OP is called the *motion-graph* of the man.]

Ex. 3. If the price of two mangoes be 5 annas, find grapically the price of 5 mangoes and the number of mangoes that can be had for Re. 1. 4 as.

Suppose x mangoes cost y annas.

Here the price of 2 mangoes$=5$ annas.

$\therefore$ the price of 1 mango$=\frac{5}{2}$ annas,

$\therefore$ the price of x mangoes$=\dfrac{5x}{2}$ annas.

$\therefore$ here $y=\dfrac{5x}{2}$ represents the equation of the price-graph of mangoes

Now, let the length of a sid of the small square measured along OX represent one mango and let an equal length measured along OY represent 1 anna. Now, draw the graph OP of the equation $y=\dfrac{5x}{2}$. It is evident from the graph that (*i*) the abscissa of the point A is 5 units and its ordinate$=12\frac{1}{2}$ units. $\therefore$ 5 mangoes cost $12\frac{1}{2}$ annas. (*ii*) Again, Re. 1. 4 as.$=20$ annas. The ordinate of the point B on the graph$=20$ units and its abscissa$=8$ units,

$\therefore$ we can have 8 mangoes for 20 annas or for Re. 1. 4 as.

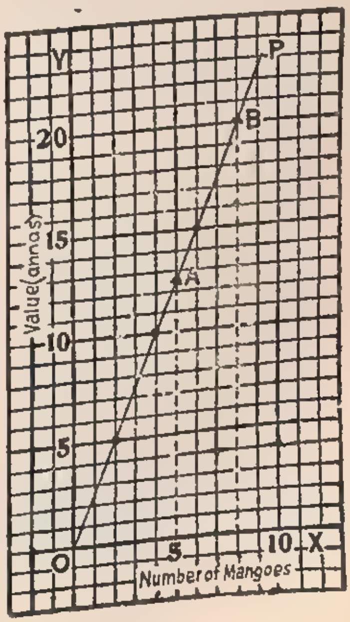


Fig. 12

Ex. 4. A starts walking at 8 A. M. at the rate of 3 miles an hour. After 2 hours B runs after him at the rate of 5 miles an hour. Find graphically when and where B will overtake A.

Let a small division on the graph paper measured along OX represent 1 hour and that along OY represent 1 mile. Suppose A was walking at the rate of 3 miles per hour starting from the

origin O. Let P be a point whose co-ordinates are (1, 3). Now the 1 unit of the abscissa of the point P represents 1 hour and 3 units of its ordinate represent 3 miles. $\therefore$ the point P is on the motion graph of A. $\therefore$ the straight line OP produced will be the motion-graph required.

Again, B has started 2 hours after A and therefore B has not walked at all for these two hours. Two hours have passed, but B's motion is 0 mile during that period. Plot the point Q whose co-ordinates are (2, 0); $\therefore$ OQ will be the motion-graph of B for those two hours (from 8 A.M. to 10 A.M.). Thereafter B has walked at the rate of 5 miles per hour. Now, taking Q as the new origin, plot a point R whose coordinates are (1, 5). Now, the straight line QR produced will be the motion-graph of B subsequent to those 2 hours.

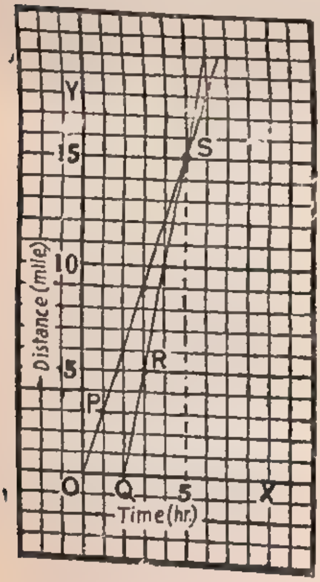


Fig. 13

Let the graphs OP and QR intersect each other at S. With reference to the origin O, we have (5, 15) as the co-ordinates of S. The abscissa 5 units represent 5 hours and the ordinate 15 units represent 15 miles. $\therefore$ B will overtake A, 5 hours after A has started, *i.e.*, at 1 P. M. at a distance of 15 miles from the starting point.

Ex. 5. A and B are at a distance of 14 miles and start at the same time cycling towards each other at the rates of 4 and 3 miles per hour respectively. Find by a graph when and where they will meet.

Draw the x-axis and the y-axis with the origin O on the graph paper. Suppose one side of the small square measured along OX represents 1 mile and an equal length measured along OY represents 1 hour.

Take a point P on the x-axis at a distance of 14 units from the origin O. Now, OP represents 14 miles.

Suppose A is cycling towards P from O at the rate of 4 miles per hour and B is cycling towards O from P at the rate of 3 miles per hour. $\therefore$ the motion-graph of A will be a straight line drawn through the point O and that of B will be a straight line through

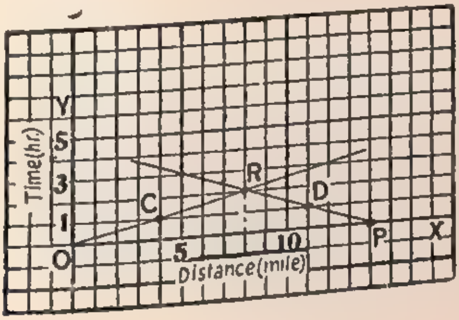


Fig. 14

the point P. Now, $\because$ A cycles 4 miles per hour, $\therefore$ his motion-graph will pass through the point $C(4, 1)$. $\therefore$ the straight line OC produced is the motion-graph of A.

Again, taking P for the origin let the abscissa and ordinate of the point D be 3 units and 1 unit respectively. $\therefore$ the straight line PD produced is the motion-graph of B. Let the two graphs intersect at R. Taking O for the origin we have (8, 2) as the co-ordinates of R and they represent 8 miles and 2 hours.

$\therefore$ A and B will meet at a distance of 8 miles from the starting point 2 hours after they have started.

Ex. 6. A mail train starts from Howrah at 9 P. M. and reaches Kharagpur, a distance of 72 miles, at 11 P. M. A passengor train starts from Kharagpur at 7 P. M. and reaches Howrah at 11 P. M. Find graphically when and where they meet. [W. B. S. F. 1955 (Addl.)]

Suppose M denotes the mail train and P the passenger train. The train-M goes $(72 \div 2)$ or 36 miles per hour and the train-P goes $(72 \div 4)$ or 18 miles per hour.

Draw the x-axis and the y-axis and the origin O on the graph paper. Let three small divisions on the graph paper measured

along OX represent 1 hour and one such division measured along OY represent 6 miles.

Suppose the point O represents the position of Howrah and the point K at a distance of 72 miles (*i.e.*, 12 small divisions) from O on the line OY represents the location of Kharagpur.

The two trains run at uniform rates, $\therefore$ their motion-graphs will be two straight lines. The train P has started at 7 P. M. from the point K. $\therefore$ the point K will be on its motion-graph. Again, the train has gone 18 miles per hour towards Howrah (*i.e.*, in opposite direction). Taking K as the origin plot a point A whose co-ordinates represnt 1 hour and 18 miles, *i.e.* whose abscissa$=3$ small divisions and ordinate$=3$ small divisions (downwards). The point A will lie on the motion-graph of P. Join KA. The straight line KA produced will be the motion-graph of P.

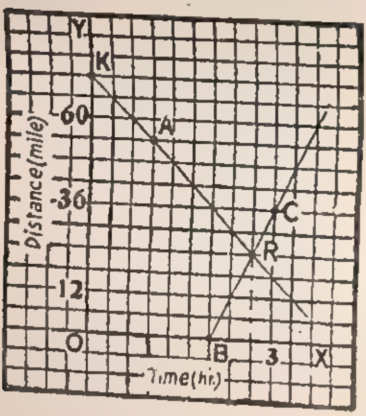


Fig. 15

Again, the train-M starts at 9 P.M. and therefore it has run no distance during the first 2 hours (from 7 P. M. to 9 P. M.). Take a point B on OX so that OB represents 2 hours (*i.e.* six small divisions). This point B will represent the starting point of the M-train and it will lie on the motion-graph of the M train. The M train runs 36 miles per hour. Taking B as the origin plot a point C whose co-ordinates represent 1 hour and 36 miles, *i.e.* whose abscissa$=3$ small divisions and ordinate$=6$ small divisions. The point C is on the motion-graph of the M train, $\therefore$ the straight line BC produced is the motion-graph of this train.

With reference to the origin O the point of intersection of the two graphs (R) has its abscissa$=8$ small divisions (representing $\frac{8}{3}$ hrs. or 2 hours 40 minutes) and ordinate$=4$ small divisions (representing 4×6 or 24 miles).

$\therefore$ the two trains will meet at a distance of 24 miles from Howrah 2 hrs. 40 minutes after 7 P.M., *i.e.*, at 9-40 P.M.

Ex. 7. P and Q are two places 29 miles apart. A starts from P and walks towards Q at $2\frac{1}{2}$ miles an hour. After 4 hours he takes rest for 2 hours and then resumes his journey at the rate of 2 miles an hour. B starts from Q 3 hours after A leaves P and walks towards P at the rate of 3 miles an hour. Find graphically when and where they will meet.

Let the two straight lines PQ and PR at right angles to each other on the graph paper represent the x-axis and the y-axis respectively. And let one side of the small square measured along PQ represent 1 mile and an equal length measured along PR represent 1 hour. Here $PQ=29$ times the side of the small square (29 miles).

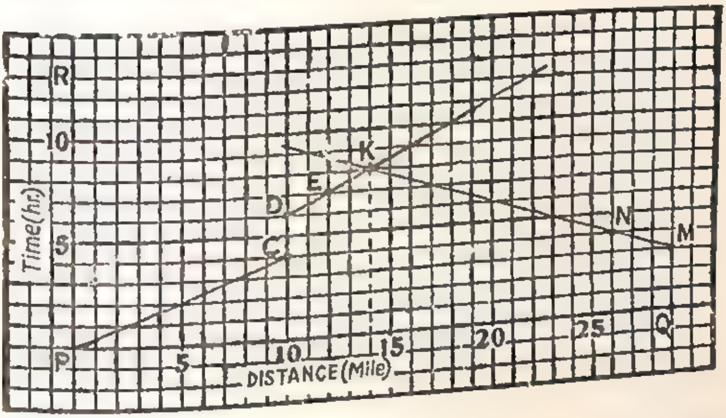


Fig. 16

(i) P has gone $(2\frac{1}{2}\times 4)$ or 10 miles in 4 hours starting from P at the rate of $2\frac{1}{2}$ miles per hour. Plot a point C whose co-ordinates are (10, 4), *i.e.*, whose abscissa 10 units represent 10 miles and ordinate 4 units represent 4 hours. Join PC. The straight line PC is the motion-graph of A for the first 4 hours. Then A takes rest for 2 hours and therefore he has gone no distance during this period. $\therefore$ the motion-graph of A for those 2 hours of rest will be such that the abscissa of any point on it will represent 10 miles. Now taking C as the origin plot a point D whose co-ordinates are (0, 2); $\therefore$ the straight line CD is the motion-graph of A for the 2 hrs. of rest.

After this A goes towards Q starting from D at the rate of 2 miles per hour. Now taking D as the origin plot a point E whose co-ordinates are (2, 1); then the straight line DE produced is the motion-graph of A after his period of rest.

(ii) Again, B has started from the point Q 3 hours later than A, $\therefore$ B has gone no distance during this period of 3 hours. Now taking Q as the origin, place a point M whose abscissa will represent 0 mile and ordinate 3 hours. The straight line QM is the motion-graph of B for his 3 hours of rest. Then B goes towards P from M at the rate of 3 miles per hour. Taking M as the origin plot the point N whose abscissa and ordinate represent 3 miles and 1 hour respectively. Then the straight line MN produced is the motion-graph of B after his period of rest.

Here let the graphs DE and MN intersect each other at K. With reference to P as the origin the abscissa and ordinate of K represent 14 miles and 8 hours respectively. $\therefore$ A and B will meet at a distance of 14 miles from P, 8 hours after A has started.

Ex. 8 A man's salary after 5 years of service is 25 rupees, after 15 years it is 45 rupees and after 20 years it is 55 rupees. If his salary was increasing at a uniform rate, show by graph his starting salary and the salary he will get after 18 years' service.

[Pat. U. 1946]

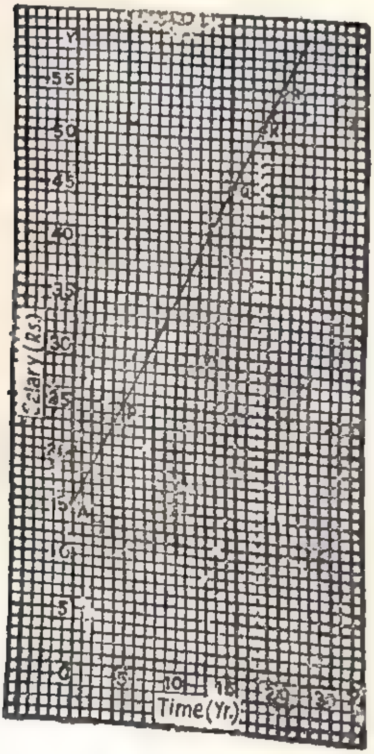


Fig. 17

Here the salary increases at a uniform rate, $\therefore$ its graph will be a straight line and the points representing 5 yrs. and Rs. 25, 15 yrs. and Rs. 45, 20 yrs. and Rs. 55 will be on that straight line.

Suppose one side of the small square measured along the x-axis represents 1 year and an equal length measured along the y-axis represents 1 rupee.

Now, plot the point P whose co-ordinates are (5, 25) *i.e.*, whose abscissa (5 times the side of the small square) represents 5 years and ordinate (25 times the side of a small square) represents Rs. 25.

Thus plot the points Q (15, 45) and R (20, 55). Join RQP and produce it so as to cut the y-axis (OY) at A. The straight line AR is the required graph. Now, it is evident from the graph that the abscissa of A$=0$ and the ordinate$=15$ times the side of the small square. $\therefore$ the starting salary of the man was Rs. 15.

Again, the salary he will get after 18 years' service will be represented by the ordinate of a point on the graph whose abscissa$=18$ units. From the graph it is found that the co-ordinates of the point K are (18, 51). $\therefore$ the salary of the man after 18 years' service will be Rs. 51.

Ex. 9. The population of a certain town is given by the following table:

Year	...	1905	1915	1925	1935	1945
Population in thousands	...	15	20	25	30	35

[Pat. U. '46]

Read the population of the town in 1920.

Here the population increases uniformly;

$\therefore$ the graph will be a straight line.

Let one side of the small square measured along the x-axis represent 5 years and an equal length measured along the y-axis represent 5 thousand population. Let the year 1905 be taken as the starting year. Now plot the point P (0, 3). The abscissa 0 unit of the point P represents the starting year 1905 and the ordinate 3 units represent 3×5 or 15 thousand population. Now plot the point Q (2, 4) so that its abscissa 2 units will represent 10 years and ordinate 4 units will represent 20 thousand population. Similarly plot the points R(4, 5), S(6, 6), T(8, 7). These represent the population of the years 1925, 1935, 1945 respectively. $\therefore$ the straight line PT is the required graph. Now, if we want to read the population of the town in 1920 from this graph,

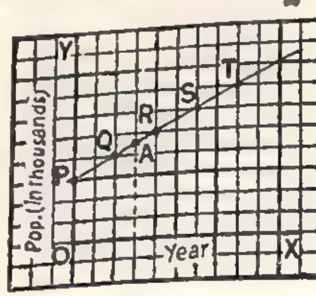


Fig. 18

we have to find the ordinate of the point whose abscissa $=3$ units $(1920-1905=15$ years $=3$ units).

Here the co-ordinates of the point A are $(3, 4\frac{1}{2})$;

$\therefore$ its ordinate $4\frac{1}{2}$ units represent $(5\times 4\frac{1}{2})$ thousand or 22500 population. $\therefore$ the population in $1920=22500$.

Exercise 11

Draw the graphs of :—

1. $y=4x+3$; 2. $3x-7y=0$; 3. $\frac{x}{4}+\frac{y}{5}=1$. [C. U. '12]

4. (i) $y=x-2$ [C. U. '13] ; (ii) $y=\frac{3-x}{4}$. [D. B. '32]

5. (i) $y=4x$; (ii) $\frac{x}{3}+\frac{y}{4}=2$. [C. U. '35]

6. (i) $2y-3x=6$; (ii) $x=5y$; (iii) $5x+3y=8$. [C. U. '40]

7. (i) $x=\frac{2y+6}{3}$; (ii) $\frac{x}{4}+\frac{y}{3}=1$; (iii) $6x-7y=12$ [C. U. '41] ; (iv) $\frac{x}{2}-\frac{y}{3}=1$. [C. U. '11]

8. $\frac{2x-3}{2}$. 9. (i) $x=-5$; (ii) $2x=7$; (iii) $y=-6$; (iv) $2y=5$; (v) $x=0$; (vi) $y=0$.

10. (i) $\frac{x}{5}+\frac{y}{6}=1$; (ii) $2x+3y=6$; (iii) $x=7(y+1)$. [C. U. '42]

11. (i) $y=2x$; (ii) $y=7$; (iii) $\frac{x}{5}+\frac{y}{7}=1$. [C. U. '44]

12. (i) $x-y=3$; (ii) $\frac{x}{2}+\frac{y}{3}=1$; (iii) $7x-3y=21$; (iv) $2x+3y=1$. [C. U. '45]

13. (i) $3x-4$; (ii) $x+y+3=0$; (iii) $3x=5y$. [C. U. '17]

14. Draw the graph of $2y-3x=7$. From the graph find, by measurement, the value of y when $x=2\frac{1}{3}$ and the value of x when $y=3\frac{1}{2}$. [W. B. S. F. '53]

[Hints : Draw the graph on plotting the points (1, 5), $(-1,2)$ and (3, 8) etc. Do not take the values $x=2\frac{1}{3}$ and $y=3\frac{1}{2}$ in the

table. From the graph the value of y is to be measured when $x=2\frac{1}{3}$ and the value of x when $y=3\frac{1}{3}$. So it is convenient here to draw the graph with six times the side of a small square as the unit of length. The value of y, when $x=2\frac{1}{3}$ and that of x, when $y=3\frac{1}{3}$, should be indicated by a small mark on the graph.

[Ans. : $x=0$, $y=7$]

15. Draw the graph of the expression $\frac{2x+7}{3}$ and read off its value when $x=4$. Find from the graph the value of x for which the given expression is 0. [D. B. '28]

Draw the graphs of the following and find the co-ordinates of their points of intersection :—

16. $x+y=2$ and $x-y=0$ [C. U. '18]
17. $3x+4y=25$ and $4x-3y=0$ [C. U. '14]
18. $y=2x$ and $3x-2y+2=0$ [C. U. '34]
19. $y=5$ and $5x+6y=30$ [C. U. '43]
20. $3x-5y=16$ and $2x-9y=5$ [P. U. '20]

Solve graphically the following equations :—

21. $4x+3y=15$ and $x-y=2$ [P. U. '24]
22. $2x+3y=13$ and $3x-2y=13$
23. $\frac{3x-4}{2}=3x-\frac{1}{3}$ [P. U. '25] 24. $\frac{2x+4}{6}=2x-1$ [P. U. '32]
25. $3x+2y=5$ and $5x-2y=3$ [D. B. '40]
26. $y-x=2$ and $8x-2y=5$

27. Draw the graph of $3y-2x=4$ and plot the points on the graph of which $x=-2$ and 4. [C. U. '16]

28. With the same axes of co-ordinates draw the graphs of (*i*) $3x-2y=0$, (*ii*) $y-3=0$, (*iii*) $2x-y=1$. [C. U. '33]

29. Draw the graph of $\frac{6x\times7}{2}$ and read off its value when $x=1{\cdot}5$, and find from the graph the value of x for which the given expression is $10{\cdot}5$.

30. Draw the graph of $3x-2y-4=0$ and find from the graph the value of y, when $x=2$. [D. B. '36]

31. Using the same axes and unit draw the graphs of (*i*) $y+x=5$, (*ii*) $x=2y-3$, (*iii*) $x=7$ and find the co-ordinates of the vertices of the triangle formed by them. [D. B. '27]

32. Draw the graphs of $2x-5y=0$ and $5x+2y=7$ and measure their angle of intersection.

33. Draw the graph of $\frac{x}{5}+\frac{y}{12}=1$ and measure its intercept between the two axes.

34. Find the area of the triangle formed by joining the points (8, 9), (2, 6) and (9, 2), 1 inch being the unit of length.

35. Find the area of the triangle formed by the graphs of (*i*) $x+y=0$, (*ii*) $3x=5y$ and (*iii*) $y=3x+12$.

36. A point moves so that twice its ordinate exceeds its abscissa by 3 ; show that its locus passes through the pt. (3, 3) and verify the result graphically.

[**Hints :** The locus of the point is a straight line which is the graph of the equation $2y-x=3$].

37. Join successively the points (2, 0), (4, 3), (2, 5), (0, 2), (2, 0), and calculate the area of the quadrilateral so formed. [C. U. '31, Addl.]

38. Find the perimeter of the triangle whose vertices are (5, 15), (10, 3), and (– 5, – 5).

39. Find the equation of the st. line which passes through each of the following pairs of points :—

(*i*) (2, 3), (0, 6) ; (*ii*) (0, 0), (2, – 4) ; (*iii*) (6, 8), (– 7, 5) ; (*iv*) (2, 3), (3, $2\frac{1}{2}$).

40. Draw the graphs of (*a*) $y-2x=7$, (*b*) $y=5$, (*c*) $x=6y$. [C. U. '46]

41. Draw the graphs of (*a*) $4x+3y=12$, (*b*) $y+x=0$, (*c*) $3y-5x=1$. [C. U. '47 Suppl.]

42. Draw the graphs of (*a*) $2y-x=7$, (*b*) $x=5$, (*c*) $4x=3y$. [C. U. '48]

43. Draw the graphs of $x+y=2$ and $x=y$. Measure their angle of intersection and determine the point where they meet. [W. B. S. F. '52]

44. A cyclist starts at 8 A. M. on a ride of 20 miles at 5 miles an hour. Draw a graph showing the relation between the distance travelled and the time taken to cover that distance.

45. If two oranges cost 3 annas, find graphically (1) the cost of 7 oranges and (2) how many oranges can be had for Re. 1.5a.

46. A man starts walking at 10 A.M. at the rate of 5 miles an hour. After 2 hours his son cycles after him at the rate of 7 miles an hour. Find graphically when and where the son will overtake the father. [A. U. '43]

47. A train P starts from Howrah and runs at 30 miles an hour. Another train Q starts 20 minutes after P and runs on a parallel line at 50 miles an hour. (1) When and where will Q overtake P? (2) How far will they be apart 12 minutes after Q starts?

48. A walks at the rate of 4 miles an hour and takes rest for 18 minutes at the end of every hour. Two hours later B runs in the same direction at the rate of 6 miles an hour. Find graphically when and where they will meet. [N. U. '47]

49. A train starts from Howrah for Magra (30 miles distant) at 8-30 A.M. and travels 40 miles an hour. Another train starts from Magra for Howrah at 8-45 A.M. and travels at 10 miles an hour. When and where do they meet?

50. Two friends X and Y leave their places A and B respectively to meet each other. X starts at 9 A. M. and travels at 12 miles per hour, while Y starts at 11 A. M. and travels at 30 m. p. h. They meet at 12-20 P. M. Draw graphs of their travels and read from them the distance between A and B. [N. U. '48]

51. Two pipes can fill a cistern in 4 and 6 hours respectively. How long will they take to fill it if they are opened together?

52. At noon A starts to cycle from P to Q a distance of 40 miles. He rides 6 miles an hour, resting for an hour after riding 12 miles. At 3 P. M. B starts from P at 10 miles an hour. Find graphically (a) when and where B overtakes A and (b) their distance apart at 5 P. M. [A. U. '45]

53. The salary of an officer increased each year by a fixed sum. After 5 years of service his salary is raised to Rs. 120 and after 12 years to Rs. 176. Draw a graph from which his salary may be read off for any year and determine from it (i) his initial salary and (ii) the salary he should receive for his 21st year. [D. B. '45]

54. The population of a town is given by the following table :—

Year	1920	1930	1940	1950
Population in thousands	12	17	22	27

Find graphically the population of the town in 1945.

QUADRATIC EQUATIONS

Definition. Any equation which contains the square of the unknown quantity, but no higher power, is called a **Quadratic equation or an equation of the second degree.** Thus $x^2=25$, $3x^2+5x=4$, $ax^2+bx+c=0$ etc. are quadratic equations.

Pure and adfected quadratic equation. If an equation contains only the second power of the unknown quantity (and not the first) it is called a **pure quadratic** ; if it contains the second as well as the first power of the unknown quantity it is called an **adfected quadratic.**

Thus $x^2=4$, $ax^2-5=0$ are pure quadratic equations and $3x^2-4x=15$ is an adfected quadratic equation.

Solution of a pure quadratic

In solving a pure quadratic we have to transpose the unknown quantity to the left and the known quantities to the right just in the same way as simple equations are solved. Thus we have the value of the square of the unknown quantity. Then the square root of the value so found will give the roots of the equation.

Examples [12]

Ex. 1. Solve : $3x^2-36=64-x^2$.

$3x^2-36=64-x^2$,

or, $3x^2+x^2=64+36$ [by transposition]

or, $4x^2=100$, or, $x^2=25$,

$\therefore\ x=\pm 5$. (*i.e.*, $x=5$ or -5)

Ex. 2. Solve : $2x-\frac{3}{x}=\frac{x}{2}$.

Multiplying both sides by $2x$, the L.C.M. of the denominators, we have

$4x^2-6=x^2$, or $3x^2=6$, or, $x^2=2$, $\therefore\ x=\pm\sqrt{2}$.

[Another Method] $2x-\frac{3}{x}=\frac{x}{2}$, or, $\frac{2x^2-3}{x}=\frac{x}{2}$

or, $4x^2-6=x^2$ [by cross multiplication]

[Now proceed as before]

Ex. 3. Solve : $\frac{1}{x+1}+\frac{2}{x+5}=\frac{1}{2}$. [C. U. '19]

$\frac{1}{x+1}+\frac{2}{x+5}=\frac{1}{2}$, or, $\frac{x+5+2x+2}{(x+1)(x+5)}=\frac{1}{2}$

or, $\frac{3x+7}{x^2+6x+5}=\frac{1}{2}$, or, $x^2+6x+5=6x+14$,

or, $x^2=14-5=9$, $\therefore$ $x=\pm 3$.

Ex. 4. Solve : $\frac{3x+4}{x+2}=\frac{x+5}{x+1}$.

By cross multiplication we have, $3x^2+7x+4=x^2+7x+10$,

or, $2x^2=6$, or, $x^2=3$, $\therefore$ $x=\pm\sqrt{3}$.

Ex. 5. Solve : $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$. [C.U. '12 ; D.B. '22]

$\frac{(x+4)^2+(x-4)^2}{(x-4)(x+4)}=\frac{10}{3}$, or, $\frac{2x^2+32}{x^2-16}=\frac{10}{3}$,

or, $10x^2-160=6x^2+96$, or, $4x^2=256$, or, $x^2=64$,

$\therefore$ $x=\pm 8$.

Solution of an adfected quadratic equation.

(1) By factorisation

Solve the following equations : [D. B. '27]

Ex. 6. $4x^2+25x-351=0$.

$4x^2+25x-351=0$,

or, $4x^2+52x-27x-351=0$,

or, $4x(x+13)-27(x+13)=0$, or, $(x+13)(4x-27)=0$.

Here, $\because$ the product of the two factors is 0, $\therefore$ one of them must be 0.

If $x+13=0$, then $x=-13$. If $4x-27=0$, then $4x=27$,

$\therefore$ $x=\frac{27}{4}=6\frac{3}{4}$. $\therefore$ $x=-13$ or $6\frac{3}{4}$.

Ex. 7. $(x-7)(x-19)=64$. [C. U. '18]

(Removing brackets) $x^2-26x+133=64$,

or, $x^2-26x+69=0$, or, $x^2-23x-3x+69=0$,

or, $x(x-23)-3(x-23)=0$, or, $(x-23)(x-3)=0$.

$\therefore$ either $x-23=0$, or, $x-3=0$, $\therefore$ $x=23$ or 3.

Ex 8. $\frac{x-2}{x+2}+\frac{6(x-2)}{x-6}=1.$ [C. U. '51]

Here, $\frac{6(x-2)}{x-6}=1-\frac{x-2}{x+2}$, or, $\frac{6(x-2)}{x-6}=\frac{x+2-x+2}{x+2}$

or, $\frac{6(x-2)}{x-6}=\frac{4}{x+2}$, or, $\frac{3(x-2)}{x-6}=\frac{2}{x+2}$

or, $3(x^2-4)=2(x-6)$, or, $3x^2-12=2x-12$,

or, $3x^2-2x-12+12=0$, or, $3x^2-2x=0$,

or, $x(3x-2)=0$, $\therefore$ either $x=0$ or $3x-2=0$.

$\therefore$ $x=0$ or $\frac{2}{3}$.

Ex. 9. $x+\frac{1}{x}=25\frac{1}{25}$. [C.U. '14, '39 Suppl., D.B. '25]

$x+\frac{1}{x}=25\frac{1}{25}$, or, $\frac{x^2+1}{x}=\frac{626}{25}$, or, $25x^2+25=626x$,

or, $25x^2-626x+25=0$, or, $25x^2-625x-x+25=0$,

or, $25x(x-25)-1(x-25)=0$, or, $(x-25)(25x-1)=0$,

$\therefore$ either $x-25=0$, or, $25x-1=0$, $\therefore$ $x=25$ or $\frac{1}{25}$.

Ex. 10. $(x-3)(x-4)=\frac{34}{33^2}$.

Suppose $a=33$; then we have, $(x-3)(x-4)=\frac{34}{a^2}$

or, $a^2(x-3)(x-4)=34$, or, $a^2(x-3)(x-4)=33+1$

or, $a(x-3).a(x-4)=a+1$

or, $(ax-3a)(ax-4a)-a-1=0$

or, $(ax-3a)(ax-4a)+(ax-4a)-(ax-3a)-1=0$

[$\because$ $(ax-4a)-(ax-3a)=-a$.]

or, $(ax-4a)(ax-3a+1)-1(ax-3a+1)=0$,

or, $(ax-3a+1)(ax-4a-1)=0$,

$\therefore$ either $ax-3a+1=0$, or, $ax-4a-1=0$,

$\therefore$ either $x=\frac{3a-1}{a}=3-\frac{1}{a}=3-\frac{1}{33}=2\frac{32}{33}$,

or, $x=\frac{4a+1}{a}=4+\frac{1}{a}=4+\frac{1}{33}=4\frac{1}{33}$. $\therefore$ $x=2\frac{32}{33}$ or $4\frac{1}{33}$.

Ex. 11. $\frac{1}{x}-\frac{1}{x+b}=\frac{1}{a}-\frac{1}{a+b}$. [C. U. '21]

Here, $\frac{x+b-x}{x(x+b)}=\frac{a+b-a}{a(a+b)}$, or, $\frac{b}{x^2+bx}=\frac{b}{a^2+ab}$,

or, $x^2+bx=a^2+ab$, or, $x^2+bx-a^2-ab=0$,

or, $(x+a)(x-a)+b(x-a)=0$, or, $(x-a)(x+a+b)=0$,

$\therefore$ $x=a$ or $-(a+b)$.

Ex. 12. $x=\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}}$. [C. U. '30]

Here $x=\cfrac{1}{2-\cfrac{1}{\cfrac{4-2x-1}{2-x}}}$, or, $x=\cfrac{1}{2-\cfrac{1}{\cfrac{3-2x}{2-x}}}$, or, $x=\cfrac{1}{2-\cfrac{2-x}{3-2x}}$

or, $x=\cfrac{1}{\cfrac{6-4x-2+x}{3-2x}}$, or, $x=\cfrac{1}{\cfrac{4-3x}{3-2x}}$, or, $x=\frac{3-2x}{4-3x}$

or, $4x-3x^2=3-2x$, or, $-3x^2+6x-3=0$,

or, $x^2-2x+1=0$ [dividing by -3]

or, $(x-1)^2=0$, $\therefore$ $x=1, 1$.

[N. B. $(x-1)^2=0$ or $(x-1)(x-1)=0$, $\therefore$ the value of $x=1, 1$]

Ex. 13. $\frac{(x+1)^3-(x-1)^3}{(x+1)^2-(x-1)^2}=2$. [D. B. '49]

or, $\frac{\{(x+1)-(x-1)\}\{(x+1)^2+(x+1)(x-1)+(x-1)^2\}}{\{(x+1)+(x-1)\}\{(x+1)-(x-1)\}}=2$

[From the formulas of a^3-b^3 and a^2-b^2.]

or, $\frac{(x+1)^2+(x+1)(x-1)+(x-1)^2}{(x+1)+(x-1)}=2$,

or, $\frac{x^2+2x+1+x^2-1+x^2-2x+1}{2x}=2$, or, $\frac{3x^2+1}{2x}=2$,

or, $3x^2+1=4x$, or, $3x^2-4x+1=0$, or, $3x^2-3x-x+1=0$,

or, $(x-1)(3x-1)=0$, $\therefore$ $x=1$ or $\frac{1}{3}$.

Ex. 14. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$. [D.B. '40, '43, '48]

or, $\dfrac{1}{a+b+x}=\dfrac{bx+ax+ab}{abx}$, or, $(a+b+x)(bx+ax+ab)=abx$,

or, $(a+b+x)(bx+ax+ab)-abx=0$,

or, $(a+b)(a+x)(b+x)=0$, or, $(a+x)(b+x)=0$,

$\therefore$ $x=-a$ or $-b$.

(2) **The ordinary method of solving an adfected quadratic equation by making the left-hand side a complete square.**

Bring the terms containing the unknown quantities to the left-hand side of the equation and the known quantities to the right-hand side; if the coefficient of x^2 be negative change the sign of evry term of the equation and then divide every term by the coefficient of x^2. Then add the square of half the coefficient of x to both sides. Thus the left-hand side becomes a complete square. See the examples below :—

Ex. 15. Solve $x^2-26x=407$. [D. B. '29]

$x^2-26x=407$, or, $x^2-26x+(13)^2=407+(13)^2$,

or, $(x-13)^2=407+169=576$, or, $(x-13)=\pm\sqrt{576}$,

or, $x-13=\pm 24$, $\therefore$ $x=13\pm 24=37$ or -11.

[N. B. Here the coefficient of $x=26$, half of which $=13$, $\therefore$ $(13)^2$ is added to both sides. At last one value of $x=(13+24)$, and the other value of $x=(13-24)$.]

Ex. 16. Solve $10x^2-69x+45=0$. [D. B. '30]

$10x^2-69x+45=0$, or, $x^2-\frac{69}{10}x+\frac{45}{10}=0$, or, $x^2-\frac{69}{10}x=-\frac{9}{2}$,

or, $x^2-\frac{69}{10}x+(\frac{69}{20})^2=(\frac{69}{20})^2-\frac{9}{2}$,

or, $(x-\frac{69}{20})^2=\frac{4761}{400}-\frac{9}{2}=\frac{2961}{400}$.

or, $x-\dfrac{69}{20}=\pm\sqrt{\dfrac{2961}{400}}$, $\therefore$ $x=\dfrac{69}{20}\pm\dfrac{\sqrt{2961}}{20}=\dfrac{69\pm\sqrt{2961}}{20}$.

[N. B. Here first the terms are divided by 10, the coefficient of x^2 ; then the square of $\frac{69}{20}$ being half the coefficient of x is added to both sides. 2961 is not a perfect square.

so, $\sqrt{\dfrac{2961}{400}}=\dfrac{\sqrt{2961}}{\sqrt{400}}=\dfrac{\sqrt{2961}}{20}$.]

Ex. 17. Solve $ax^2+bx+c=0$. [C. U. '46]

Dividing both sides by a we have $x^2+\frac{b}{a}x+\frac{c}{a}=0$,

or, $x^2+\frac{b}{a}x=-\frac{c}{a}$, or, $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$

[Adding $\left(\frac{b}{2a}\right)^2$ to both sides.]

or, $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$, or, $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

$\therefore \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

[N. B. All quadratic equations can be reduced to the general **form $ax^2+bx+c=0$.** Then we can solve them by the help of the **formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.**

The boys should clearly understand the formula and commit it to memory. The explanation of the formula : Here the coefficient of $x^2=a$, that of $x=b$ and the term free from $x=c$ in the given equation. The numerator of the formula is the coefficient of x with its opposite sign together with the square root (under the radical sign $\pm\sqrt{}$) of the square of the coefficient of x (*i.e.*, b^2) minus 4 times the product of the coefficient of x^2 and the term free from x (*i.e.*, $-4ac$). Its denominator will be twice the coefficient of x^2. Here note the two values of x, one value $=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and the other $=\frac{-b-\sqrt{b^2-4ac}}{2a}$.

(17. *a*) Apply the above formula to find the roots of the equation $x^2-2\sqrt{3}x-13=0$. [C. U. '46]

(3) **Solution of a quadratic equation by means of formula**

Ex. 18. $x^2-2\sqrt{17}x-8=0$. [C. U. '47]

Here, $x=\frac{2\sqrt{17}\pm\sqrt{(2\sqrt{17})^2-4\times1\times-8}}{2\times1}$

$=\frac{2\sqrt{17}\pm\sqrt{68+32}}{2}=\frac{2\sqrt{17}\pm\sqrt{100}}{2}=\frac{2\sqrt{17}\pm10}{2}=\sqrt{17}\pm5.$

Ex. 19. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-3)(x-4)}=\dfrac{1}{6}.$

[C. U. '37]

or, $\dfrac{1}{x-2}-\dfrac{1}{x-1}+\dfrac{1}{x-3}-\dfrac{1}{x-2}+\dfrac{1}{x-4}-\dfrac{1}{x-3}=\dfrac{1}{6}$

or, $\dfrac{1}{x-4}-\dfrac{1}{x-1}=\dfrac{1}{6}$, or, $\dfrac{x-1-x+4}{(x-4)(x-1)}=\dfrac{1}{6}$,

or, $\dfrac{3}{x^2-5x+4}=\dfrac{1}{6}$, or, $x^2-5x+4=18$, or, $x^2-5x-14=0$,

or, $(x-7)(x+2)=0$, $\therefore$ $x=7,\ -2$.

[N. B. $\dfrac{1}{(x-1)(x-2)}$ may be written in the form $\dfrac{1}{x-2}-\dfrac{1}{x-1}$,

$\because$ $\dfrac{1}{x-2}-\dfrac{1}{x-1}=\dfrac{1}{(x-1)(x-2)}$. Thus the remaining two terms are also similarly written.]

Ex. 20. Find the roots of the equation $ax^2+2bx+c=0$.

[C. U. '45 ; G. U, '49]

[Here the two values of x are wanted, $\therefore$ the equation is to be solved.]

$ax^2+2bx+c=0$, or, $ax^2+2bx=-c$,

or, $x^2+\dfrac{2b}{a}x=-\dfrac{c}{a}$, or, $x^2+\dfrac{2b}{a}x+\left(\dfrac{b}{a}\right)^2=\dfrac{b^2}{a^2}-\dfrac{c}{a}$,

or, $\left(x+\dfrac{b}{a}\right)^2=\dfrac{b^2-ac}{a^2}$, or, $x+\dfrac{b}{a}=\pm\dfrac{\sqrt{b^2-ac}}{a}$,

$\therefore$ $x=-\dfrac{b}{a}\pm\dfrac{\sqrt{b^2-ac}}{a}=\dfrac{-b\pm\sqrt{b^2-ac}}{a}$.

[By the application of the formula]

$x=\dfrac{-2b\pm\sqrt{(2b)^2-4ac}}{2a}=\dfrac{-2b\pm\sqrt{4b^2-4ac}}{2a}$

$=\dfrac{-2b\pm2\sqrt{b^2-ac}}{2a}=\dfrac{-b\pm\sqrt{b^2-ac}}{a}$.

Ex. 21. $\dfrac{1}{x+a}+\dfrac{1}{x+2a}+\dfrac{1}{x+3a}=\dfrac{3}{x}.$ [C. U. '50]

Here, from the equation, $\dfrac{1}{x+a}-\dfrac{1}{x}+\dfrac{1}{x+2a}-\dfrac{1}{x}+\dfrac{1}{x+2a}-\dfrac{1}{x}=0,$

or, $\dfrac{x-x-a}{x(x+a)}+\dfrac{x-x-2a}{x(x+2a)}+\dfrac{x-x-3a}{x(x+3a)}=0,$

or, $\dfrac{-a}{x(x+a)}+\dfrac{-2a}{x(x+2a)}+\dfrac{-3a}{x(x+3a)}=0,$

or, $\dfrac{1}{x+a}+\dfrac{2}{x+2a}+\dfrac{3}{x+3a}=0,$ or, $\dfrac{1}{x+a}+\dfrac{3}{x+3a}=\dfrac{-2}{x+2a},$

or, $\dfrac{x+3a+3x+3a}{(x+a)(x+3a)}=\dfrac{-2}{x+2a},$ or, $\dfrac{4x+6a}{x^2+4ax+3a^2}=\dfrac{-2}{x+2a},$

or, $\dfrac{2x+3a}{x^2+4ax+3a^2}=\dfrac{-1}{x+2a},$

or, $2x^2+7ax+6a^2=-x^2-4ax-3a^2,$

or, $3x^2+11ax+9a^2=0,$

$\therefore \quad x=\dfrac{-11a\pm\sqrt{(11a)^2-4.3.9a^2}}{6}=\dfrac{-11a\pm\sqrt{13a^2}}{6}$

$=\dfrac{11a\pm a\sqrt{13}}{6}=\dfrac{-11\pm\sqrt{13}}{6}.a.$

Ex. 22. $\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}=0.$ [C. U. '26, '29 ; D. B. '50]

Multiplying both sides by $(x-a)(x-b)(x-c)$ we have

$(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0,$

or, $x^2-(b+c)x+bc+x^2-(c+a)x+ac+x^2-(a+b)x+ab=0,$

or, $3x^2-(b+c+c+a+a+b)x+ab+bc+ca=0,$

or, $3x^2-2(a+b+c)x+(ab+bc+ca)=0,$

$\therefore \quad x=\dfrac{2(a+b+c)\pm\sqrt{4(a+b+c)^2-4.3(ab+bc+ca)}}{6}$

$=\dfrac{2(a+b+c)\pm2\sqrt{(a+b+c)^2-3(ab+bc+ca)}}{6}$

$=\dfrac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}.$

Ex. 23. Solve the equation $x^2-11x=82052$ without assuming any formula. [C. U. '42]

$x^2-11x=82052$, or, $x^2-11x+(\frac{11}{2})^2=82052+(\frac{11}{2})^2$,

or, $(x-\frac{11}{2})^2=82052+\frac{121}{4}=\frac{328329}{4}$,

or, $x-\frac{11}{2}=\pm\frac{573}{2}$ [taking the square root of both sides]

$\therefore$ $x=\frac{11}{2}\pm\frac{573}{2}=292$ or -281.

Sreedharacharyya's method or Hindu method of solving a quadratic equation.

First transform the given equation into the general form of $ax^2+bx+c=0$. [As for example the equation $(x-2)^2=3x+5$ should be written as $x^2-4x+4=3x+5$, or, $x^2-7x-1=0$] Transpose the term free from x (i.e., c) to the right-hand side. Multiply both sides by 4 times the coefficient of x^2 (here by $4a$). Then add the square of the coefficient of x (here b^2) to both sides. This will give a complete square on the left-hand side.

Ex. 24. $3x^2-11x+9=0$. [C. U. '35]

Here, $3x^2-11x=-9$,

or, $36x^2-132x=-108$ [Multiplying both sides by 4×3]

or, $36x^2-132x+(11)^2=121-108$ [Adding the square of the coefficient of x or $(-11)^2$ to both sides.]

or, $(6x-11)^2=13$, or, $6x-11=\pm\sqrt{13}$, or, $6x=11\pm\sqrt{13}$,

$\therefore$ $x=\dfrac{11\pm\sqrt{13}}{6}$.

Ex. 25. $x^2-2\sqrt{7}x-2=0$. [C. U. '48]

Here, $x^2-2\sqrt{7}x=2$, or, $4x^2-8\sqrt{7}x=8$ [Multiplying both sides by 4×1]

or, $4x^2-8\sqrt{7}x+(2\sqrt{7})^2=(2\sqrt{7})^2+8$

[Adding $(2\sqrt{7})^2$ to both sides]

or, $(2x-2\sqrt{7})^2=28+8=36$,

or, $2x-2\sqrt{7}=\pm6$, or, $2x=2\sqrt{7}\pm6$, $\therefore$ $x=\sqrt{7}\pm3$.

Exercise 12

Solve :

1. $5x^2+3=128.$

2. $(2x-1)^2=5-4x.$

3. $2x-\frac{3}{x}=\frac{x}{2}.$

4. $\frac{2x+5}{x+1}=\frac{x+8}{x+4}.$ [C.U. '31]

5. $42x^2-41x-20=0.$ [C. U. '13]

6. $3x^2-10x+3=0.$ [C. U. '33]

7. $4x^2-65x+126=0.$ [C. U. '16]

8. $6x^2-11x-10=0.$ [C. U. '22]

9. $(17x-8)(x-2)=555.$ [C. U. '32]

10. $17x^2+19x=1848.$ [C. U. '11]

11. $6x^2-91x+323=0.$ [C. U. '14]

12. $(x-7)(x-19)=64.$ [C. U. '18]

13. $(x+4)(2x-3)=6.$ [B. U. '29]

14. $\frac{x}{3}+\frac{3}{x}=4\frac{1}{4}.$ [C.U. '31]

15. $x^2-6x+2=0.$ [G. U. '48]

16. $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{6}.$ [E. B. S. B. '50]

17. $x^2-2\sqrt{13}x+4=0.$ [C. U. '49]

18. $\frac{x+3}{x-3}+6\frac{x-3}{x+3}=5.$ [W. B. S. F. '52]

19. $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{6}{7}=0.$ [C. U. '11]

20. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{25}{12}.$ [C. U. '10]

21. $x^2-10x+8=0.$ [C. U. '47]

22. $\frac{12x+17}{3x+1}-\frac{2x+15}{x+7}=3\frac{1}{5}.$ [C. U. '20]

23. $\frac{40}{x-5}+\frac{27}{x}=13.$ [D. B. '26]

24. $x+\frac{1}{x}=6\frac{1}{6}.$ [C. U. '28]

25. $\frac{x-6}{x+2}+\frac{x-10}{x+6}+2=0.$

26. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

27. $\frac{x+1}{2}+\frac{2}{x+1}=\frac{x+1}{3}+\frac{3}{x+1}-\frac{5}{6}$. [C. U. '36]

28. $\frac{x-3}{x+3}+\frac{x+3}{x-3}=\frac{2(x+4)}{x-4}$.

29. $\left(\frac{x-a}{x+a}\right)^2-5\left(\frac{x-a}{x+a}\right)+6=0$. [P. U.]

30. $1+x=\cfrac{3}{4-\cfrac{3}{4-x}}$. [C. U. '44]

31. $x^2-2\sqrt{3}x-13=0$. [C. U. '46]

32. $x^2+2(b-c)x+c^2=2bc$.

33. $\frac{x-a}{x-b}+\frac{x-b}{x-a}=\frac{a}{b}+\frac{b}{a}$. [P. U. 1891]

34. $ax^2-bx-c=0$. [C. U. '44]

35. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$. [D. B. '40, '43]

Solve without assuming the formula :

36. $3x^2+4x=8$. [C. U. '51]

37. $x^2-x=1806$. [C. U. '17] 38. $63x^2-62x=221$.

39. For what value of x is $ax^2-(a+1)x+1=0$?

40. Solve $15x^2-11x-31=0$, and find the roots correct to two places of decimals. [G. U. '54]

PROBLEMS ON QUADRATIC EQUATIONS

[**N. B.** In solving problems on quadratic equations, the solutions which are found inadmissible by the condition of the problem should be rejected. Hence the two values of the unknown quantity obtained in each solution should be tested before they are taken for the answer.

Ex. 1. Find two consecutive numbers the sum of whose squares is 145. [C. U. '16]

Let x be the first number.

Then the next greater number $= x+1$

$\therefore$ by the condition of the problem we have $x^2+(x+1)^2=145$,

or $x^2+x^2+2x+1=145$, or, $2x^2+2x-144=0$,

or $x^2+x-72=0$, or, $(x+9)(x-8)=0$,

$\therefore \quad x=-9$ or 8.

$\therefore$ the required numbers $=8$ and 9, or -9 and -8.

[N. B. The first set is only an arithmetical solution and both the first and second sets of answers are algebraical solution.]

Ex. 2. Find the two consecutive odd numbers, whose product is 899. [Pat. U. '24]

Let $2x-1$ and $2x+1$ be two consecutive odd numbers, x being an integer.

$\therefore$ by the condition of the problem we have

$(2x-1)(2x+1)=899$, or, $4x^2-1=899$,

or $4x^2=900$, or, $x^2=255$, $\therefore \quad x=\pm 15$

$\therefore$ the required numbers $=29$ and 31 or -31 and -29.

[N. B. $2x$ is an even number, for any integral value of x.

$\therefore$ $2x+1$ or $2x-1$ is an odd number.]

Ex. 3. A man bought a certain number of books for Rs. 20. Had he obtained one more book for the same sum, the average price of each would have been a rupee less. Find the number of books bought.

Let x be the number of books bought.

$\therefore$ the average price of each book $=\dfrac{20}{x}$ rupees.

If $(x+1)$ books are bought for Rs. 20, the average price of each book $=\dfrac{20}{x+1}$ rupees.

$\therefore$ by the condition of the problem we have

$\dfrac{20}{x+1}=\dfrac{20}{x}-1$, or, $\dfrac{20}{x+1}=\dfrac{20-x}{x}$

or, $20x=20x+20-x^2-x$, or, $x^2+x-20=0$,

or, $(x+5)(x-4)=0$, $\therefore \quad x=-5$ or 4

$\therefore$ the required number of books $=4$.

Here the number of books cannot be negative. So the other value of $x(=-5)$ is rejected.

Ex. 4. Divide unity into two parts such that the sum of their cubes is $\frac{7}{16}$. [C. U. '15]

Let x be one part. Then $1-x$=the other part.

$\therefore$ by the condition of the problem, we have $(x)^3+(1-x)^3=\frac{7}{16}$,

or, $x^3+1-3x+3x^2-x^3=\frac{7}{16}$

or, $3x^2-3x+\frac{9}{16}=0$, or, $48x^2-48x+9=0$,

or, $16x^2-16x+3=0$, or, $16x^2-12x-4x+3=0$,

or, $(4x-1)(4x-3)=0$, $\therefore$ $x=\frac{1}{4}$ or $\frac{3}{4}$

$\therefore$ the required parts$=\frac{1}{4}$ and $\frac{3}{4}$.

Ex. 5. Divide 50 into two parts such that the sum of their reciprocals may be $\frac{1}{12}$. [C. U. '13]

Let x be one part. Then $50-x$ is the other part.

$\therefore$ by the condition of the problem we have

$\frac{1}{x}+\frac{1}{50-x}=\frac{1}{12}$, or, $\frac{50-x+x}{x(50-x)}=\frac{1}{12}$, or, $\frac{50}{50x-x^2}=\frac{1}{12}$,

or, $600=50x-x^2$, or, $x^2-50x+600=0$,

or, $x^2-30x-20x+600=0$, or, $(x-20)(x-30)=0$,

$\therefore$ $x=20$ or 30. $\therefore$ the required parts$=20$ and 30.

[N. B. If the product of two numbers is 1, then one is called the reciprocal of the other. Thus, the reciprocal of $\frac{3}{4}=\frac{4}{3}$ and that of $5=\frac{1}{5}$ etc.]

Ex. 6. The difference between a proper fraction and its reciprocal is $\frac{9}{20}$. Find the fraction. [C. U. '41 ; D. B. '29]

Suppose x is the proper fraction. Then $\frac{1}{x}$ is its reciprocal.

$\because$ x is a proper fraction, $\therefore$ its denominator is greater than its numerator.

$\therefore$ the numerator of $\frac{1}{x}$ is greater than its denominator. $\therefore$ $\frac{1}{x}>x$.

$\therefore$ by the condition of the problem we have $\frac{1}{x}-x=\frac{9}{20}$

or, $20-20x^2=9x$, or, $20x^2+9x-20=0$,

or, $20x^2+25x-16x-20=0$, or, $(4x+5)(5x-4)=0$,

$\therefore$ $x=-\frac{5}{4}$ or $\frac{4}{5}$. $\therefore$ the required fraction$=\frac{4}{5}$; $-\frac{5}{4}$ not being a proper fraction is rejected.

Ex. 7. The sum of two numbers is 45 and the mean proportional between them is 18; find the numbers. [B.U. '29]

Let x be one of the numbers, then $45-x$ is the other number.

$\because$ their mean proportional$=18$, $\therefore$ $x(45-x)=18^2$,

or, $45x-x^2=324$, or, $-x^2+45x-324=0$,

or, $x^2-45x+324=0$, or, $(x-36)(x-9)=0$, $\therefore$ $x=9, 36$.

$\therefore$ the required numbers$=9$ and 36.

Ex. 8. A certain number exceeds its reciprocal by 1. How many such numbers are there? Find them. [C. U. '34]

Let the number be x, then its reciprocal$=\frac{1}{x}$.

$\therefore$ by the condition of the problem we have

$x-\frac{1}{x}=1$, or, $x^2-1=x$, or, $x^2-x-1=0$,

$$\therefore \quad x=\frac{1\pm\sqrt{(-1)^2-4(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}$$

$\therefore$ There are two such numbers, viz., $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Ex. 9. Show that the product of any four consecutive numbers increased by unity is a perfect square.

Suppose x, $x+1$, $x+2$ and $x+3$ are the four consecutive numbers. Adding 1 to their product we have

$$x(x+1)(x+2)(x+3)+1=x(x+3)(x+1)(x+2)+1$$
$$=(x^2+3x)(x^2+3x+2)+1=a(a+2)+1 \text{ [Putting } a \text{ for } x^2+3x]$$
$$=a^2+2a+1=(a+1)^2=(x^2+3x+1)^2, \text{ which is a perfect}$$

square. Thus the problem is proved.

Ex. 10. The hypotenuse of a right-angled triangle is 20 cms. If the difference between its other two sides be 4 cms., find the sides. [G. U. '49]

Suppose the smaller side$=x$ cms.,

then the greater side$=(x+4)$ cms.

$\therefore$ $x^2+(x+4)^2=20^2$, or, $x^2+x^2+8x+16=400$,

or, $2x^2+8x-384=0$, or, $x^2+4x-192=0$,

or, $(x+16)(x-12)=0$, $\therefore$ $x=-16$ or 12.

$\therefore$ the smaller side$=12$ cms. and the greater side$=(12+4)$ or 16 cms., the other value of x being negative is inadmissible.

Ex. 11. The area of a rectangular plot of land fenced all round is 2000 sq. metres and the total length of fencing is 180 metres. Obtain a quadratic equation to determine the length of the plot.

2(length + breadth) = perimeter = the length of the fencing = 180 metres. $\therefore$ Length + breadth = 90 metres.

Suppose the length to be x metres, $\therefore$ the breadth $=(90-x)$ metres. From the first condition we have $x(90-x)=2000$,

or, $x^2-90x+2000=0$, this is the required equation.

On solving this equation we have $x=40$ or 50.

$\therefore$ the required length $=50$ metres [$\because$ length $>$ breadth]

Ex. 12. A cyclist travels 84 miles and finds that he would have made the journey in 5 hours less if he had travelled 5 miles an hour faster. At what rate did he travel? [C. U. '50]

Suppose the cyclist travels at the rate of x miles per hour. $\therefore$ he takes $\frac{84}{x}$ hours to travel 84 miles. If the speed be $(x+5)$ miles per hour, he takes $\frac{84}{x+5}$ hours to travel 84 miles,

$\therefore$ $\frac{84}{x}-\frac{84}{x+5}=5$, or, $\frac{84x+420-84x}{x(x+5)}=5$, or, $\frac{420}{x^2+5x}=5$,

or, $5(x^2+5x)=420$, or, $x^2+5x=84$, or, $x^2+5x-84=0$,

or, $(x+12)(x-7)=0$, $\therefore$ $x=-12$ or 7.

$\because$ the speed cannot be negative,

$\therefore$ the required speed $=7$ miles per hour.

Ex. 13. The perpendicular drawn from the centre of a circle to a certain chord is 3 centimetres less than half of the chord. If the radius of the circle be 15 centimetres, find the length of the chord.

The perpendicular drawn from the centre of a circle to a chord bisects the chord. Suppose OD is drawn perpendicular on the chord AB from the centre O. Now, OAD is a right-angled triangle. Let OD $=x$ cm.

$\therefore$ AD $=x+3$ cm. and the hypotenuse OA $=15$ cm.

$\therefore$ $x^2+(x+3)^2=15^2$,

or, $2x^2+6x+9=225$, or, $2x^2+6x-216=0$,

or, $x^2+3x-108=0$, or, $(x+12)(x-9)=0$, $\therefore$ $x=-12$ or 9.

$\because$ the measure of any length cannot be negative,

$\therefore$ the perpendicular $=9$ cm.

$\therefore$ the length of the given chord $=2(9+3)$ cm. or 24 cm.

Ex. 14. A boatman can row 7 km. down a river and back in $4\frac{2}{3}$ hours. If the river runs at the rate of 2 km. an hour, find the rate of the pull in still water.

Let x kilometres per hour be the rate of the pull, if there were no current.

$\therefore$ with the current the boat goes $(x+2)$ km. per hour and against the current $(x-2)$ km. per hour.

$\therefore \quad \frac{7}{x+2}+\frac{7}{x-2}=4\frac{2}{3}$, or, $\frac{1}{x+2}+\frac{1}{x-2}=\frac{2}{3}$

[dividing both sides by 7]

or, $\frac{x-2+x+2}{x^2-4}=\frac{2}{3}$, or, $\frac{2x}{x^2-4}=\frac{2}{3}$, or, $x^2-3x-4=0$,

or, $(x-4)(x+1)=0$, $\therefore \quad x=4$ or -1.

$\because$ the rate of the boat cannot be -1, $\therefore$ the required rate of the pull $=4$ km. per hour.

Ex. 15. A regiment of soldiers when formed into a solid square, has 16 men fewer in the front than when formed into a hollow square 4 deep. Find the number of soldiers. [D.B. '40]

Let x be the number of soldiers in the front row of the hollow square; then the whole number of soldiers$=x^2-(x-4\times 2)^2$ $=x^2-(x-8)^2$.

Again, the number of soldiers in the front row of the solid square$=x-16$.

$\therefore$ the whole number of soldiers$=(x-16)^2$.

$\therefore$ by the condition of the problem we have

$(x-16)^2=x^2-(x-8)^2$,

or, $x^2-32x+256=x^2-x^2+16x-64$, or, $x^2-48x+320=0$,

or, $(x-40)(x-8)=0$, $\therefore \quad x=40$ or 8.

$\therefore$ the required number of soldiers

$=(x-16)^2=(40-16)^2=24^2=576$.

Here the value of $x=8$ is not admissible, because in that case the number of men in the front row cannot be 16 men fewer than 8 men.

Ex. 16. What is the price of eggs per dozen, if one more for 6 annas, reduces the price by 1 anna per dozen ?

[D. B. '39, '41, '46]

Suppose the price of x eggs$=6$ annas.

$\therefore$ the price of 12 eggs$=\frac{72}{x}$ annas......(1)

Now, if the price of $(x+1)$ eggs$=6$ annas,

then the price of 12 eggs$=\frac{72}{x+1}$ annas......(2)

$\therefore$ by the condition of the problem we have $\frac{72}{x}-\frac{72}{x+1}=1$

[$\because$ the second price is less than the first price by 1 anna.]

or, $\frac{72x+72-72x}{x(x+1)}=1$, or, $x^2+x=72$,

or, $x^2+x-72=0$, or, $(x+9)(x-8)=0$, $\therefore$ $x=-9$ or 8.

$\because$ the number of eggs cannot be negative (-9),

$\therefore$ the required number of eggs$=8$.

Now, from (1) the price of one dozen of eggs$=\frac{72}{8}$as.$=9$ annas.

Ex. 17. Find a fraction which assumes twice or thrice its original value when 2 or 3 respectively is added to both its numerator and denominator.

[D. B. '49]

Let $\frac{x}{y}$ be the given fraction.

Now, by the first condition of the problem we have

$\frac{x+2}{y+2}=\frac{2x}{y}$, or, $xy+2y=2xy+4x$, or, $2y-4x=xy$,

or, $\frac{2y}{xy}-\frac{4x}{xy}=1$, or, $\frac{2}{x}-\frac{4}{y}=1$......(1)

Again, by the second condition of the problem we have

$\frac{x+3}{y+3}=\frac{3x}{y}$, or, $3y-9x=2xy$, or, $\frac{3}{x}-\frac{9}{y}=2$......(2)

Now, from (1)$\times$3 and (2)$\times$2 we have,

$$\frac{6}{x}-\frac{12}{y}=3$$

and $$\frac{6}{x}-\frac{18}{y}=4$$

$\therefore$ (subtracting) $\frac{6}{y}=-1$, $\therefore$ $y=-6$.

Putting the value of y in (1) we have $\frac{2}{x}+\frac{4}{6}=1$, or, $\frac{2}{x}=\frac{1}{3}$,

$\therefore$ $x=6$. $\therefore$ the required fraction $=\frac{6}{-6}$.

[N.B. Here, the answer must be $\frac{6}{-6}$. It cannot be $\frac{1}{-1}$ or, -1, because in that case the conditions will not be satisfied.]

Ex. 18. The product of two numbers is 24. If the sum of their squares be added to their sum the result is 62. Find the two numbers. [E.B.S.B. '58]

Let the numbers be x and y.

$\therefore$ $xy=24$......(1) and $x^2+y^2+x+y=62$......(2)

From (2) we have $(x+y)^2-2xy+x+y=62$

or, $(x+y)^2-2\times24+(x+y)=62$,

or. $(x+y)^2+(x+y)-110=0$,

or, $a^2+a-110=0$ [Putting a for $x+y$]

or, $(a+11)(a-10)=0$, $\therefore$ $a=-11$ or 10.

(i) If $a=-11$, then $x+y=-11$.

Now $\because$ $xy=24$,

$\therefore$ $(x-y)^2=(x+y)^2-4xy=(-11)^2-4\times24=25$

$\therefore$ $x-y=\pm5$

Now, $x+y=-11$

$x-y=\ \ 5$ [Taking $x-y=5$]

$\therefore$ $2x=-6$, $\therefore$ $x=-3$, $\therefore$ $y=-8$.

$\therefore$ the two numbers are -3 and -8.

[The same answer will be obtained if $x-y=-5$.]

(ii) Again, if $a=10$, then $x+y=10$, $\therefore$ $y=10-x$,

Now, $\because$ $xy=24$, $\therefore$ $x(10-x)=24$, or, $x^2-10x+24=0$,

or, $(x-4)(x-6)=0$, $\therefore$ $x=4$ or 6 ;

$\therefore$ $y=6$ or 4. $\therefore$ the two numbers$=4$, 6.

$\therefore$ the required numbers$=-3$ and -8, or, 4 and 6.

[Two different solutions are shown in (i) and (ii).]

Exercise 13

1. Find a number such that its square added to its cube is 16 times the next number. [A. U. '16]

2. The sum of two numbers is 2 and the sum of their two reciprocals is $2\frac{1}{4}$. Find the two numbers. [C. U. '36]

3. Find two consecutive odd numbers whose product is 35.

4. Find two consecutive numbers such that the difference of their two reciprocals is $\frac{1}{110}$.

5. What number when added to 30 will be less than its square by 12.
[E. B. S. B. '50]

6. The sum of the squares of two consecutive odd integers is 290 ; find the integers.

7. A number is greater than its square root by 110. Find it.

8. The sum of the squares of two consecutive even numbers is 100. Find the numbers.
[A.U. '24]

9. In a right-angled triangle the hypotenuse is 17″ and the sum of the other two sides is 23″. Find the two sides. [G.U. '51]

10. The hypotenuse of a right-angled triangle is 13 inches. Find the length of each of the remaining two sides, if their sum is 17 inches.
[C. U. 45]

11. A and B can do a piece of work in 72 minutes ; but B alone takes 1 hour more than A to do it. In what time can each do it ?

12. Find the price of eggs per dozen when two more eggs for a shilling would reduce the price by one penny per dozen.
[E.B.S.B. '51]

13. A man bought a certain number of goats for Rs. 420 ; had he obtained one more for the same sum, the price of each would have been Re. 1 less. Find the number of goats bought.

14. A number is less than twice the product of its two digits by 8 ; if the digit in the tens' place is greater than the digit in the units' place by 1, find the number.

15. A man takes one hour less to ride 24 kilometres if he increases his speed by 2 km. per hour. Find his speed per hour.

16. The perpendicular from the centre of a circle to a chord is less than half the chord by 1 cm. If the radius of the circle be 5 centimetres, find the length of the chord.

17. The area of a rectangular field is 260 sq. metres. If its length be diminished by 5 m. and the breadth increased by 2 m., it becomes a square field. Find its length and breadth.

18. A boatman can row in $1\frac{1}{2}$ hours 9 kilometres down a river but 3 km. up stream. Find the speed of the current and of the boat per hour.

19. Find two consecutive positive numbers the sum of whose squares is 761.
[G. U. '52]

20. The hypotenuse of a right-angled triangle is 25 cm. and its perimeter is 56 cm. ; find the length of its smaller side.

MENSURATION

RECTANGLE

The perimeter of a rectangle $=$ the sum of its four sides $=2(\text{length}+\text{breadth})$. The area of a rectangle $=\text{length}\times\text{breadth}$;

$\therefore$ its length $=\text{area}\div\text{breadth}$, and breadth $=\text{area}\div\text{length}$,

The diagonal of a rectangle $=\sqrt{\text{length}^2+\text{breadth}^2}$.

Square

The perimeter of a square $=$ the length of a side $\times 4$ ($\because$ all sides of a square are equal.)

The area of a square $=(\text{length})^2=(\text{breadth})^2$;

The side of a square $=\sqrt{\text{area of the square}}$, and the diagonal of a square $=$ a side $\times\sqrt{2}$; $\therefore$ the area of a square $=\frac{1}{2}(\text{diagonal})^2$.

You know that the floor, roof and walls of a room are all rectangular.

$\therefore$ the area of a floor or a roof $=\text{length}\times\text{breadth}$.

The area of the four walls of a room $=2(\text{length}+\text{breadth})\times\text{height}$.

N.B. These have already been discussed. Below are given a few examples.

Examples [1]

1. A square field contains 31 acres 10·25 square poles ; find the length of a side. [R. E.]

The area of the square $=31$ ac. 10·25 sq. po. $=4970{\cdot}25$ sq. po.

$\therefore$ the length of a side $=\sqrt{\text{area}}=\sqrt{4970{\cdot}25}$ pole $=70{\cdot}5$ poles.

2. A rectangular grass-plot, the sides of which are as 3 : 2 costs £14. 8*s*. for turfing it at 4*d*. per square yard, find the lengths of the sides.

Here, length : breadth $=3:2$, $\therefore$ length $=\frac{3}{2}\times$ breadth.

$\therefore$ $2\times\text{length}=3\times\text{breadth}$,

Now, the total cost $= £14.\ 8s. = 288s.$

and the cost of 1 sq. yard $= 4d. = \frac{1}{3}s.$

$\therefore$ the area $= (288 \div \frac{1}{3})$ sq. yds. $= 864$ sq. yds.

$\therefore$ length $\times$ breadth $= 864$ sq. yds.

or, $\frac{3}{2}$ breadth $\times$ breadth $= 864$ sq. yds.

or, $\frac{3}{2}$ (breadth)$^2 = 864$ sq. yds.

or, (breadth)$^2 = \frac{864 \times 2}{3}$ sq. yds. $= 576$ sq. yds.

$\therefore$ breadth $= \sqrt{576}$ yds. $= 24$ yds.

and length $= \frac{3}{2} \times$ breadth $= \frac{3}{2} \times 24$ yds. $= 36$ yds.

3. There is a garden 140 ft. long and 120 ft. wide and a gravel walk of uniform breadth is to be made all round it so as to take up just one-fourth of the garden. What must be the breadth of the path ? [R.E.]

Let the breadth of the walk be x feet.

The area of the walk $= 2 \times 140 \times x + 2\,(120 - 2x)x$

$= 280x + 240x - 4x^2 = 520x - 4x^2.$

Again, that area $= \frac{1}{4} \times$ (the area of the garden)

$= \frac{1}{4} \times 140 \times 120$ sq. ft. $= 4200$ sq. ft. $\quad \therefore \quad 4200 = 520x - 4x^2$

or, $4x^2 - 520x + 4200 = 0$, or, $x^2 - 130x + 1050 = 0$

$$\therefore \quad x = \frac{130 \pm \sqrt{130^2 - 4 \times 1050}}{2} = \frac{130 \pm \sqrt{16900 - 4200}}{2}$$

$$= \frac{130 \pm 112{\cdot}694}{2} = 121{\cdot}347 \text{ or } 8{\cdot}65$$

As here the walk cannot possibly be 121·347 ft. wide,

$\therefore$ the required breadth of the path $= 8{\cdot}65$ ft. (nearly).

4. Find the cost of lining a rectangular cistern with lead at Re. 1. 2 as. per sq. ft. of surface, the inside dimensions of the cistern being as follows—the length 3 ft. 2 in., the breadth 2 ft. 10 in. and the depth 2 ft. 6 in. [R. E.]

The area of the floor of the cistern $=$ length $\times$ breadth

$= \frac{19}{6}$ ft. $\times \frac{17}{6}$ ft. $= \frac{323}{36}$ sq. ft.

The total area of its four walls $= 2$ (length $+$ breadth) $\times$ height

$= 2(3$ ft 2 in. $+ 2$ ft. 10 in.$) \times 2$ ft. 6 in.

$= 2 \times 6$ ft $\times \frac{5}{2}$ ft. $= 30$ sq. ft.

$\because$ the cost per sq. ft. $=$ Re. 1. 2 as. $=$ Rs. $\frac{9}{8}$,

$\therefore$ the required cost $=$ Rs. $(\frac{323}{36} \times \frac{9}{8}) +$ Rs. $30 \times \frac{9}{8}$

$=$ Rs. 10. 1 a. 6 p. $+$ Rs. 33. 12 as. $=$ Rs. 43. 13 as. 6 p.

Exercise 1

1. Each student requires a space of 4 ft. by 30 in. How many students can be seated in a room 20 yards by 28 feet ?

2. Find the cost of putting a fence around a square field whose area is 13·225 acres at Rs. 1. 12 as. per yard. [C.U. 1890]

3. Two square fields jointly contain 6 acres and the side of one is three-fourths as long as that of the other ; how many acres are in each ?

4. A square grass-plot contains 10 acres. There is a path 30 feet wide surrounding it. How many stones each 1 ft. by 9 in., will be required to pave the path ? [C. U. 1907]

5. A room is 30 ft. long, 22 ft. wide and $18\frac{1}{2}$ ft. high and has 5 doors and 3 windows ; find the cost of colouring the walls at 3 as. per square yard deducting 30 sq. feet for each door and window. [C. U. 1875]

6. Find the cost of paving a pathway 6 ft. wide round and immediately outside a flower garden 21 yds. long and 10 yds. broad at $5\frac{4}{7}$ pies a sq. yard. [C.U. 1905 ; D.B. '33]

7. A postage stamp is $\frac{23}{24}$ of an inch long and $\frac{4}{5}$ of an inch wide. How many stamps will be required to cover a board 1 ft. 11 in. long and 1 ft. wide ?

8. A rectangular court, three times as long as it is broad, is paved with 2028 stones, each $1\frac{1}{3}$ feet square. Find the length of the court. [C.U. 1912]

TRIANGLE

The perpendicular drawn from the vertex to the base of a triangle is called its height or altitude.

(1) In a **right-angled triangle**

$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{perpendicular})^2$ [i.e. = the sum of the squares on its other two sides.]

(2) **The area of the triangle** $= \frac{1}{2} \times \text{base} \times \text{height}$,

$\therefore$ base $= \dfrac{2 \times \text{area}}{\text{height}}$; height $= \dfrac{2 \times \text{area}}{\text{base}}$.

(3) **The area of a right-angled triangle** $=\frac{1}{2}$ base $\times$ perpendicular $=\frac{1}{2}\times$ the product of the sides containing the right angle.

(4) In the right-angled isosceles triangle
$(\text{hypotenuse})^2=2\,(\text{ base })^2=2\,(\text{ perpendicular })^2$,

$\therefore$ the hypotenuse $=\sqrt{2}\times$ base $=\sqrt{2}\times$ perpendicular.

(5) **The height of an equilateral triangle** $=$ side $\times\dfrac{\sqrt{3}}{2}$,

$\therefore$ the **area of an equilateral triangle** $=\dfrac{\sqrt{3}}{4}\times(\text{side})^2$.

(6) The height of an isosceles triangle
$=\sqrt{\text{square on any one equal side}-\text{square on half the base}}$.

The area of a triangle in terms of its sides :

The letter s is used to denote the semi-perimeter of a triangle, that is to say, half the sum of its sides. If the sides are a, b, c units in length then $2s=a+b+c$, and $s=\frac{1}{2}(a+b+c)$. Δ denotes the area of a triangle.

With these notations the area of a triangle is given by the formula $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$.

If each of the equal sides of an isosceles triangle be a and the third side be b units of length, the area of the isosceles triangle $=\dfrac{b}{4}\sqrt{4a^2-b^2}$ units of area.

Examples [2]

1. The area of an equilateral triangle is 25 square inches ; find its perimeter. [A. U.]

The area of an equilateral triangle $=\dfrac{\sqrt{3}}{4}\times(\text{side})^2$,

$\therefore$ here $\dfrac{\sqrt{3}}{4}\times(\text{side})^2=25$ sq. in.,

$\therefore$ $(\text{ side })^2=\dfrac{25\times4}{\sqrt{3}}$ sq. in. $=\dfrac{100}{\sqrt{3}}$ sq. in. $=\dfrac{100\sqrt{3}}{3}$ sq. in.

$=\dfrac{100\times1{\cdot}732}{3}$ sq. in. $=\dfrac{173{\cdot}2}{3}$ sq. in. $=57{\cdot}7$ sq. in.

$\therefore$ side $=\sqrt{57{\cdot}7}$ in. (nearly) $=7{\cdot}6$ inch (nearly).

$\therefore$ the required perimeter $=7{\cdot}6$ in. $\times3=22{\cdot}8$ inches (nearly).

2. One side of a right-angled triangle is 588 cm. and the sum of its hypotenuse and the third side is 882 cm. Find the hypotenuse and the third side.

Let the hypotenuse be x cm., $\therefore$ the third side $=(882-x)$ cm.

Now, $x^2=(588)^2+(882-x)^2=(588)^2+(882)^2-2\times 882.x+x^2$,

or, $1764x=(588)^2+(882)^2=1123668$, $\therefore$ $x=637$.

$\therefore$ the required hypotenuse $=637$ cm.

and the third side $=(882-637)$ cm. $=245$ cm.

3. What must be the side of an equilateral triangle so that its area may be equal to that of a given square of which the diagonal is 120 ft.? [R. U. S.]

$\because$ the diagonal of the square $=120$ ft.

$\therefore$ the area of the square $=\frac{1}{2}\times(120)^2$ sq. ft. $=60\times 120$ sq. ft.

Let each side be a feet.

$\therefore$ its area $=\dfrac{\sqrt{3}}{4}a^2$. $\therefore$ $\dfrac{\sqrt{3}}{4}a^2=60\times 120$

or, $a^2=\dfrac{60\times 120\times 4}{\sqrt{3}}=\dfrac{60\times 120\times 4\times\sqrt{3}}{3}=16627\cdot 2$.

$\therefore$ $a=\sqrt{16627\cdot 2}=128\cdot 9$. $\therefore$ the required side $=128\cdot 9$ ft. (App.)

4. The perimeter of a right-angled isosceles triangle is $(\sqrt{2}+1)$ feet, find its hypotenuse. [P. U.]

Let each of the equal sides of the right-angled isosceles triangle be x ft.

$\therefore$ its hypotenuse $=\sqrt{2}x$ ft. and the given perimeter $=(\sqrt{2}+1)$ ft.

$\therefore$ $\sqrt{2}x+2x=$ perimeter $=\sqrt{2}+1$, or, $x(2+\sqrt{2})=\sqrt{2}+1$,

$$\therefore\quad x=\frac{\sqrt{2}+1}{2+\sqrt{2}}.$$

$\therefore$ the required hypotenuse

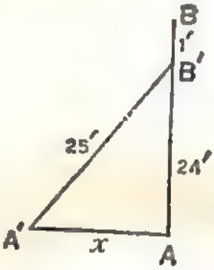
$$=\sqrt{2}.x \text{ ft.}=\sqrt{2}\times\frac{\sqrt{2}+1}{2+\sqrt{2}}\text{ ft.}=\frac{2+\sqrt{2}}{2+\sqrt{2}}=1\text{ ft.}$$

5. A ladder 25 ft. long stands upright against a wall, how far must the bottom of the ladder be pulled out so as to lower the top by 1 ft. ?

The ladder AB is standing erect against the wall (AB). Suppose, if its lower end be drawn off x feet from A to A′, its top descends 1 foot from its original position B to B′. $\because$ AB=25 ft.,

$\therefore$ AB′=(25−1) or 24 ft.

and A′B′=AB=25 ft.



$\therefore x^2=(A'B')^2-(AB')^2=25^2-24^2=49,$

$\therefore$ $x=\sqrt{49}=7$. $\therefore$ the lower end of the ladder should be drawn off 7 ft. from the wall.

6. A tower, which stands on a horizontal plane, subtends a certain angle at a point 160 ft. from the foot of the tower. On advancing 100 ft. towards it, the tower is found to subtend an angle twice as great as before. What is the height of the tower ? [A. U.]

Let the height AB=h.

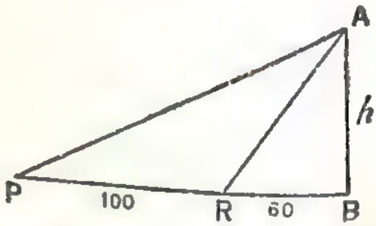
Here, BP=160 ft. ; PR=100 ft.

$\therefore$ BR=60 ft.

Again, $\angle ARB=2\angle P$ (by hypothesis).

But $\angle ARB=\angle P+\angle PAR$.

$\therefore$ $2\angle P=\angle P+\angle PAR$.



$\therefore$ $\angle P=\angle PAR$. $\therefore$ AR=PR=100 ft.

$\therefore$ from the right-angled triangle ABR, we have

$h^2=AR^2-BR^2=100^2-60^2=(100+60)(100-60)=6400.$

$\therefore$ $h=\sqrt{6400}=80$. $\therefore$ the required height=80 feet.

7. Two sides of a triangle are 85 metres and 154 metres respectively and its perimeter is 324 metres. Find the area of the triangle.

The third side of the triangle=(324−85−154)m.=85m.

and its semi-perimeter=$\frac{1}{2}\times 324$m.=162 m.

$\therefore$ the required area= $\sqrt{s(s-a)(s-b)(s-c)}$ sq. m.

$=\sqrt{162(162-85)(162-154)(162-85)}$ sq. m.

$=\sqrt{162\times 77\times 8\times 77}$ sq. m.$=\sqrt{81\times 2\times 77\times 8\times 77}$ sq. m.

$=9\times 77\times 4$ sq. m.=2772 sq. m.

8. The sides of a triangle are 13, 14 and 15 metres, find the perpendicular from the opposite angle on the side 14 metres long. [P.U.]

Let h metres be the required perpendicular.

$\therefore$ the area of the triangle
$=\frac{1}{2}\times \text{base}\times \text{height}=\frac{1}{2}\times 14 \text{ m.}\times h \text{ m.}=7 \text{ m.}\times h \text{ m.}$

Again, its semi-perimeter$(s)=\frac{1}{2}(13+14+15)\text{m.}=21 \text{ m.}$

$\therefore$ the area of the triangle

$=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21(21-13)(21-14)(21-15)}$ sq. m.

$=\sqrt{21\times 8\times 7\times 6}$ sq. m. $=\sqrt{7\times 3\times 8\times 7\times 2\times 3}$ sq. m.

$=7\times 3\times 4$ sq. m. $\therefore$ $7 \text{ m.}\times h \text{ m.}=7\times 3\times 4$ sq. m.

$\therefore$ $h=\frac{7\times 3\times 4}{7}\text{m.}=12 \text{ m.}$ $\therefore$ the required height $=12$ metres.

9. The cost of paving a triangular lawn is £200 at 2*s*. 6*d*. per square foot. If one side be 24 yards long, find the length of the other two equal sides.

Let each of the two equal sides of the lawn be a yards in length. $\therefore$ the semi-perimeter(s) of the triangle
$=\frac{1}{2}(24+a+a)$ yds. $=(12+a)$ yds.

$\therefore$ the area of the triangle $=\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{(12+a)(12+a-a)(12+a-a)(12+a-24)}$ sq. yds.

$=\sqrt{(a+12).\ 12.\ 12\ (a-12)}$ sq. yds. $=12\sqrt{a^2-144}$ sq. yds.

Again, $\because$ the total cost $=$ £200, and the cost per sq. ft. $=\frac{5}{2}$s. $=$ £$\frac{1}{8}$,

$\therefore$ the area of the lawn $=(200\div\frac{1}{8})$ sq. ft.
$=1600$ sq. ft. $=\frac{1600}{9}$ sq. yds. $\therefore$ $12\sqrt{a^2-144}=\frac{1600}{9}$

or, $\sqrt{a^2-144}=\frac{400}{27}$ or, $a^2-144=\frac{160000}{729}$

or, $a^2=\frac{160000}{729}+144=219{\cdot}4787\cdots+144=363{\cdot}4787\cdots$,

$\therefore$ $a=\sqrt{363{\cdot}4787\cdots}=19{\cdot}06$ (nearly).

$\therefore$ the required length of the side $=19{\cdot}06$ yards (Approx.)

10. The sides of a triangle are in the ratio of 3 : 4 : 5 and the perimeter is 432 ft. Find the area of the triangle.

$3+4+5=12$, $432 \text{ ft.}\div 12=36$ ft.

$\therefore$ the lengths of the sides are $36 \text{ ft.}\times 3$, $36 \text{ ft.}\times 4$ and $36 \text{ ft.}\times 5$ or 108 ft., 144 ft. and 180 ft.

The semi-perimeter $=\frac{1}{2}\times 432 \text{ ft.}=216$ ft.

$\therefore$ the required area

$=\sqrt{216(216-108)(216-144)(216-180)}$ sq. ft.

$=\sqrt{2\times 108\times 108\times 72\times 36}$ sq. ft. $=108\times 72$ sq. ft.

$=\frac{108\times 72}{9}$ sq. yds. $=864$ sq. yds.

11. If the side of an equilateral triangle be increased by 1 metre, its area is increased by $\sqrt{3}$ square metres. Find the length of its side.

Let each side of the equilateral triangle be a metres.

$\therefore$ its area$=\frac{\sqrt{3}}{4}a^2$ sq. m.

If the side be 1 m. longer, it is $(a+1)$ m.

$\therefore$ then its area$=\frac{\sqrt{3}}{4}(a+1)^2$ sq. m. and by the condition this area is $\sqrt{3}$ sq. m. more than the previous area.

$$\therefore \quad \frac{\sqrt{3}}{4}(a+1)^2=\frac{\sqrt{3}}{4}a^2+\sqrt{3}$$

$$\text{or} \quad \frac{\sqrt{3}}{4}a^2+\frac{\sqrt{3}}{2}a+\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{4}a^2+\sqrt{3}$$

$$\text{or} \quad \frac{\sqrt{3}}{2}a=\sqrt{3}-\frac{\sqrt{3}}{4}=\frac{3\sqrt{3}}{4}, \quad \therefore\ a=\frac{3\sqrt{3}}{4}\times\frac{2}{\sqrt{3}}=\frac{3}{2}.$$

$\therefore$ the required length of the side$=\frac{3}{2}$ m.$=1$ m. 5 dm.

12. What is the area of a right-angled triangle whose hypotenuse and one side is 6·5 ft. and 5·2 ft. respectively. Find the ratio of the segments of the hypotenuse into which it is divided by the perpendicular drawn from the right angle on it. [C. S.]

Suppose, the third side of the triangle is a ft.

$\therefore$ $a^2=(6{\cdot}5)^2-(5{\cdot}2)^2=42{\cdot}25-27{\cdot}04=15{\cdot}21$

$\therefore$ $a=\sqrt{15{\cdot}21}=3{\cdot}9.$ $\quad\therefore$ the third side$=3{\cdot}9$ ft.

$\therefore$ the area of the right-angled triangle

$=\frac{1}{2}\times3{\cdot}9\times5{\cdot}2$ sq. ft.$=10{\cdot}14$ sq. ft.

Again, suppose that the hypotenuse of the triangle is divided by the perpendicular into two segments x and y.

Now, from Geometry we have $(3{\cdot}9)^2=x\times$hypotenuse and $(5{\cdot}2)^2=y\times$hypotenuse.

$$\therefore \quad \frac{x\times\text{hypotenuse}}{y\times\text{hypotenuse}}=\frac{(3{\cdot}9)^2}{(5{\cdot}2)^2}=\frac{15{\cdot}21}{27{\cdot}04}, \quad \therefore\ \frac{x}{y}=\frac{15{\cdot}21}{27{\cdot}04}=\frac{9}{16}.$$

$\therefore$ the required ratio$=9:16.$

Exercise 2

1. In a right-angled triangle the sides containing the right angle are 15 cm. and 20 cm. respectively. Find the hypotenuse and the area.

2. A man, 6 ft. high, stands at a distance of 90 ft. from a tower 126 ft. high. Find the distance of the top of the tower from his head.

3. One side of a right-angled triangle is 36 cm. and the sum of the hypotenuse and the other side is 54 cm. Find the hypotenuse and the other side.

4. One side of a right-angled triangle is 3925 ft., the difference between the hypotenuse and the other side is 625 ft.; find the hypotenuse and the other side. [R. U. S.]

5. A post 32 metres high is broken by the wind, its top touching the ground at a distance of 8 m. from the foot of the post. Find the height at which the post is broken.

6. In a certain lake, the tip of a bud of lotus was seen a foot above the surface of water. Forced by the wind, it gradually advanced, and was just submerged at a distance of 4 ft. Find the depth of water in the lake. [P.U. 1921]

7. Find the height of an equilateral triangle standing on the base 8 cm. long.

8. A ladder, 24 ft. long, stands upright against a wall, how far must the bottom of the ladder be pulled out so as to lower the top by 3 ft.? [A. U.]

9. A cow is tethered to a post 8 ft. from a long straight hedge with a rope 17 ft. long. How much of the hedge can she nibble?

10. The diagonal of a square court is 300 ft.; find its area in square yards. [R. U. S.]

11. The area of a right-angled triangle is 2 acres and one of the sides containing the right angle is 110 yds., find the other side.

12. Find the area of the triangle whose sides are 1 ft. 8 in., 2 ft. 10 in. and 3 ft. 6 in.

13. The sides of a triangle are in the ratio of 3 : 4 : 5 and its perimeter is 24 cms. Find its area.

14. The cost of paving a triangular court came to £100 at 1 s. 3 d. per sq. ft.; if one of the sides be 24 yds. long, find the length of the other two equal sides. [R. U. S.]

15. If the perpendiculars drawn to the sides of an equilateral triangle from any point within it measure 8, 10 and 12 ft. respectively, find its area and the length of each side.

16. The area of an equilateral triangle is equal to that of a square whose diagonal is $10\sqrt{3}$ ft. Find the area of the square drawn on a side of the triangle.

17. The sides of a triangle are as 6 : 7 : 8 and its perimeter is 462 feet. Find the area of the triangle.

CIRCLE

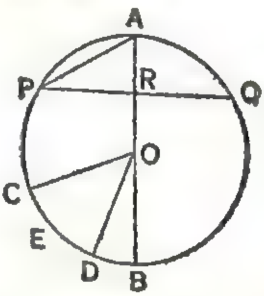
You are already acquainted with the definitions of circle, centre, diameter, radius, circumference, arc, chord, etc.

A chord of the arc is the straight line joining the two extremities of an arc.

The straight line which joins one extremity of an arc and the middle point of the arc is called the chord of half an arc.

A segment of a circle. A segment of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.

In the given figure, O is called the centre, the curved line PAQB is the circumference, AB the diameter, OA, OB, OC or OD ($\frac{1}{2}$ of the diameter) radius, PAQ the arc and PQ the chord.



The figure PAQP which is bounded by the arc PAQ and the chord PQ is called a segment of the circle. The figure PBQP is also another segment of the circle.

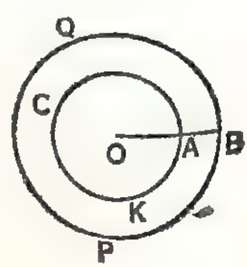
$\therefore$ a circle is divided into two segments by any chord.

Sector of a circle. A sector of a circle is a figure bounded by two radii and the arc intercepted between them.

In the given diagram, the figure OCEDO, which is bounded by the two radii OC and OD and the arc CED, is called the sector. The angle COD is called the *angle of that sector.*

Circular ring. Circles that have the same centre are said to be concentric circles.

The figure bounded by the circumferences of two concentric circles is said to be a *circular ring*.



In the given diagram, the figure bounded by the circumferences PBQ and ACK is a circular ring. The external circumference BPQ is the *outer ring* and the internal circumference AKC is the *inner ring.*

Circumference

It can be proved by Geometry that the ratio of the circumference and diameter of a circle i.e., $\frac{\text{circumference}}{\text{diameter}}$ is a constant quantity, that is to say, that this is the same for all circles. This constant ratio of circumference to diameter in any circle is denoted by the Greek letter π (Pi); so that $\frac{\text{circumference}}{\text{diameter}}=\pi$ (generally π is taken to be equal to $\frac{22}{7}$), $\pi=\frac{22}{7}$, or $\pi=3\cdot1415926\ldots$

The circumference of any circle $=\pi\times$ diameter $=2\pi\times$ radius.

$\therefore\quad c=2\pi r$ (here c is the circumference and r is the radius)

$\therefore\quad$ Diameter $=\frac{c}{\pi}$, radius $=\frac{c}{2\pi}$.

The area of a circle $=\pi\times(\text{radius})^2=\pi r^2$ [r is the radius of the circle].

If the radii of two concentric circles be R and r respectively, then the area of the circular ring bounded by the circumferences of two circles $=\pi R^2-\pi r^2=\pi(R^2-r^2)=\pi(R+r)(R-r)$.

Examples [3]

[If not otherwise stated, take $\pi=\frac{22}{7}$]

1. The diameter of a circle is 2 dm. 1 cm., find the circumference.

The required circumference $=\pi\times$ diameter $=\frac{22}{7}\times21$ cm. $=66$cm. $=6$ dm. 6 cm.

2. Two wheels make 32 and 48 revolutions in going over a distance of one mile. Find the difference between their radii.

Suppose, the radii of the two wheels are R and r respectively.

The circumference of the first wheel $=2\pi R=2\times\frac{22}{7}\times R=\frac{44}{7}R$.

Again, it revolves 32 times to go 1 mile,

$\therefore\quad$ its circumference $=\frac{1 \text{ mile}}{32}=\frac{1760}{32}$ yds $=55$ yds.

$\therefore\quad \frac{44R}{7}=55$ yds., $\therefore$ R $=\frac{55\times7}{44}$ yds $=\frac{35}{4}$ yds.

Again, the circumference of the second wheel $=2\pi r=2\times\frac{22}{7}\times r=\frac{44}{7}r$. $\because$ it revolves 48 times to go a mile,

$\therefore\quad$ its circumference $=\frac{1760}{48}$ yds. $=\frac{110}{3}$ yds.

$\therefore\quad \frac{44}{7}r=\frac{110}{3}$, $\therefore\ r=\frac{110}{3}\times\frac{7}{44}$ yds. $=\frac{35}{6}$ yds.

$\therefore\quad R-r=\frac{35}{4}$ yds. $-\frac{35}{6}$ yds. $=\frac{35}{12}$ yds. $=2$ yds. 2 ft. 9 in.

3. It cost Rs. 16. 8 as. to fence a circular lawn at 6 as. per yard. Find the radius of the lawn.

Here, circumference=(Rs. 16. 8 as.÷6 as.) yds.$=\frac{33}{2}\times\frac{8}{3}$ yds. $=44$ yds. $\therefore$ $2\pi r=44$ yds., or, $\frac{2\times 22}{7}r=44$ yds.

$\therefore$ r (radius)$=\frac{44\times 7}{2\times 22}$ yds.$=7$ yds.

4. The circumference of a circle is half as long again as that of another circle. If the difference of their radii is 1 ft. 9 in.; find them.

Let the radii of the two circles be R and r respectively.

$\because$ the circumference of the first circle is $1\frac{1}{2}$ times that of the second circle, $\therefore$ the radius of the first circle must be $1\frac{1}{2}$ times the radius of the second circle.

$\therefore$ $R=\frac{3}{2}r$. Again, $R-r=1$ ft. 9 in.$=21$ in. (by hypothesis),

$\therefore$ $\frac{3}{2}r-r=21$ in. or, $\frac{1}{2}r=21$ inches.

$\therefore$ $r=42$ in.$=3$ ft. 6 in. and $R=\frac{3}{2}r=5$ ft. 3 in.

5. The difference between the circumference and diameter of a circle is 60 cm.; find the radius. [M. T.]

Let r be the radius, then diameter$=2r$ and circumference$=2\pi r=2\times\frac{22}{7}r=\frac{44}{7}r$. $\therefore$ from the given condition $\frac{44}{7}r-2r=60$ cm. or, $\frac{30}{7}r=60$ cm. $\therefore r=\frac{60\times 7}{30}$ cm.$=14$ ft., $\therefore$ the reqd. radius$=14$ cm.

6. The diameters of two given circles are as 3 : 4, and the sum of the areas of the circles is equal to the area of a circle whose diameter measures 30″. Find the diameters of the given circles.

$\because$ The ratio of the diameters of the two circles$=3:4$,

$\therefore$ the ratio of their radii$=3:4$. Suppose, the radius of the smaller circle$=r$, $\therefore$ the radius of the other circle$=\frac{4}{3}r$.

$\therefore$ the sum of the areas of the two given circles

$$=\pi r^2+\pi\left(\tfrac{4}{3}r\right)^2=\pi r^2+\tfrac{16}{9}\pi r^2=\tfrac{25}{9}\pi r^2.$$

$\because$ the diameter of the third circle$=30$ in., $\therefore$ its radius$=15$ in.

$\therefore$ its area$=\pi(15)^2=225\pi$.

$\therefore$ by the given condition, $\dfrac{25}{9}\pi r^2=225\pi$, or, $r^2=\dfrac{225\pi\times 9}{25\pi}=81$,

$\therefore$ $r=9$, i.e., the radius of the first circle$=9$ inches.

$\therefore$ the diameter of the first circle$=18$ inches and that of the second circle$=\frac{4}{3}\times 18$ in.$=24$ inches.

7. The sum of two radii of two circles is 7 cm. and the difference of their circumferences is 8 cm. Find the two circumferences.

Suppose, the radius of one circle is r cm.,

$\therefore$ the radius of another circle $=(7-r)$ cm.

Now, the circumference of the first circle $=2\pi r$ cm. and the circumference of the second circle $=2\pi(7-r)$ cm.

$\because$ the difference of the two circumference $=8$ cm.,

$\therefore$ $2\pi r-2\pi(7-r)=8$, or, $2\times 2\pi r-14\pi=8$,

or, $2\times 2\pi r-14\times\frac{22}{7}=8$, or, $2\times 2\pi r=8+44=52$. $\therefore$ $2\pi r=26$.

$\therefore$ the required circumference of the first circle $=26$ cm. and that of the second circle $=2\pi(7-r)$ cm. $=(44-2\pi r)$ cm. $=(44-26)$cm. $=18$ cm.

8. A man by walking diametrically across a circular grass-plot finds it has taken him 45 sec. less than if he had kept to the path round the outside; if he walks 80 yards a minute, what is the diameter of the grass-plot? [C. U.]

Suppose, the radius of the circular plot $=r$ yds.

$\therefore$ its diameter $=2r$ yds., and circumference $=2\pi r=\frac{44}{7}r$ yds.

$\therefore$ the time taken for crossing the field along the diameter

$=(2r\div 80)$ minutes $=\frac{r}{40}$ minutes.

Again, the time taken to walk round the circumference $=(\frac{44}{7}r\div 80)$ minutes $=\frac{11}{140}r$ minutes.

The difference of these two times $=45$ seconds $=\frac{3}{4}$ minutes (by hypothesis)

$\therefore$ $\frac{11}{140}r-\frac{r}{40}=\frac{3}{4}$, or, $\frac{3}{56}r=\frac{3}{4}$, $\therefore$ $r=\frac{3}{4}\times\frac{56}{3}=14$,

$\therefore$ the required diameter $=2r$ yds. $=28$ yds.

9. A road runs round a circular grass-plot of which the outer circumference is 500 yds. and the inner circumference is 300 yds.; find the width of the road. [P. U.]

Suppose, the radius of the outer ring is R and the radius of the inner ring is r. $\therefore$ the width of the path $=R-r$.

$\because$ the circumference of the outer ring $=500$ yds. $\therefore$ $2\pi R=500$,

$\therefore$ $R=\frac{500}{2\pi}=\frac{500\times 7}{44}=\frac{875}{11}$ (yds.)

Again, the circumference of the inner ring $=300$ yds.

$\therefore$ $2\pi r=300$, $\therefore$ $r=\frac{300\times 7}{44}=\frac{525}{11}$ (yds.)

$\therefore$ the required width of the path $=\frac{875}{11}$ yds. $-\frac{525}{11}$ yds.

$=\frac{350}{11}$ yds. $=31\frac{9}{11}$ yds.

10. A wire can be bent into a circle of diameter of 4 ft. 8 in. If the wire can be bent into the form of a square, what will be the length of its side ?

Here, the diameter of the circle $=4$ ft. 8 in.

$\therefore$ the radius $=2$ ft. 4 in. $=28$ in.

$\therefore$ the length of the wire $=$ circumference of the circle

$$=2\pi r=2\times\frac{22}{7}\times 28 \text{ in.}=176 \text{ in.}$$

$\therefore$ the perimeter of the square $=176$ inches.

$\therefore$ the length of each side of the square $=176$ in. $\div 4=44$ in.

$=3$ ft. 8 in.

11. The hands of a clock are 5 in. and 4 in. long respectively. Find the difference of the distances traversed by their extremities in 2 days 6 hours.

The extremity of the minute hand of the clock will move in 1 hour a distance equal to the circumference of the circle having a radius of 5 inches.

$\therefore$ in 1 hour the minute hand will move $2\pi\times 5$ inches.

$\therefore$ in 2 days 6 hours or in 54 hours it will move

$2\times\frac{22}{7}\times 5\times 54$ in. or $\frac{220\times 54}{7}$ inches.

Again, in 12 hours the extremity of the hour hand will move a distance equal to the circumference of the circle having 4 inches radius, *i.e.*, $2\pi\times 4$ inches.

$\therefore$ it will travel in 54 hours $\dfrac{2\pi\times 4\times 54}{12}$ or $\dfrac{36\times 22}{7}$ inches.

$\therefore$ the difference of the two distances $=\left(\frac{220\times 54}{7}-\frac{22\times 36}{7}\right)$ inches $=\frac{22}{7}(540-36)$ inches $=\frac{22}{7}\times 504$ inches $=44$ yards.

[**Relating to area of the circle**]

12. Find the area of the circle whose diameter is 12·6 cm.

Here, diameter $=12·6$ cm., $\therefore$ the radius $=6·3$ cm.

The area of the circle $=\pi r^2=\frac{22}{7}\times(6·3)^2$ sq. cm.

$=\frac{22}{7}\times\frac{63\times 63}{10\times 10}$ sq. cm. $=124·74$ sq. cm.

13. The area of a circle is 2 sq. yds. 8 sq. ft. 106 sq. in., find its radius.

The area of the circle $=2$ sq. yds. 8 sq. ft. 106 sq. in.
$=3850$ sq. inches.

$\therefore$ $\pi r^2=3850$ sq. in., or, $\frac{22}{7}\times r^2=3850$ sq. in.,

or, $r^2=\frac{3850\times7}{22}$ sq. in. $=5^2\times7^2$ sq. in., $\therefore r=5\times7$ in. $=35$ in.

$\therefore$ the required radius $=35$ inches $=$ 2 ft. 11 inches.

14. The area of a circle is 385 acres, find the circumference of the circle. [P. U.]

The area of the circle $=\pi r^2=385$ acres (by hypothesis),

$\therefore$ $\pi r^2=385\times4840$ sq. yds.

or, $r^2=\frac{385\times4840\times7}{22}$ sq. yds. $=35\times7\times2420$ sq. yds.

$\therefore$ $r=\sqrt{35\times7\times2420}$ yds. $=770$ yds.

$\therefore$ the required circumference $=2\pi r=2\times\frac{22}{7}\times770$ yds.
$=4840$ yards.

15. The radius of the outer circle of a ring is 342 ft. and the radius of the inner circle is half of that ; find the area of the ring? [P. U.]

The radius of the outer circle $=342$ ft.

$\therefore$ the area of that circle $=\pi r^2=\frac{22}{7}\times(342)^2$ sq. ft.

Again, the radius of the inner circle $=\frac{1}{2}\times342$ ft. $=171$ ft.

$\therefore$ the area of that circle $=\frac{22}{7}\times(171)^2$ sq. ft.

$\therefore$ the area of the circular ring $=$ the difference of the areas of the two circles $=\frac{22}{7}(342)^2$ sq. ft. $-\frac{22}{7}\times(171)^2$ sq. ft.

$=\frac{22}{7}\{(342)^2-(171)^2\}$ sq. ft. $=\frac{22}{7}(342+171)(342-171)$ sq. ft.

$=\frac{1929906}{7}$ sq. ft. $=30633$ sq. yds. $3\frac{6}{7}$ sq. ft.

16. A circular silver plate costs £13. 15s. at 3s. 6d. per square inch, find the radius of the plate.

The total cost of the silver plate $=$ £13. 15s. $=275$s.,

and the cost of 1 sq. inch of the plate $=$ 3s. 6d. $=\frac{7}{2}$s.

$\therefore$ the area of the silver plate $=(275\div\frac{7}{2})$ sq. in. $=\frac{275\times2}{7}$ sq. in.

i.e., $\pi r^2=\frac{275\times2}{7}$ sq. in., or, $\frac{22}{7}\times r^2=\frac{275\times2}{7}$ sq. in.

or, $r^2=\frac{275\times2}{7}\times\frac{7}{22}$ sq. in. $=25$ sq. in. $\therefore$ $r=5$ inches.

$\therefore$ the required radius $=5$ inches.

17. A road runs round a circular grass-plot, the outer circumference is 500 yards and the inner circumference is 300 yds. Find the area of the road. [P. U.]

Here, the path is bounded by two concentric circles.

Suppose, the radii of the outer and inner boundaries are R and r respectively.

$\therefore\ 2\pi R=500$, or, $\frac{44}{7}R=500$, $\therefore\ R=\frac{500\times7}{44}$;

and $2\pi r=300$, or, $\frac{44}{7}r=300$, $\therefore\ r=\frac{300\times7}{44}$.

$\therefore$ the required area of the path $=\pi(R^2-r^2)$ sq. yds.

$=\frac{22}{7}\{(\frac{500\times7}{44})^2-(\frac{300\times7}{44})^2\}$ sq. yds.

$=\frac{22}{7}\times(\frac{7}{44})^2\{(500)^2-(300)^2\}$ sq. yds.

$=\frac{22}{7}\times\frac{7\times7}{44\times44}\times800\times200$ sq. yds.

$=\frac{140000}{11}$ sq. yds. $=12727\frac{3}{11}$ sq. yds.

18. Find the radius of a circle which is equal in area to a rectangle, 5 ft. 3 in. by 1 ft. 10 in.

The area of the circle = the area of the rectangle

$=5$ ft. 3 in. $\times1$ ft. 10 in. $=63$ in. $\times22$ in.

Again, the area of the circle $=\pi r^2$ (r being radius) $=\frac{22}{7}r^2$.

$\therefore\ \frac{22}{7}r^2=63\times22$ sq. in., or, $r^2=\frac{63\times22\times7}{22}$ sq. in. $=441$ sq. in.

$\therefore\ r=\sqrt{441}$ in. $=21$ in.

$\therefore$ the required radius $=21$ in. $=1$ ft. 9 in.

19. A road runs round a circular plot of ground, the outer circumference of the road is 44 metres longer than the inner; find the breadth of the road. [R. U. S.]

Let the radius of the outer circle be R and the radius of the inner circle be r. $\therefore$ the breadth of the road $=(R-r)$ metres.

Now, the circumference of the first circle $=2\pi r=\frac{44}{7}R$ metres and the circumference of the second circle $=2\pi r=\frac{44}{7}r$ metres.

$\therefore$ from the given condition, $\frac{44}{7}R-\frac{44}{7}r=44$ m.

or $\frac{44}{7}(R-r)=44$ m., $\therefore\ R-r=\frac{44\times7}{44}$ m. $=7$ m.

$\therefore$ the required breadth $=7$ metres.

20. A cow is tethered to a post by a rope so that she may graze over 38 sq. ft. 72 sq. in. of a grass plot. Find the length of the rope.

The cow can graze moving round in a circle and so the length of the rope is equal to the radius of that circle.

Here, the area of that circle is 38 sq. ft. 72 sq. in. or $\frac{77}{2}$ sq. ft.

Suppose, the length of the rope (*i.e.*, the radius of the circle) is r ft.

$\therefore \quad \pi r^2 = \frac{77}{2}$ sq. ft., or, $\frac{22}{7} r^2 = \frac{77}{2}$ sq. ft. or, $r^2 = \frac{49}{4}$ sq. ft.,

$\therefore \quad r = \sqrt{\frac{49}{4}}$ ft. $= \frac{7}{2}$ ft. $\therefore$ the required length of the rope $= 3\frac{1}{2}$ ft.

21. A circular grass-plot 40 ft. in radius is surrounded by a path; find the width of the path so that the area of the grass-plot and the path may be equal. [A. U.]

The radius of the circular grass-plot = 40 ft.

Let R be the radius of the whole circle including the path.

$\therefore$ the breath of the path $= (R - 40)$ ft.

Now, the area of the circular grass-plot $= \pi(40)^2$ and the area of the path $= \pi(R^2 - 40^2)$.

$\because$ these two areas are equal, $\therefore \quad \pi(R^2 - 40^2) = \pi . 40^2$

or, $R^2 - 40^2 = 40^2$, or, $R^2 = 2 \times 40^2$, $\therefore \quad R = 40\sqrt{2}$.

$\therefore$ the required breadth of the path $= (R - 40)$ ft.

$= (40\sqrt{2} - 40)$ ft. $= 40(\sqrt{2} - 1)$ ft. $= 40 \times {\cdot}414$ ft. $= 16{\cdot}56$ ft. (App.)

22. A circular grass-plot whose diameter is 40 yards contains a gravel path 1 yard wide running round it one yard from the edge. What will be the cost to turf the grass-plot at 4d. per square yard? [R. E.]

Here, the radius of the whole plot $= \frac{1}{2} \times 40$ yds. $= 20$ yards. The radius of the outer circle of the path is 19 yds., because the path is within the field and 1 yd. away from the boundary line of the plot.

Again, the path is 1 yd. wide, $\therefore$ the radius of its inner circle $= 18$ yds. [Draw the figure]

$\therefore$ the area of the path $= \pi(19^2 - 18^2)$ sq. yds. $= 37\pi$ sq. yds.

Again, the area of the whole plot including the path

$= \pi . (20)^2$ sq. yds. $= 400\pi$. sq. yds.

$\therefore$ the area of the remaining portion of the plot which is to be turfed $= (400\pi - 37\pi)$ sq. yds. $= \pi \times 363$ sq. yds. $= \frac{22}{7} \times 363$ sq. yds.

$\because$ the cost of 1 sq. yd. $= 4$d. $= \frac{1}{3}$s.

$\therefore$ the required cost $= \frac{22}{7} \times \frac{363}{3}$s. $= \frac{2662}{7}$s. $=$ £19. $3\frac{3}{7}$d.

23. A plot of land has the shape formed by a square having a semi-circle on the outside described on each of the four sides as diameter. If a side of the square be 54 feet, what is the rent of the land at Rs. 750 per acre ?

The radius of each semi-circle $=\frac{1}{2}\times 54$ ft. $=27$ ft.

$\therefore$ its area $=\frac{1}{2}\pi\times(27)^2$ sq. ft.

$\therefore$ the area of the whole plot of land = the area of the square + area of the 4 semi-circles $=(54^2+4\times\frac{1}{2}\times\pi\times 27^2)$ sq. ft.

$=(54^2+\frac{44}{7}\times 27^2)$ sq. ft. $=(54^2+\frac{11}{7}\times 54^2)$ sq. ft.

$=54^2(1+\frac{11}{7})$ sq. ft. $=54\times 54\times\frac{18}{7}$ sq. ft. $=\frac{54\times 54\times 18}{9\times 4840\times 7}$ acres.

$\therefore$ the required rent $=\frac{54\times 54\times 18}{9\times 4840\times 7}\times$ Rs. 750 = Rs. $\frac{109350}{847}$

= Rs. 129. 2 as. (App.).

24. A circular track 10 ft. wide has an outer radius 61 ft. Find the cost of paving it at $3\frac{3}{4}$ as. per square foot. What is the average length of the track ? What time is taken by a man running 10 times round the track (of average length) at the rate of 20 ft. per second ? [C. U. '52]

The outer radius of the track = 61 ft.

The track is 10 ft. wide, $\therefore$ the inner radius of the track $=(61-10)$ or 51 ft.

$\therefore$ the area of the track $=\pi(61^2-51^2)$ sq. ft. $=\frac{22}{7}\times 112\times 10$ sq. ft.

$\therefore$ the required cost of paving the track

$=\frac{22}{7}\times 1120\times\frac{15}{4}$ as. = Rs. 825.

Again, the track is 10 ft. wide, $\therefore$ its average length = the circumference of the circle having a radius equal to $(51+\frac{10}{2})$ or 56 ft.

$=2\times\pi\times 56$ ft. $=2\times\frac{22}{7}\times 56$ ft. $=352$ ft.

$\therefore$ the time taken to make 10 revolutions round the track $=\frac{352\times 10}{20}$ seconds = 2 minutes 56 seconds.

Exercise 3

[Take $\pi=\frac{22}{7}$]

1. The radius of a circle is 1 ft. 2 in. ; find its circumference.
2. The circumference of a wheel is 1 yd. 2 ft. 6 in., find its diameter.

3. The radius of a wheel is 21 ft. How many revolutions will it make in passing over $4\frac{1}{2}$ miles ?

4. What is the diameter of a wheel that makes 128 revolutions in going over 2 miles ?

5. Find the cost of fencing a circular courtyard 10 ft. 6 in. in diameter at 1 s. 6 d. per yard.

6. The difference between the circumferences of two circles is 22 ft. and the sum of their two radii is 10 ft. 6 in. ; find each circumference.

7. The diameter of a wheel is 63 in. and it makes 960 revolutions in 6 minutes. Find its speed per hour.

8. The difference between the circumference and the diameter of a circle is 45 inches ; find its radius.

9. The diameter of a wheel is 2 ft. 11 in. and it makes 96 revolutions per minute ; find its speed per hour.

10. The sum of the diameter and circumference of a circle is 87 inches ; find its radius and circumference.

11. A horse runs round a circular field in $1\frac{1}{2}$ seconds at the rate of 66 ft. per second. Find the radius of the field.

12. A road runs round a circular plot of land ; the outer circumference of the road is 912 metres and the inner 868 metres. Find the width of the path.

13. A cyclist takes 46 and 44 seconds respectively in going round the outer and inner edges of a circular path, 7 ft. 6 in. wide. Find the diameter of the circle forming the inner edge of the path.

14. The hands of a clock are 4 in. and 3 in. respectively. Find the difference of the distances traversed by their extremities in 30 hours.

15. Find the area of a circle whose diameter is 14 cm.

16. The area of a circle is 4 sq. ft. 40 sq. in. ; find its radius.

17. Find the circumference of a circle whose area is 1 sq. ft. 10 sq. in.

18. Find the area of a circle whose circumference is 66 inches.

19. The radii of the outer circle and the inner circle of a ring are 35 in. and 21 in. respectively ; find the area of the ring.

20. The area of a circle is 385 acres ; find its circumference. [P. U.]

21. A circular pond is to be dug ; what must be the length of the cord with which the circumference is to be described, so that it shall just occupy half an acre ? [R. U. S.]

22. The area of a circular ring is 352 sq. cm. and the radius of its outer circle is 16 cm. ; find the radius of the inner circle.

23. The diameter of a circular courtyard is 37 yds. 1 ft. ; find the cost of paving it at $4\frac{1}{2}$ s. per sq. yd.

24. Find the length of the rope by which a cow must be tethered so that she may graze over 616 sq. metres.

25. A circular silver plate costs Rs. 38. 8 as. at Rs. 36 per sq. ft. ; find the radius of the plate.

26. Find the side of a square which is equal in area to a circle of radius 4 ft. 8 in.

27. A road runs round a circular garden, the outer circumference of the road being 22 ft. longer than the inner. Find the width of the road.

28. A plot of land has the shape formed by a square having a semi-circle about each of the four sides as diameter. If the length of the side of the square be 60 ft., calculate the amount of rent realisable when the plot is let out at Rs. 1100 an acre per annum. [C. U. I. A. '55]

29. A circular track 6 ft. wide has an outer radius 52 ft. Find the cost of paving it at 5 as. per sq. ft. What is the average length of the track ? What time is taken by a boy in going round the track (of average length) at 22 ft. per second ?

30. The inner diameter of a circular building is 68 ft. 10 in. and the thickness of the wall is 22 inches ; find how many square ft. of ground the base of the wall occupies. [R. U. S.]

RECTANGULAR PARALLELOPIPED

Cubes and Parallelopipeds have been dealt with in the chapter of the cubic measure.

If the length, breadth and thickness or height be a, b, c units of length respectively,

The **volume of a rectangular parallelopiped**
$=$length $\times$ breadth $\times$ thickness $= abc$ cubic units (units of volume)

The **total surface of a rectangular parallelopiped**
$=2$ (length $\times$ breadth $+$ breadth $\times$ height $+$ length $\times$ height)
$=2\,(ab+bc+ca)$ units of area.

The **volume of a cube** $=(\text{edge})^3=(\text{side})^3=a^3$ units of volume.

The **total surface of the cube** $=6\times(\text{side})^2=6a^2$ units of area.

The length of the **diagonal** of a rectangular parallelopiped
$=\sqrt{a^2+b^2+c^2}$ units of length,

The length of the **diagonal of a cube**
$=\sqrt{a^2+a^2+a^2}=\sqrt{3a^2}=a\sqrt{3}$ units of length.

Examples 4

1. The length, breadth and height of a rectangular parallelopiped are 16 m., 12 m. and 4·5 m. respectively. Find the whole surface and volume.

Here, the length $a=16$m., breadth $b=12$m. and height $c=\frac{9}{2}$m.
whole surface $=2(ab+bc+ca)$

$\therefore$ the required whole surface $=2(16\times 12+12\times\frac{9}{2}+16\times\frac{9}{2})$ sq. m. $=636$ sq. m.

and the required volume $=abc$
$=16$ m. $\times 12$ m. $\times\frac{9}{2}$ m. $=864$ cu. metres.

2. The volume of a cube is 216 cubic cm.; find its edge, the length of the diagonal and the whole surface.

$\because$ the volume of the cube $=(\text{edge})^3$,

$\therefore$ here $(\text{edge})^3=216$ cu. cm., $\therefore$ each edge $=\sqrt[3]{216}$ cm. $=6$cm.

The length of the diagonal of the cube $=a\sqrt{3}=6\sqrt{3}$ cm.;

and the whole surface $=6a^2=6\times(6)^2$ or 216 sq. cm.

3. The length, breadth and height of a rectangular parallelopiped are in the ratio of 4 : 3 : 2 and its whole surface is 468 sq. cm. ; find the dimensions of the rectangular parallelopiped.

Here, length : breadth : height $=4:3:2$;

Suppose, length $=4a$ cm., breadth $=3a$ cm. and height $=2a$ cm.

$\therefore$ the whole surface $=2(4a\times 3a+4a\times 2a+3a\times 2a)=52a^2$.

$\therefore$ here $52a^2=468$, or, $a^2=9$, $\therefore$ $a=3$.

$\therefore$ the required length $=4\times 3$ cm. $=12$ cm. ; breadth $=3\times 3$ cm. $=9$ cm. ; height $=2\times 3$ cm. $=6$ cm.

4. The diagonal of each surface of a cube is $4\sqrt{2}$ cm. Find the diagonal and the volume of the cube.

Let the edge (side) be a cm., $\therefore$ the square of the diagonal of each surface $=2a^2$. $\therefore$ the diagonal of each surface $=\sqrt{2a^2}=a\sqrt{2}$ cm. $\therefore$ here $a\sqrt{2}=4\sqrt{2}$, $\therefore$ $a=4$ cm.

$\therefore$ the length of the diagonal of the cube $=a\sqrt{3}=4\sqrt{3}$ cm. and the volume of the cube $=a^3=(4)^3$ cu. cm. $=64$ cu. cm.

5. The diagonal of a rectangular parallelopiped is 12 cm. and the sum of its length, breadth and height is 17 cm. ; find its whole surface.

Here $a+b+c=17$ cm. and diagonal $=12$ cm.

i.e., $\sqrt{a^2+b^2+c^2}=12$, $\therefore$ $a^2+b^2+c^2=144$.

Now, $(a+b+c)^2=(17)^2$, or, $a^2+b^2+c^2+2(ab+bc+ca)=289$,

$\therefore$ $144+2(ab+bc+ca)=289$, $\therefore$ $2(ab+bc+ca)=289-144=145$.

$\therefore$ the whole surface $=2(ab+bc+ca)=145$ sq. cm.

6. A room is 12 metres long, 10 metres broad and 9 metres high ; find the longest possible rod that can be placed within it.

The room is a rectangular parallelopiped in size, $\therefore$ the length of the longest possible rod is equal to its diagonal.

$\therefore$ the length of the rod $=\sqrt{12^2+10^2+9^2}$ m.

$=\sqrt{325}$ m. $=5\sqrt{13}$ metres.

7. The whole surface of a rectangular parallelopiped which is 18 m. long and 5 m. high is 732 sq. metres ; find the breadth.

Let the breadth be a metres.

$\therefore$ the whole surface

$=2$ (length $\times$ breadth $+$ length $\times$ heigth $+$ breadth $\times$ height)

$=2(18\times a+18\times 5+a\times 5)$ sq. m. $=2(23a+90)$ sq. m.

$\therefore$ $2(23a+90)=732$, or, $23a+90=366$

or, $23a=366-90=276$, $\therefore$ $a=\frac{276}{23}=12$.

$\therefore$ the required breadth $=12$ metres.

8. The ratio of the length and breadth of a rectangular parallelopiped is 4 : 3 and its volume is 2304 cu. cm. The cost of lining its base with lead is Rs. 19. 20 P. at 10 P. per sq. cm. Find its dimensions.

The area of the base=(Rs. 19. 20 P.÷10 P.) sq. cm.

=192 sq. cm.

Let its length be $4a$ and breadth be $3a$.

$\therefore$ $4a \times 3a = 192$ sq. cm., or, $a^2 = 16$ sq. cm., $\therefore$ $a = 4$ cm.

$\therefore$ length$=4 \times 4$ cm.$=16$ cm. and breadth$=3 \times 4$ cm.$=12$ cm.

Again, $\because$ the volume of the parallelopiped=2304 cu. cm.,

$\therefore$ 16 cm.$\times$12 cm.$\times$height=2304 cu. cm.

or 192 sq. cm.$\times$height=2304 cu. cm.

$\therefore$ height$=\frac{2304}{192}$ cm.$=12$ cm.

$\therefore$ the required dimensions are :

length=16 cm., breadth =12 cm. and height=12 cm.

Exercise 4

1. The length, breadth and height of a rectangular parallelopiped are 18 cm., 12 cm. and 10 cm. respectively. Find its whole surface and volume.

2. Find the length of the edge and the diagonal of a cube whose volume is 125 cubic centimetres.

3. The edge of a cube is 4 cm. ; find its whole surface and volume.

4. The area of one surface of a cube is 25 square metres. If each cubic metre of it weighs 20 grams, find the weight of the cube.

5. The length, breadth and height of a rectangular parallelopiped are 24 m., 8 m. and 6 m. respectively ; find its volume and the length of the diagonal.

6. The length, breadth and height of a rectangular parallelopiped are as 9 : 5 : 4 and the whole surface is 1818 sq. metres ; find the dimensions.

7. The edge of a cube is 3 cm. ; find the length of its diagonal.

8. A cubic metre of copper plate is hammered into a plate 2 metres square ; find the thickness of the plate.

9. The diagonal of each surface of a cube is 8 $\sqrt{2}$ cm. ; find the length of the diagonal and the volume of the cube.

10. A room, rectangular parallelopiped in shape, is 12 m. long, 4 m. broad 3 m. high. What is the longest possible rod that can be placed within it ?

11. The whole surface of a rectangular parallelopiped, 18 m. long and 12 m. broad, is 732 sq. metres ; find its height.

12. Find the breadth of a rectangular solid whose length is 12 m., breadth 10 m. and the whole surface 592 sq. metres.

13. The sum of the length, breadth and thickness of a rectangular parallelopiped is 21 cm. and the length of the diagonal is 14 cm. ; find the whole surface.

14. The length of a rectangular parallelopiped is thrice its breadth and 5 times its height. If its volume be 14400 cu. cm., find the whole surface.

15. The length, breadth and thickness of a rectangular parallelopiped are as 5 : 4 : 3 and the length of its diagonal is 10 $\sqrt{2}$ cm. ; find the whole surface.

16. The length, height and diagonal of a rectangular parallelopiped are 36, 12 and 42 metres respectively ; find its breadth.

17. A cistern, 10 metres long and 8 metres broad, holds 20 cubic metres of water. Find the depth of water.

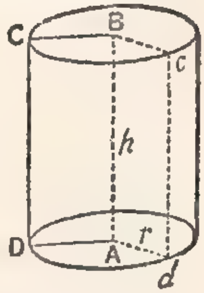
18. How many cubes whose edge measures 3 cm. can be formed by melting a cubic block of metal whose edge is 15 cm. ?

19. How many bricks, each 10 cm. by 8 cm. by 6 cm. will be required to build a wall 16 m. by 12 cm. by 8 m. ?

20. The cost of painting the six surfaces of a cube at 5 P. per square centimetre is Rs. 19. 20 P., find the volume of the cube.

A right circular cylinder is a solid generated by the revolution of a rectangle round one of its sides as its axis.

A drum, a lead pencil, a heap of pice placed one upon another or a uniformly thick spherical shell, etc. may be taken as circular cylinders.



If the rectangle ABCD is made to turn round the side AB, it generates a right circular cylinder as shown in the figure. AB is said to be the axis of the cylinder. The opposite side CD generates the curved surface of the cylinder and is called the generating line. C and D always remain equidistant from B and A respectively. So, AD, BC describe respectively two plane parallel circles. AB is perpendicular to both the ends. The end on which the cylinder stands is called its base.

Area of the curved surface of a right circular cylinder :

Draw perpendicularly a straight line along the curved surface of a cylinder and cut it along that line. If you now fully stretch the curved surface, it will turn into a plane surface. Evidently this plane surface will be a rectangle whose length and breadth are equal respectively to the circumference and height of the cylinder.

$\therefore$ **the area of the curved surface** of the right circular cylinder = circumference of the base $\times$ height.

If r be the radius of the base and h the height of a right circular cylinder,

the **area of the curved surface** of the cylinder

= (circumference of the base) $\times$ height

$= 2\pi rh$ units of area ;

and the **area of the whole surface**

= area of the curved surface + area of the two ends

$= 2\pi rh + 2\pi r^2$ ($\because$ area of a circle $= \pi r^2$)

$= 2\pi r\,(h+r)$ units of area.

The **volume** of the cylinder

= (area of the base) $\times$ height $= \pi r^2 h$ units of volume.

Examples [5]

[Take $\pi=\frac{22}{7}$]

1. The height of a right circular cylinder is 1m. 4dm. and the diameter of the base is 5m. Find the area of the curved surface of the cylinder.

The area of the curved surface of the cylinder $=2\pi rh$.

Here, r=radius$=\frac{1}{2}$ diameter$=\frac{5}{2}$m., h (height)=1m. 4dm.$=\frac{7}{5}$m.

$\therefore$ the required area$=2\times\frac{22}{7}\times\frac{5}{2}$ m.$\times\frac{7}{5}$m.=22 sq. metres.

2. The height of a cylindrical column is 9 metres and the radius of the base is 1·75 m. Find the area of the whole surface.

Here, radius (r)=1·75 m.$=\frac{7}{4}$ m., height (h)=9 m.

$\therefore$ the area of the curved surface of the column$=2\pi rh$

$=2\times\frac{22}{7}\times\frac{7}{4}$ m.$\times$9 m.=99 sq. m.

Again, the area of the two circular ends$=2\pi r^2$

$=2\times\frac{22}{7}\times(\frac{7}{4})^2$ sq. m.$=19\frac{1}{4}$ sq. m.

$\therefore$ the area of the whole surface$=(99+19\frac{1}{4})$ sq. m.

$=118\frac{1}{4}$ sq. m.=118 sq. m. 25 sq. dm.

3. Find the radius of the base of a cylindrical column, 8 metres high, whose curved surface is 2464 square metres.

The area of the curved surface of a cylinder$=2\pi rh$.

Here, the given area=2464 sq. m. and height (h)=8 m.

$\therefore$ $2\pi rh$=2464 sq. m. (r=radius),

or, $2\times\frac{22}{7}\times$8 m.$\times r$=2464 sq. m., $\therefore$ $r=\frac{2464\times7}{2\times22\times8}$ m.=49 m.

$\therefore$ the required radius=49 metres.

4. The height of a right circular cylinder is 16 metres and the radius of the base is 3 m. 5 dm. Find its volume.

The volume of the cyiinder=area of the base$\times$height$=\pi r^2h$.

Here, $r=3\frac{1}{2}$ m. and h=16 m.

$\therefore$ the required volume$=\frac{22}{7}\times(\frac{7}{2})^2\times$16 cu. m.=616 cubic metres.

5. The diameter of the base of a cylindrical pillar is 7 metres and its height is 12 metres. Find the cost of constructing it at Rs. $2\frac{1}{8}$ per cubic metre.

Here, $r=\frac{7}{2}$ m., h=12 m., $\therefore$ the volume of the pillar$=\pi r^2h$

$=\frac{22}{7}\times(\frac{7}{2})^2\times$12 cu. m.=22$\times$7$\times$3 cu. m.

Cost per cu. m.=Rs. $2\frac{1}{8}$=Rs. $\frac{7}{8}$.

$\therefore$ the required cost of construction=Rs. $\frac{22\times7\times3\times7}{8}$=Rs. 1078.

6. An iron pipe is 3 inches in bore, $\frac{1}{2}$ in. thick, and 20 feet long. Find its weight supposing that a cubic inch of iron weighs 4·526 ounces.

The length of the pipe $=20$ feet $=240$ inches.

The internal radius of the pipe is $\frac{3}{2}$ inches and the iron plate is $\frac{1}{2}$ inch thick.

$\therefore$ its radius up to the external side $=(\frac{3}{2}+\frac{1}{2})$ or 2 inches.

$\therefore$ if the pipe be solid, the area of its circular end
$$=\frac{22}{7}\times 2^2 \text{ sq. inches}=\frac{88}{7} \text{ sq. inches.}$$

The area of its hollow circular end $=\frac{22}{7}\times(\frac{3}{2})^2$ sq. inches
$$=\frac{99}{14} \text{ sq. inches.}$$

$\therefore$ the volume of the iron plate $=(\frac{88}{7}-\frac{99}{14})$ sq. inches $\times$ height
$$=\frac{11}{2} \text{ sq. in.}\times 240 \text{ in.}=1320 \text{ cubic inches.}$$

The weight of 1 cubic inch of iron plate $=4·526$ ounces,

$\therefore$ the required weight $=4·526\times 1320$ ounces
$$=\frac{4·526\times 1320}{16} \text{ pounds}=373·395 \text{ lbs.}$$

7. The curved surface of a cylinder is 100 square centimetres and the diameter of the base is 20 cms.; find the volume of the cylinder. Also find the height to the nearest millimetre. [C.U. 1934]

The circumference of the base $=2\pi r=20\pi$ cm.
[$\because$ $2r$ (diameter) $=20$ cm.]

$\therefore$ the required height of the cylinder
$$=(1000\div 20\pi)\text{ cm.}=\frac{50}{\pi}\text{ cm.}$$

$$=\frac{50}{3·1416}\text{ cm. (taking } \pi=3·1416)= 15 \text{ cm. } 9 \text{ mm. (approximately.)}$$

Again, the volume of the cylinder
$$=\pi r^2 h=\pi\times(10)^2\times\frac{50}{\pi}\text{ cu.cm.}=5000 \text{ cu. cm.}$$

8. The curved surface of a cylindrical pillar is 264 square metres and its volume is 924 cubic metres. Find the diameter and the height of the pillar.

Suppose, the radius of the pillar $=r$ metres and height $=h$ metres.

$\therefore$ by the condition of the problem,
$$2\pi rh=264\ldots\ldots(1) \quad \text{and } \pi r^2 h=924\ldots\ldots(2)$$

$\therefore$ from (2)÷(1) we get $\frac{\pi r^2 h}{2\pi rh}=\frac{924}{264}$, or, $\frac{r}{2}=\frac{7}{2}$, $\therefore$ $r=7$.

$\therefore$ the required diameter $=2r=14$ metres.

Again, from (1) we get $2\pi\times 7\times h=264$, or, $2\times\frac{22}{7}\times 7\times h=264$, or, $44h=264$, $\therefore$ $h=6$. $\therefore$ the required height $=6$ metres.

9. How many pieces of coin $\frac{1}{4}$ cm. thick and 2 cm. in diameter can be made by melting a rectangular parallelopied of metal with dimensions of 22 cm., 6 cm. and 4 cm.?

The volume of the metallic parallelopiped $=22\times6\times4$ cubic cm.

and the volume of each coin

$$=\pi r^2h=\frac{22}{7}\times1^2\times\frac{1}{4}\text{ cu. cm.}=\frac{11}{14}\text{ cu. cm.}$$

$\therefore$ the required number of coins $=(22\times6\times4)\div\frac{11}{14}=672$.

Exercise 5

[Take $\pi=\frac{22}{7}$]

1. The length of a hollow cylinder is 10 metres and the diameter of the base is 7 metres. Find the area of its curved surface.

2. The circumference of the base of a cylindrical column is 4 feet 7 inches and its height is 12 yards. Find the area of its curved surface.

3. The diameter of the ends of the cylinder is 2 m. 8 dm. and its length is 16 dm. Find the area of its two ends.

4 The height of a right circular cylinder is 12 cm. and the diameter of the base is 7 cm. Find the area of the whole surface.

5. The height of a cylindrical pillar is 14 m. and its curved surface is 264 square metres. Find the radius of its base.

6. The height of a right circular chimney is 30 ft. and the radius of the base is 1 ft. 3 in. What is the cost of painting its curved surface at 2 as. per square foot?

7. If it costs Rs. 41. 25 P. to polish the curved surface of a cylindrical pillar 15 metres high at 25 P. per square metre, find the radius of its base.

8. The diameter of a circular cylinder 14 m. high is 6 metres. Find its volume.

9. The volume of a cylindrical pillar 1 Dm. 4 m. high is 539 cubic metres. Find the diameter of the base.

10. The diameter of the base of a cylindrical pillar is 4 metres and its height is 21 metres. Find the cost of constructing the pillar at 1·6 rupees per cubic metre.

11. The external and internal radii of the base of a hollow circular cylinder are 14 cm. and 7 cm. respectively. Find the area of one of its ends.

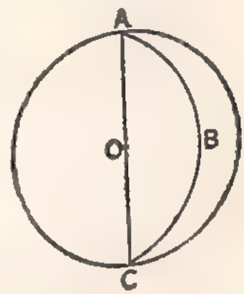
12. Find the weight of a cast-iron pipe whose length is 9 feet, the bore 3 in. and thickness of the metal is 1 inch. A cubic inch of cast-iron weighs $\frac{1}{4}$ lb. [R. U. S.]

13. 11 cubic centimetres of iron is drawn into a wire 56 cm. long. Find the radius of the end of the wire.

14. Find the cubic inches of a material in a cylindrical tube, the radius of the outer surface being 10 inches, the thickness 2 inches and the height 9 inches. [R. U. S.]

SPHERE

A **sphere** is a solid bounded by one surface and it may be seen to be generated by the revolution of a semi-circle about its diameter as axis. The surface described by the semi-circumference is called the surface of the sphere. The radius of this semi-circle is the radius of the sphere.



A tennis ball, a marble, etc. are familiar examples of a sphere.

If the radius of a sphere be r, then

(*a*) the **area of the sarface** of the sphere $=\pi\times(\text{diameter})^2=4\pi r^2$ units of area,

or, the area of the surface = the circumference of the generating circle × diameter $=2\pi r\times 2r$ units of the area.

(*b*) The **volume of the sphere** $=\frac{4}{3}\pi r^3$ units of volume.

Examples [6]

[Take $\pi=\frac{22}{7}$, if not otherwise stated]

1. Find the surface and the volume of a sphere whose diameter is 14 metres.

Here, r (radius) = 7m., ∴ the required area of the surface $=\pi r^2=4\times\frac{22}{7}\times(7)^2$ sq. m. = 616 sq. m.

The required volume $=\frac{4}{3}\pi r^3=\frac{4}{3}\times\frac{22}{7}\times(7)^3$ cu. m. $=1437\frac{1}{3}$ cu.m.

2. The surface of a sphere is 9856 sq. cm. Find its diameter.
$\because$ the area of the surface $=4\pi r^2$, $\therefore$ here, $4\times\frac{22}{7}\times r^2=9856$,
$\therefore r^2=\frac{9856\times7}{4\times22}=784$, $\therefore$ $r=\sqrt{784}$ cm. $=28$ cm.

$\therefore$ the required diameter $=2r=56$ cm.

3. There are as many cubic inches in the volume of a sphere as there are square inches in the area of its curved surface. Find the radius of the sphere.

Suppose, the radius of the sphere $=r$ inches.

$\therefore$ the area of its surface $=4\pi r^2$ square inches,

and its volume $=\frac{4}{3}\pi r^3$ cubic inches.

$\therefore$ by the condition of the problem, $4\pi r^2=\frac{4}{3}\pi r^3$

or, $\frac{r^3}{r^2}=\frac{4\pi}{\frac{4}{3}\pi}$, $\therefore r=3$. $\therefore$ the required radius $=3$ inches.

4. Find the radius of a sphere whose surface is equal to the curved surface of a right circular cylinder having height and diameter each 10 metres in length.

The radius of the cylinder $=5$ m.

The curved surface of the cylinder $=2\pi rh=2\pi\times5\times10=100\pi$;

Again, the surface of the sphere $=4\pi r^2$ (taking r for radius of the sphere).

$\therefore$ $4\pi r^2=100\pi$, or, $r^2=25$, $\therefore$ $r=5$.

$\therefore$ the required radius $=5$ metres.

5. A leaden sphere 1 inch in diameter is beaten into a circular sheet of uniform thickness equal to $\frac{1}{100}$ inch. Find the radius of the sheet. The radius of the sphere $=\frac{1}{2}$ inch.

$\therefore$ the volume of the sphere $=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\times(\frac{1}{2})^3=\frac{1}{6}\pi$

Let the radius of the sheet be r inches.

$\therefore$ the area of the sheet $=\pi r^2$ and it is $\frac{1}{100}$ inch thick.

$\therefore$ its volume $=\pi r^2\times\frac{1}{100}=\frac{\pi r^2}{100}$. $\therefore$ $\frac{\pi r^2}{100}=\frac{\pi}{6}$,

$\therefore$ $r^2=\frac{100}{6}=\frac{50}{3}$ $\therefore$ $r=\sqrt{\frac{50}{3}}=4\cdot0825$.

$\therefore$ the required radius of the sheet $=4\cdot0825$ inches (App.)

6. How many spherical bullets each 5 dm. in diameter can be cast from a rectangular block of lead 11 m. by 10 m. by 5 m. ? $[\pi=\frac{22}{7}]$

The volume of the block of lead $=11\text{m.}\times10\text{m.}\times5\text{m.}=550$ cu.m.

$\therefore$ the total volume of all the bullets $=550$ cu.m.

The radius of one bullet $=\frac{5}{2}$dm. $=\frac{1}{4}$m.

$\therefore$ the volume of 1 bullet $=\frac{4}{3}\pi r^3=\frac{4}{3}\times\frac{22}{7}\times(\frac{1}{4})^3=\frac{11}{7\times24}$ cu.m.

$\therefore$ the number of bullets $=(550$ cu.m. $\div\frac{11}{7\times24}$ cu. m.$)=8400$.

7. The external and internal radii of a sphere are 6 cm. and 3 cm. respectively ; find the volume.

Here, the difference of the volumes of two concentric spheres having 6 cm. and 3 cm. as their respective radius is the volume of the sphere.

$\therefore$ the required volume of the sphere $= \frac{4}{3}\pi \times (6)^3 - \frac{4}{3}\pi \times (3)^3$

$= \frac{4}{3}\pi(6^3 - 3^3) = \frac{4}{3} \times \frac{22}{7} \times 189$ cu. cm. $= 792$ cu. cm.

8. The external diameter of a sphere, made of iron sheet 2 in. thick, is one foot. If one cubic foot of iron weighs 450 lbs., find the weight of the sphere. $[\pi = \frac{22}{7}]$

The external radius of the sphere $= 6$ inches.

$\therefore$ its internal radius $= 6$ inches $- 2$ inches $= 4$ inches.

$\therefore$ the volume of the sphere $= \frac{4}{3}\pi(6^3 - 4^3) = \frac{4}{3} \times \frac{22}{7} \times 152$ cu.in.

$= \dfrac{4 \times 22 \times 152}{3 \times 7 \times (12)^3}$ cu.ft. $\because$ the weight of 1 cu. ft. of iron $= 450$ lbs.,

$\therefore$ the required weight of the sphere

$= \dfrac{4 \times 22 \times 152 \times 450}{3 \times 7 \times (12)^3}$ lbs. $= \dfrac{10450}{63}$ lbs. $= 165{\cdot}87$ lbs. (Approximately)

9. Three solid spheres of gold whose radii are 1 cm., 6 cm. and 8 cm. respectively are melted into a single solid sphere. Find the radius of the sphere so formed. [C. U. '56]

Let R be the radius of the new sphere.

$\therefore$ the volume of the new sphere $= \frac{4}{3}\pi R^3$.

Again, the volume of the first sphere $= \frac{4}{3}.\pi.1^3$; that of the second sphere $= \frac{4}{3}.\pi.6^3$ and the volume of the third sphere $= \frac{4}{3}.\pi.8^3$

$\therefore$ the total volume of the three spheres $= \frac{4}{3}\pi(1^3 + 6^3 + 8^3)$ cu.cm.

$= \frac{4}{3}.\pi \times (1 + 216 + 512)$ cu.cm. $= \frac{4}{3}.\pi \times 729$ cu.cm.

$\therefore$ $\frac{4}{3}.\pi.R^3 = \frac{4}{3}.\pi \times 729$, $\quad \therefore R^3 = 729 = 9 \times 9 \times 9$, $\quad \therefore R = 9$.

$\therefore$ the required radius of the new sphere $= 9$ cm.

Exercise 6

$[\text{Take } \pi = \frac{22}{7}]$

1. Find the surface of a sphere whose diameter is 5 dm. 6 cm.

2. The radius of a sphere is $3\frac{1}{2}$ dm. ; find the area of its surface.

3. Find the volume of a sphere having a diameter of 1 dm. 4 cm.

4. The surface of a sphere is 154 sq. cm., find its radius.

5. The surface of a globe is $\frac{11}{28}$ sq. m. Find its diameter.

6. The volume of a sphere is $1437\frac{1}{3}$ cu. m., find its radius.

7. A sphere is 36 inches in diameter, find its volume in cubic feet.

[P. U.]

8. The height and diameter of the base of a circular cylinder are each 6 metres. Find the radius of the sphere whose surface is equal to the curved surface of the cylinder.

9. How many spherical bullets, each 1 dm. in diameter, can be formed from an iron ball whose diameter is 6 decimetres ?

10. How many spherical bullets each $\frac{1}{2}$ cm. in radius can be cast from a rectangular block of lead 10 cm. long, 8 cm. broad and $5\frac{1}{2}$ cm. thick ?

11. The external and internal diameters of a shell are respectively $15\frac{1}{2}$ in. and $10\frac{3}{4}$ in., find the volume. [R. U. S.]

12. An iron sphere, 4 cm. in diameter, is beaten into a circular sheet, $\frac{2}{3}$ cm. thick ; find the radius of the sheet.

13. If r_1 and r_2 be the radii of two solid spheres of gold and if they are melted into one solid sphere, prove that the radius of the new sphere is $\left(r_1^3+r_2^3\right)^{\frac{1}{3}}$.

14. Find the weight of a hollow iron shell, if the exterior diameter is 13 inches and thickness of the iron be 2 inches. (Iron weighs 4·2 ozs. per cubic inch.)

[R. E.]

15. A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere.

[C. U. '49]

16. Find the surface of a sphere 10 cm. in radius.

GEOMETRY

[THEOREMS & PROBLEMS]

Theorem 1

If two straight lines intersect, the vertically opposite angles are equal.

Let the st. lines AB and CD intersect at O.

To prove that

(i) $\angle AOC$ = the vert. opp. $\angle BOD$;

(ii) $\angle AOD$ = the vert. opp. $\angle BOC$.

Fig. 1

Proof: $\because$ AO meets the st. line CD at O, $\therefore$ $\angle AOC + \angle AOD$ = 2 right angles.

Again, $\because$ DO meets the st. line AB at O, $\therefore$ $\angle AOD + \angle BOD$ = 2 right angles.

$\therefore$ $\angle AOC + \angle AOD = \angle BOD + \angle AOD$.

Taking away the common $\angle AOD$ from these equals, we have $\angle AOC = \angle BOD$.

Similarly it can be proved that $\angle AOD = \angle BOC$.

Theorem 2

If a st. line, cutting two other st. lines, makes (1) the alternate angles equal, or (2) the interior angles on the same side of the cutting line together equal to two right angles, then the two st. lines are parallel.

Let the st. line EF cut the st. lines AB, CD at G and H respectively.

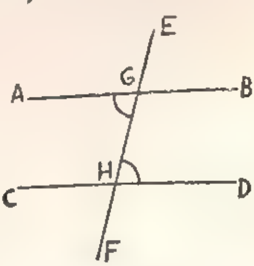


Fig. 2

If (1) $\angle AGH$ = the alternate $\angle GHD$, or if (2) $\angle BGH + \angle GHD$ = 2 right angles, it is to be proved that AB and CD are parallel.

(1) **Proof:** $\angle AGH$ = the vert. opp. $\angle EGB$,

but $\angle AGH = \angle GHD$ (Hyp.);

$\therefore$ $\angle EGB = \angle GHD$, and these are corresponding angles.

$\therefore$ AB is parallel to CD [Axiom.]

(2) **Proof :** $\angle BGH + \angle EGB = 2$ right angles (Axiom).

Also $\angle BGH + \angle GHD = 2$ right angles (Hyp.)

$\therefore$ $\angle BGH + \angle EGB = \angle BGH + \angle GHD$,

$\therefore$ $\angle EGB = \angle GHD$, and these are corresponding angles,

$\therefore$ AB is parallel to CD.

Theorem 3

If a st. line cuts two parallel st. lines, it makes (1) the alternate angles equal, (2) the corresponding angles equal, and (3) the two interior angles on the same side of the cutting line together equal to two right angles.

Let the st. line EF cut the parallel st. lines AB and CD at G and H respectively.

To prove that

(1) $\angle AGH =$ the alt. $\angle GHD$,

(2) $\angle EGB =$ the corresponding $\angle GHD$,

and (3) $\angle BGH + \angle GHD = 2$ right angles.

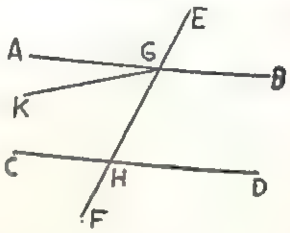


Fig. 3

Proof : (1) If $\angle AGH$ and $\angle GHD$ be not equal, let $\angle KGH$ be equal and alternate to $\angle GHD$.

Now, $\because \angle KGH =$ alt. $\angle GHD$, $\therefore$ KG $\parallel$ CD ; but AB $\parallel$ CD (Hyp.),

$\therefore$ two intersecting st. lines AB and KG are both parallel to CD, which is impossible. [Playfair's Axiom]

$\therefore$ $\angle AGH$ and $\angle GHD$ cannot be unequal,

$\therefore$ $\angle AGH =$ alternate $\angle GHD$.

(2) **Proof :** $\angle EGB =$ the vert. opp. $\angle AGH$.

Again, $\angle AGH = \angle GHD$ [proved], $\therefore$ $\angle EGB = \angle GHD$.

(3) **Proof :** $\angle BGH + \angle AGH = 2$ right angles [Axiom],

But $\angle AGH =$ alt. $\angle GHD$ [proved],

$\therefore$ $\angle BGH + \angle GHD = 2$ right angles.

Theorem 4

Straight lines which are parallel to the same st. line are parallel to one another.

Let the st. lines AB, CD be each parallel to the st. line XY.

To prove that AB ∥ CD.

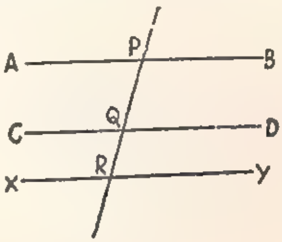


Fig. 4

Proof: If AB, CD are not parallel, they will intersect, if produced.

Then two intersecting st. lines would be both parallel to the st. line XY, which is impossible [Playfair's Axiom.]

∴ AB and CD are parallel.

[You have already learnt its *alternative proof*.]

Theorem 5

The three angles of a triangle are together equal to two right angles.

Let ABC be triangle.

To prove that $\angle ABC + \angle BCA + \angle CAB = 2$ right angles.

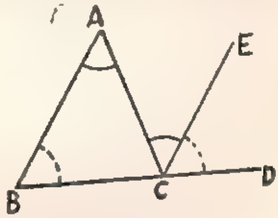


Fig. 5

Constrction: Produce BC to any pt. D and draw CE ∥ BA.

Proof: ∵ BA ∥ CE and AC meets them, ∴ $\angle CAB =$ alt. $\angle ACE$.

Again ∵ BA ∥ CE and BCD cuts them,

∴ $\angle ABC =$ corresponding $\angle ECD$.

∴ $\angle ABC + \angle CAB = \angle ACE + \angle ECD = \angle ACD$.

Adding $\angle BCA$ to these equals, we have

$\angle ABC + \angle CAB + \angle BCA = \angle ACD + \angle BCA = 2$ right angles.

Theorem 6

If one side of a triangle is produced the exterior angle so formed is equal to the sum of the interior opposite angles.

[Draw Fig. 5. It is to be proved that $\angle ACD = \angle ABC + \angle BAC$.

Now proceed as in Theorem 5 up to...

∴ $\angle ABC + \angle CAB = \angle ACD$.]

Theorem 7

The sum of the interior angles of a convex polygon of n sides is $(2n-4)$ right angles.

Let ABCDE be a convex polygon of n sides.

To prove that the sum of the n interior angles of this polygon $=(2n-4)$ rt. angles.

Construction : Take any point O within the polygon and join O to each vertex. Then the polygon is divided into n triangles.

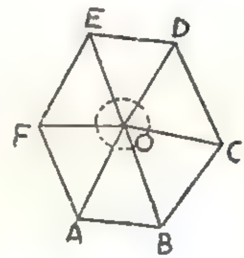


Fig. 6

Proof : $\because$ the sum of the angles, of each triangle $=2$ right angles,

$\therefore$ the sum of all the angles of n triangles $=2n$ right angles. Again, all the interior angles of the polygon + the angles at O = all the angles of the n triangles $=2n$ rt. angles. But the angles at O = 4 right angles,

$\therefore$ the interior angles of the polygon + 4 rt. angles $=2n$ right angles.

$\therefore$ the sum of the interior angles of the polygon $=(2n-4)$ right angles.

Theorem 8

If the sides of a convex polygon are produced in order, the sum of the exterior angles is equal to four right angles.

Let ABCDE be a convex polygon of n sides and let its sides be produced in order as indicated by the arrow marks.

To prove that the exterior angles thus formed = 4 right angles.

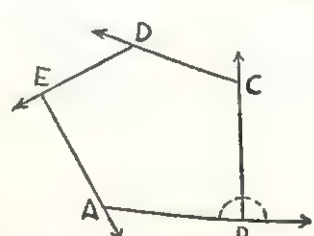


Fig. 7

Proof : As the polygon has n sides, it has n vertices.

Now, at each vertex, the interior angle + the exterior angle = 2 right angles,

$\therefore$ n interior angles + n exterior angles $=2n$ right angles.

Again, the sum of the n interior angles of the polygon $+4$ rt. angles $=2n$ right angles,

$\therefore$ The sum of the interior angles + the sum of the exterior angles = the sum of the interior angles + 4 right angles.

$\therefore$ The sum of the exterior angles of the polygon = 4 rt. angles.

Theorem 9

If two sides of a triangle are equal, then the angles opposite to these two sides are equal.

Let ABC be a triangle, in which AB = AC.

To prove that $\angle ABC = \angle ACB$.

Construction: Let the st. line AD bisect $\angle BAC$ and meet BC in D.

Proof: In $\triangle ABD$ and $\triangle ACD$, AB = AC [Hyp.], AD is common to both the triangles, and the included $\angle BAD$ = the included $\angle CAD$.

$\therefore$ The triangles are congruent,

$\therefore$ $\angle ABD = \angle ACD$, *i. e.*, $\angle ABC = \angle ACB$.

Fig. 8

Theorem 10

If two angles of a triangle are equal, then the sides opposite to these angles are also equal.

[It is evidently converse of Theorem 9.]

Let ABC be a triangle, in which $\angle ABC = \angle ACB$.

To prove that AB = AC.

Construction: [Draw Fig. 8] Let the st. line AD bisect $\angle BAC$ and meet BC in D.

Proof: In $\triangle$s ABD and ACD, $\angle ABD = \angle ACD$ (Hyp.), $\angle BAD = \angle CAD$(cons.) and AD is common to both $\triangle$s,

$\therefore$ The two triangles are congruent. $\therefore$ AB = AC.

Theorem 11

If the three sides of one triangle are respectively equal to the three sides of another triangle, the two triangles are congruent.

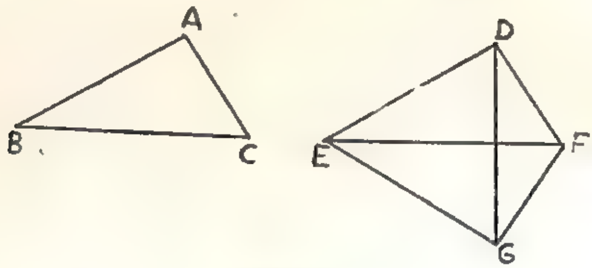


Fig. 9

Let ABC, DEF be two triangles, in which AB=DE, BC=EF and AC=DF.

To prove that △ˢABC, DEF are congruent.

Proof : Of the sides of the two triangles, let BC and EF be not less than any other side.

Apply the △ABC to △DEF so that B falls on E and BC along EF, and let A fall at G on the side of EF opposite to D.

Then ∵ BC=EF, ∴ C must fall on F. Thus △GEF is the new position of △ABC. Join DG.

Now, in △DEG, ∵ ED=AB=EG, ∴ ∠EDG=∠EGD.

Again, ∵ FD=CA=FG, ∴ ∠FDG=∠FGD.

∴ The whole ∠EDF=the whole ∠EGF=∠BAC.

Then, in △ˢ ABC and DEF, AB=DE, AC=DF and the included ∠BAC=the included ∠EDF (Proved),

∴ △ABC and △DEF are congruent.

Theorem 12

If two right-angled triangles have their hypotenuses equal and one side of the one equal to one side of the other, then the triangles are equal in all respects.

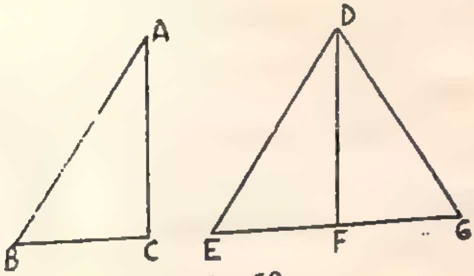


Fig. 10

Let ABC and DEF be two right-angled triangles, in which angles at C and F are right angles, the hypotenuse AB = the hypotenuse DE, and the side AC = the side DF.

To prove that $\triangle$ABC and $\triangle$DEF are equal in all respects.

Proof : Apply $\triangle$ABC to $\triangle$DEF so that A falls on D and AC on DF, and let B fall at G on the side of DF opposite to E.

Then $\because$ AC = DF, C must fall on F.

Thus $\triangle$DGF is the new position of $\triangle$ABC.

Now, $\because$ $\angle$DFG = $\angle$ACB = 1 right angle,

$\therefore$ $\angle$DFE + $\angle$DFG = 2 right angles.

$\therefore$ FE and FG are in the same st. line. Hence DEG is a triangle, in which DE = AB = AG, $\therefore$ $\angle$DEF = $\angle$DGF = $\angle$ABC.

Then, in $\triangle$'ABC, DEF, $\because$ $\angle$ACB = $\angle$DFE (being rt. angles), $\angle$ABC = $\angle$DEF (Proved), and AC = DF (Hyp.)

$\therefore$ $\triangle$'ABC and DEF are equal in all respects.

Theorem 13

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.

Let ABC be a triangle in which AB is greater than AC.

To prove that $\angle ACB > \angle ABC$.

Construction : From AB, cut off AD equal to AC, and join CD.

Proof : $\because$ AD = AC (cons.),

$\therefore$ $\angle ADC = \angle ACD$.

$\because$ the side BD of $\triangle BDC$ is produced to A, $\therefore$ the exterior $\angle ADC > \angle DBC$. $\therefore$ $\angle ACD > \angle DBC$, *i.e.,* $\angle ACD > \angle ABC$.

$\therefore$ The whole $\angle ACB$ is still more greater than $\angle ABC$.

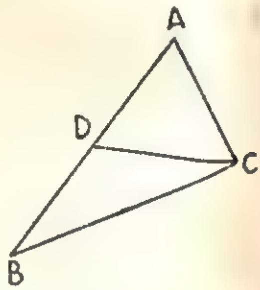


Fig. 11

Theorem 14

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less.

Let ABC be a triangle, in which $\angle ABC$ is greater than $\angle ACB$. *To prove that* AC > AB.

Proof : If AC is not greater than AB, then AC is *either* equal to AB *or* less than AB.

Now, if AC = AB, then $\angle ABC = \angle ACB$, but by hypothesis the angles are not equal.

$\therefore$ AC is not equal to AB.

Again, if AC < AB, then $\angle ABC$ is less than $\angle ACB$, but by hypothesis it is not so.

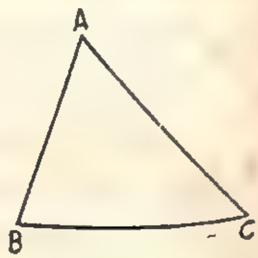


Fig. 12

$\therefore$ AC is not less than AB.

Thus AC is neither equal to AB nor less than AB.

Hence AC is greater than AB.

Theorem 15

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle. *To prove that* any two of its sides are together greater than the third side.

Construction : Produce BA to D so that AD = AC. Join DC.

Proof : $\because$ AD = AC, $\therefore$ $\angle$ACD = $\angle$ADC.

But $\angle$BCD > its part $\angle$ACD,

$\therefore$ $\angle$BCD > $\angle$ADC, *i.e.*, $\angle$BCD > $\angle$BDC.

Now, in $\triangle$BCD, $\because$ $\angle$BCD > $\angle$BDC,

$\therefore$ BD > BC.

But BD = BA + AD = BA + AC,

$\therefore$ BA + AC > BC.

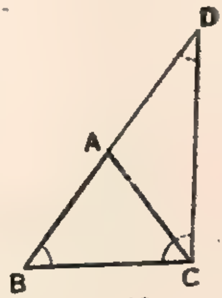


Fig. 13

Similarly it can be proved that AB + BC > AC and AC + BC > AB.

Theorem 16

Of all straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

Let AB be a given st. line and O be a given point outside it.

To prove that of all the st. lines that can be drawn from O to AB, the perpendicular will be the shortest.

Construction : Draw OP perpendicular to AB and let OQ be any other st. line drawn from O to AB.

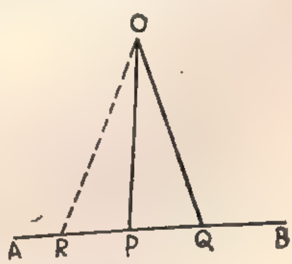


Fig. 14

Proof : In $\triangle$OPQ, $\because$ $\angle$OPQ is a right-angle,

$\therefore$ $\angle$OQP is an acute angle. $\therefore$ $\angle$OQP < $\angle$OPQ.

$\therefore$ OP < OQ.

Thus the perpendicular OP is less than any straight line drawn from O to AB.

$\therefore$ the perpendicular OP is the shortest of all these st. lines.

PRACTIAL GEOMETRY

You have already known that the instruments, viz., a flat ruler, a pair of compasses, a semi-circular protractor, a pair of dividers, etc. are used in pratical geometry.

Now you are to know that a straight ruler and a pair of compasses are the only instruments that can be used in all purely geometrical constructions. The use of no other instruments is permitted.

The **ruler** is used (i) to draw a straight line, (ii) to join two points, (iii) to produce a st. line either way or both ways.

The **compasses** are used to draw circles or arcs of circles with given centres and radii.

Problems : In constructing figures in problems, (i) all the traces of construction should be neatly and accurately shown in the diagram, (ii) the statement of construction and (iii) the theoretical proof of the problem should be given.

Problem 1

To bisect a given angle.

Let BAC be the given angle to be bisected.

Construction : With centre A and with any radius draw an arc of a circle, cutting AB in D and AC in E.

With centres D and E and with radius DE draw two arcs cutting each other at O. Join AO. Then AO bisects $\angle$BAC.

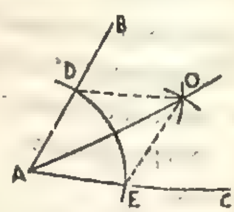


Fig. 15

Proof : Join DO, EO.

In $\triangle^s$ ADO and AEO, AD $=$ AE (radii of a circle), DO $=$ EO (radii of equal circles) and AO is common.

$\therefore$ the triangles are congruent. $\therefore$ $\angle$DAO $=$ $\angle$EAO.

Hence AO bisects $\angle$BAC.

[N. B The two arcs with centres D and E may be drawn with radius DE or any convenient radius such that the two arcs intersect.]

Problem 2.

To bisect a given finite straight line.

Let AB be the given straight line to be biseected.

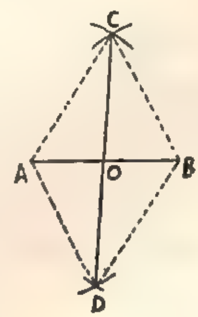


Fig 16

Construction : With centre A and with radius AB draw arcs of two equal circles, one on each side of AB.

Again, with centre B and the same radius draw two arcs, one on each side of AB, cutting the former arcs in C and D. Join CD cutting AB at O. Then AB is bisicted at O.

Proof : Join CA, CB, DA, DB. In $\triangle$ACD and $\triangle$BCD,

AC = BC (radii of equal circles),
AD = BD (,, ,, ,,)

and CD is common to both ;

$\therefore$ the triangles are congruent ; $\therefore$ $\angle$ACD = $\angle$BCD.

Again, in $\triangle^s$ ACO and BCO, AC = BC, CO is common to both and the included $\angle$ACO = the included $\angle$BCO ;

$\therefore$ the triangles are congruent.

$\therefore$ AO = BO, that is, AB is bisected at O.

[N. B. (1) In drawing arcs with centres A and B, any convenient radius not less than $\frac{1}{2}$ AB, may be taken. If the radius be less than $\frac{1}{2}$ AB, the arcs will not intersect.

(2) In the above problem, CD is the bisector of AB.

Again, $\because$ $\triangle$ACO $\equiv$ $\triangle$BCO, $\therefore$ $\angle$AOC = $\angle$BOC, $\therefore$ CO $\perp$ AB.

Hence, CD is the perpendicular bisector of AB.]

Problem 3.

To draw the perpendicular to a given straight line at a given point in it.

Let AB be the given straight line and C the given point in it.

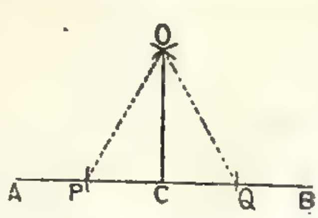


Fig. 17

To draw a perpendicular to AB at C.

Construction : With centre C and with the same radius draw two arcs cutting AB at P and Q.

Again, with centres P and Q and with any radius greater than PC draw two arcs cutting each other at O. Join OC.

Then CO is perpendicular to AB at C.

Proof : Join OP, OQ. In $\triangle^s$ OPC and OQC,

CP = CQ (radii of a circle),

OP = OQ (radii of equal circles), and OC is common ;

$\therefore$ the triangles are congruent.

$\therefore$ $\angle OCP = \angle OCQ$; and these being adjacent angles, each is a right angle. $\therefore$ OC is perpendicular to AB.

[**Second method**]

Construction : Take any pt. O outside AB. With centre O and radius OC draw a circle cutting AB at P. Join PO and produce it to cut the circle at Q. Join CQ.

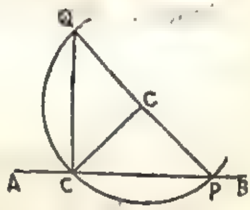


Fig. 18

Then CQ is perpendicular to AB.

Proof : Join OC. $\because$ OC = OP (radii of a circle). $\therefore$ $\angle OCP = \angle OPC$.

Again $\because$ OC = OQ (radii of a circle)

$\therefore$ $\angle OCQ = \angle OQC$.

$\therefore$ The whole $\angle PCQ = \angle CPQ + \angle CQP = \frac{1}{2}$ of 2 rt. angles

= 1 right angle. $\therefore$ $CQ \perp AB$.

[**Another method**]

Construction : With centre C and any radius draw an arc DEF cutting AB at D. With centre D and with the same radius draw an arc cutting the arc DEF at E. With centre E and the same radius as before draw an arc cutting the arc DEF at F. Now, with centres E and F and with any radius draw two arcs cutting each other at O. Join OC. Then $OC \perp AB$.

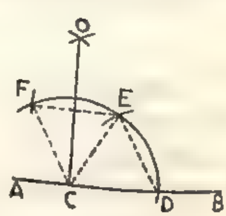


Fig. 19

Proof : Join DE, CE, EF, CF. CD, CE, DE, EF and CF are equal being equal to the same radius.

$\therefore$ the $\triangle^s$ CED, CEF are congruent. $\therefore$ $\angle DCE = \angle ECF = 60^\circ$.

Again, by construction $\angle ECF$ has been bisected by CO.

$\therefore \quad \angle ECO = 30°$.

$\therefore \quad \angle OCD = \angle DCE + \angle ECO = 60° + 30° = 90°, \quad \therefore$ OC$\perp$AB.

Problem 4.

To draw the perpendicular to a given straight line from a given point outside it.

Let AB be the given straight line and C the given point outside it.

To draw the perpendicular from C to AB.

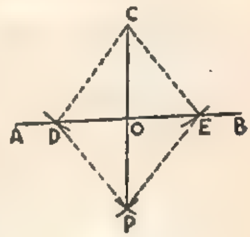


Fig. 20

Construction : With centre C and any convenient radius draw two arcs of a circle cutting AB at D and E. With centres D and E and with any radius greater than half of DE draw two arcs cutting each other at P. Join CP, cutting AB in O.

Then CO is perpendicular to AB.

Proof : Join CD, CE, PD, PE. In Δ^s CPD and CPE, $\because$ CD = CE (radii of a circle), PD = PE (radii of equal circles) and CP is common to both,

$\therefore$ the triangles are congruent. $\therefore \angle PCD = \angle PCE$.

Again, in Δ^s COD and COE, $\because$ CD = CE, CO is common and the included $\angle OCD$ = the included $\angle OCE$,

$\therefore$ the triangles are congruent,

$\therefore \quad \angle COD = \angle COE$; but these being adjacent angles are right angles. $\therefore$ CO$\perp$AB.

[N.B. (1) In drawing the arcs with centre C, the radius should be such as to cut AB. You may also take a pt. (say Q) on the other side of AB and then draw the circle with centre C and radius CQ. For, in that case the circle must cut AB at two points. If the radius be less than the perpendicular distance of C from AB, then the circle will not cut AB.

(2) In drawing the circles with centres D and E, if the radius be not greater than $\frac{1}{2}$ DE, the circles will not intersect.]

[Another method]

Construction : Take any convenient point D in AB. Join CD and bisect CD at P. With centre P and radius PC draw an arc of a circle cutting AB at O. Join CO.

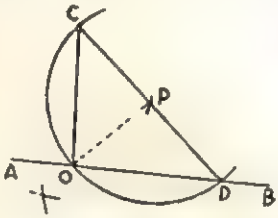


Fig. 21

Then CO is perpendicular to AB.

Proof : Join OP.

$\because$ OP = PD (radii of a circle)

$\therefore$ $\angle$POD = $\angle$PDO.

Again, $\because$ CP = OP, $\therefore$ $\angle$COP = $\angle$OCP.

$\therefore$ the whole $\angle$COD = $\angle$PDO + $\angle$OCP = $\frac{1}{2}$ of 2 rt. angles = 1 right angle. $\therefore$ CO $\perp$ AB.

Problem 5

At a given point in a given straight line to draw an angle equal to a given angle.

Let BAC be the given angle and D be the given point in the st. line XY.

To draw an angle at D equal to $\angle$BAC.

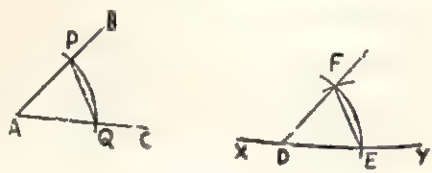


Fig. 22

Construction : With centre A and any radius draw an arc of a circle cutting AB in P and AC in Q. With centre D and radius AP draw an arc EF cutting XY in E.

With centre E and radius equal to PQ draw an arc cutting the arc EF at F. Join DF.

Then $\angle$FDE is the angle required.

Proof : Join PQ, EF. In Δ^s APQ and DFE,

AP = DF (radii of equal circles)

AQ = DE („ „ „)

and PQ = FE („ „ „)

$\therefore$ the triangles are congruent ;

$\therefore$ $\angle$FDE = $\angle$PAQ = $\angle$BAC.

Problem 6

Through a given point to draw a straight line parallel to a given straight line.

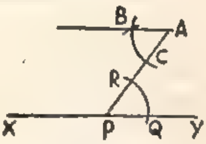


Fig. 23

Let XY be the given st. line and A the given point, through which a st. line parallel to XY is to be drawn.

Construction : Take any point P in XY and join AP. At the point A in AP, draw the $\angle$PAB equal to $\angle$APY and alternate to it [problem 5]. Then AB is parallel to XY.

Proof : $\angle BAP = \angle APY$ and these are alternate angles, $\therefore$ AB and XY are parallel.

[Construction of triangles]

Problem 7

To construct a triangle having given its three sides.

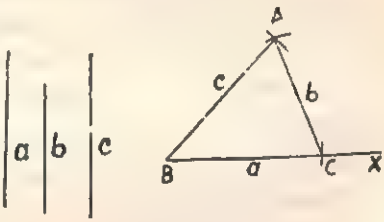


Fig. 24

Let a, b, c be the lengths of three sides of the triangle. To construct the triangle.

Construction : Take any st. line BX. From BX cut off BC equal to a. With centre B and radius c draw an arc of a circle. Again, with centre C and radius b draw an arc cutting the former arc at A. Join AB, AC. Then $\triangle$ABC is the triangle required.

Proof : By construction $BC = a$, $AC = b$, and $AB = c$.

[N. B. (i) To cut off BC equal to a from BX, draw an arc with centre B and radius a cutting BX at C.

(iii) If two arcs be drawn with centres B and C on the other side of BX, then another required triangle may be obtained.

(iii) You know that any two sides of a triangle are together greater than the third. So the lengths a, b, c must

be such that any two of them are greater than the third. Otherwise the construction will fail.

(iv) It is usual to represent the angles of a $\triangle ABC$ by the letters A, B and C, and the sides opposite to them by the letters a, b and c respectively. Thus the side opposite to $\angle A$ is denoted by a and so on. Also the angles opposite to the sides a, b and c are denoted by $\angle A$, $\angle B$ and $\angle C$.]

Problem 8

To construct a triangle having given two sides and the included angle.

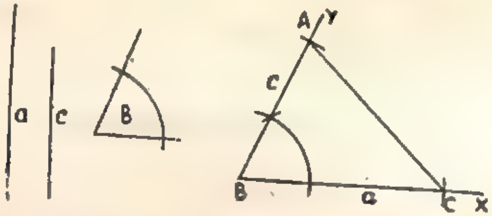


Fig. 25

Let a and c be the lengths of two sides of the triangle and $\angle B$ be the included angle. To construct the triangle.

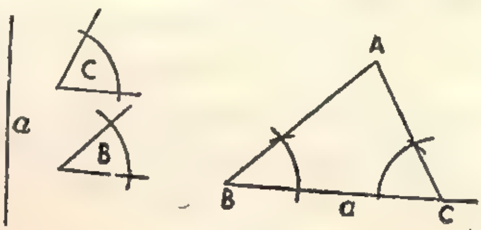
Construction : Take any st. line BX and cut off from it $BC = a$. At B in BC draw the $\angle CBY = \angle B$. From BY cut off $BA = c$. Join AC. Then $\triangle ABC$ is the triangle required.

Proof : By construction $BC = a$, $AB = c$ and the included $\angle ABC = \angle B$.

Problem 9

To construct a triangle having given a side and the angles adjacent to it.

Let a be the given side of the triangle and $\angle B$ and $\angle C$ be the angles adjacent to it. To construct the triangle.



Construction : Take any st. line BC equal to a.

Fig. 26

At B and C and on the same side of BC, draw $\angle CBA$ and

$\angle BCA$ equal respectively to $\angle B$ and $\angle C$. Let the arms BA and CA intersect at A. Then ABC is the triangle required.

Proof: By construction $BC=a$, and the adjacent $\angle ABC=\angle B$ and $\angle ACB=\angle C$.

[N. B. If $\angle A$, $\angle B$ and a be given, then $\angle C$ is known from $\angle C=180°-(\angle A+\angle B)$.]

Problems 10

To construct a triangle having given two sides and the angle opposite to one of them.

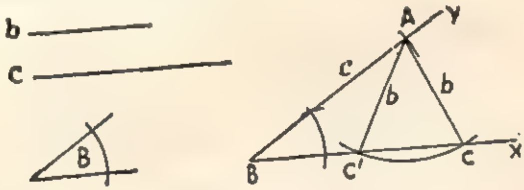


Fig 27

Let the sides b, c and $\angle B$ of a triangle be given.

To construct the triangle.

Construction: Take any st. line BX. At B in BX, draw $\angle XBY=$ given $\angle B$. From BY cut off BA equal to c. With centre A and radius b draw an arc cutting BX at C and C'. Join AC and AC'. Then both the $\triangle^s$ ABC and ABC' are the triangles required.

Proof: By construction in $\triangle ABC$, $AC=b$, $AB=c$ and $\angle ABC$ opposite to AC is equal to the given $\angle B$.

Also in $\triangle ABC'$, $AC'=b$, $AB=c$ and $\angle ABC'=\angle B$.

[Note: Here we have *two* triangles having the three given parts. So this solution is known as the **ambiguous** case. We may have the following cases of solution from the data given here.]

Case 1. If the perpendicular from A to BX be less than the given length b, then the circle will not cut BX and so no triangle can be drawn.

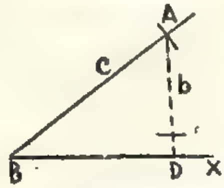


Fig. 28

Case 2. If $b=AD$, i.e., if $b=$ the

perpendicular from A to BX, then the circle drawn with centre A and radius b will meet BX in one point only.

It is to be observed that in this case the points C and C′ coincide with D. So we have only **one** triangle in this case and this is a **right-angled** triangle, ∠D being a right angle.

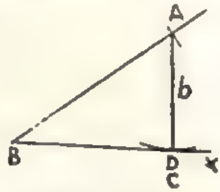


Fig. 29

Case 3. If $b=c$, the circle drawn with radius b will cut BX at C and will pass through B. Here C′ coincides with B. Hence here we have only one triangle ABC, which is an isosceles triangle having AB = AC.

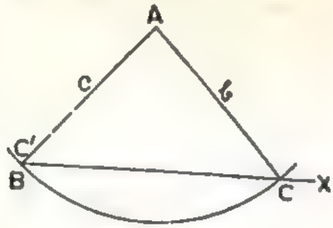


Fig. 30

Case 4. If $b<c$, then we have two triangles. See Problem 10. This is known as **ambiguous case.**

Case 5. If $b>c$, the circle will cut BX at C and CB produced at C′. The pt. B is here within the circle and C, C′ are on opposite sides of B. But here we have only *one* triangle ABC.

The △ABC′ does not satisfy the given data, because its ∠ABC′ is not equal to the given ∠B, but is equal to the supplement of ∠B.

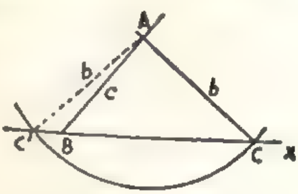


Fig. 31

Problem 11

To construct a triangle having given two angles and a side opposite to one of them.

Let ∠A and ∠B of a triangle and the side a opposite to ∠A be given. To construct the triangle.

Construction : Take any st. line BD and from it cut off BC = a. At C in BD, draw ∠DCE = ∠B and ∠ECA = ∠A.

At B draw ∠CBA = ∠B, so that BA and CA intersect at A. Then ABC is the triangle required.

Proof : The exterior $\angle ACD = \angle ABC + \angle BAC$.

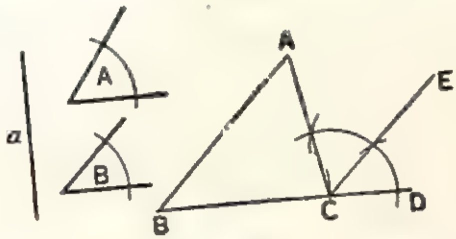


Fig. 32

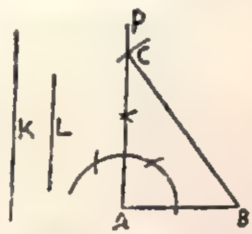
But $\angle ABC = \angle B = \angle DCE$, $\therefore$ $\angle BAC = \angle ACE = \angle A$, and also $BC = a$.

Problem 12

To construct a right-angled triangle having given the hypotenuse and a side.

Let K be the hypotenuse and L be a side of a right-angled triangle.

To construct the triangle.



Construction : Take a st. line AB equal to L. At A draw AP perpendicular to AB. With centre B and radius K draw an arc of a circle cutting AP at C. Join BC. Then ABC is the triangle required.

Proof : By construction $\angle BAC$ is a right-angle. $AB = L$ and the hypotenuse $BC = K$.

[**N. B.** $\because$ the circle with centre B and radius K cuts PA produced at another point, we may have another solution.]

[Another Method]

Construction : Take a st. line $BC = K$, and bisect BC at O. With centre O and radius OB draw a semi-circle. With centre B and radius L draw an arc cutting the semi-circle at A. Join AB, AC. Then ABC is the triangle required.

Proof : Join AO. $\because$ $OA = OB$,

$\therefore$ $\angle OAB = \angle OBA$.

Again, $\because$ $OA = OC$,

$\therefore$ $\angle OAC = \angle OCA$.

$\therefore$ The whole $\angle BAC$

$= \angle OBA + \angle OCA = \angle ABC + \angle ACB$

$= \frac{1}{2}$ of 2 rt. angles $= 1$ rt. angle, and by construction the hypotenuse $BC = K$ and the side $AB = L$.

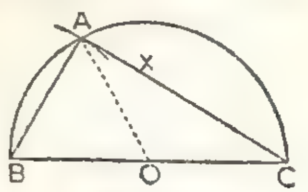


Fig. 34

Theorem 17

The opposite sides and angles of a parallelogram are equal and each diagonal bisects the parallelogram.

Let ABCD be a parallelogram of which AC is a diagonal.

To prove that (1) AB = CD, BC = AD ; (2) $\angle$ABC = $\angle$ADC, $\angle$BAD = $\angle$BCD ; (3) AC bisects the parallelogram, *i.e.*, $\triangle$ABC and $\triangle$ADC are equal.

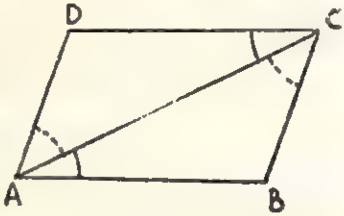


Fig. 35

Proof : $\because$ AB and CD are parallel and AC meets them, $\therefore$ $\angle$BAC = alt. $\angle$ACD.

Again, $\because$ BC and AD are parallel and AC meets them, $\therefore$ $\angle$ACB = alt. $\angle$DAC.

Then in $\triangle^s$ ABC and ADC, $\angle$BAC = $\angle$ACD, $\angle$ACB = $\angle$DAC, and AC is commmon to both ;

$\therefore$ the triangles are congruent.

$\therefore$ AB = CD, BC = AD···(1), $\angle$ABC = $\angle$ADC···(2) and AC bisects the parallelogram.

Similarly it can be proved that the diagonal BD also bisects the parallelogram···(3)

Again, $\because$ $\angle$BAC = $\angle$ACD and $\angle$DAC = $\angle$ACB,

$\therefore$ the whole $\angle$BAD = the whole $\angle$BCD.

Theorem 18

The diagonals of a parallelogram bisect each other.

Let ABCD be a parallelogram, of which the diagonals AC and BD intersect at O.

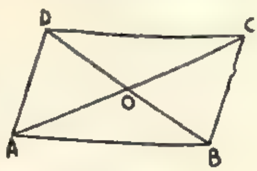


Fig. 36

To prove that AO = CO and BO = DO.

Proof : $\because$ AB ∥ CD and AC meets them, $\therefore$ $\angle$BAC = alt. $\angle$ACD.

Also $\because$ AB ∥ CD and BD meets them,

$\therefore$ $\angle$ABD = alt. $\angle$BDC.

Now, in $\triangle^s$AOB and COD, $\angle$OAB = $\angle$OCD, $\angle$OBA = $\angle$ODC, and AB = CD ;

$\therefore$ the triangles are congruent ;

$\therefore$ AO = CO and BO = DO.

Theorem 19

A quadrilateral is a parallelogram, if both pairs of its opposite sides are equal.

Let ABCD be a quadrilateral in which AB = CD and BC = AD.

To prove that ABCD is a parallelogram.

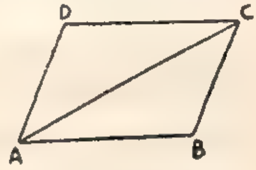


Fig. 37

Construction : Join AC.

Proof : In Δ'ABC and ADC, AB = CD, BC = AD, and AC is common to both.

$\therefore$ the triangles are congruent. $\therefore$ $\angle$BAC = $\angle$ACD, and $\angle$ACB = $\angle$DAC.

Now, $\because$ $\angle$BAC = $\angle$ACD and these are alternate angles,

$\therefore$ AB and CD are parallel.

Again, $\because$ $\angle$ACB = $\angle$DAC and these are alternate angles,

$\therefore$ BC and AD are parallel. Now $\because$ AB $\parallel$ CD and BC $\parallel$ AD,

$\therefore$ ABCD is a parallelogram.

Theorem 20

A quadrilateral is a parallelogram, if its opposite angles are equal.

Let ABCD be a quadrilateral, in which $\angle$A = $\angle$C and $\angle$B = $\angle$D.

To prove that ABCD is a parallelogram.

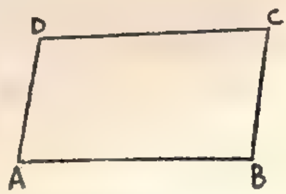


Fig. 38

Proof : The four angles of a quadrilateral are together equal to 4 right angles.

Now, $\because$ $\angle$A = $\angle$C

and $\angle$B = $\angle$D,

$\therefore$ $\angle$A + $\angle$B = $\angle$C + $\angle$D

= $\frac{1}{2}$ of the angles of the quadrilateral

= 2 right angles, $\therefore$ AD $\parallel$ BC ($\because$ the sum of the interior angles on the same side of AB = 2 rt. angles).

Again, $\because$ $\angle$A = $\angle$C and $\angle$D = $\angle$B,

$\therefore$ $\angle A + \angle D = \angle C + \angle B = \frac{1}{2}$ of 4 right angles = 2 right angles, $\therefore$ AB $\parallel$ CD.

Now $\because$ AB $\parallel$ CD and AD $\parallel$ BC,

$\therefore$ ABCD is a parallelogram.

Theorem 21

A quadrilateral is a parallelogram, if one pair of its opposite sides are equal and parallel.

Let ABCD be a quadrilateral of which AB and CD are equal and parallel.

To prove that ABCD is a parallelogram.

Construction : Join AC.

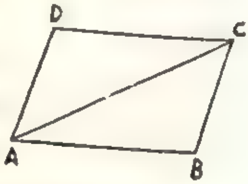


Fig. 39

Proof : $\because$ AB $\parallel$ CD and AC meets them, $\therefore$ $\angle BAC$ = alt. $\angle ACD$.

Now, in $\triangle ABC$ and $\triangle ACD$,

AB = CD, AC is common to both and the included $\angle BAC$ = the included $\angle ACD$,

$\therefore$ the triangles are congruent.

$\therefore$ $\angle ACB = \angle DAC$, but these are alternate angles, $\therefore$ AD $\parallel$ BC.

$\because$ AB $\parallel$ CD and AD $\parallel$ BC, $\therefore$ ABCD is a parallelogram.

Theorem 22

A quadrilateral is a parallelogram if its diagonals bisect each other.

Let ABCD be a quadrilateral, in which the diagonals AC, BD bisect each other at O, so that AO = CO and BO = DO.

To prove that ABCD is a parallelogram.

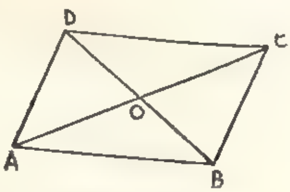


Fig. 40

Proof : In $\triangle$ˢ AOB and COD, AO = CO, BO = DO (Hyp.) and $\angle AOB$ = vert. opp. $\angle COD$,

$\therefore$ the triangles are congruent.

$\therefore$ AB = CD, and $\angle OAB = \angle OCD$, but these are alternate angles, $\therefore$ AB $\parallel$ CD.

$\therefore$ The opposite sides AB and CD are equal and parallel.

$\therefore$ ABCD is a parallelogram.

Theorem 23

If three or more parallel straight lines make equal intercepts on any transversal, they make equal intercepts on any other transversal.

Let the parallel st. lines AB, CD, EF make equal intercepts PQ and QR on the transversal PQR, and let them make the intercepts XY and YZ on the transversal XYZ.

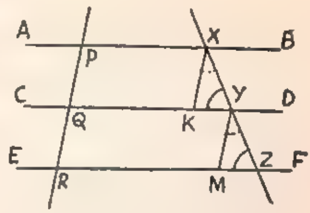


Fig. 41

To prove that XY = YZ.

Construction : Through X and Y draw XK and YM parallel to PQR meeting CD and EF in K and M respectively.

Proof : By construction PQKX and QRMY are each a parallelogram,

∴ XK = PQ and YM = QR, but PQ = QR (Hyp), ∴ XK = YM.

Again, ∵ both XK and YM are parallel to PQR, ∴ XK ∥ YM.

Now, in Δ^s XYK and YZM,

∠XYK = corresponding ∠YZM (∵ CD ∥ EF),

∠KXY = corresponding ∠MYZ (∵ XK ∥ YM),

and XK = YM (proved),

∴ the triangles are congruent. ∴ XY = YZ.

Theorem 24

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

Let ABC be a triangle, in which D is the middle point of AB, and let DE be drawn parallel to BC meeting AC in E.

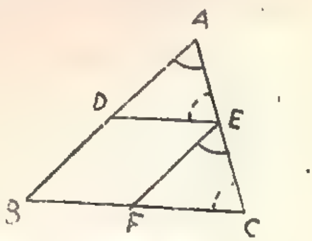


Fig. 42

To prove tha AE = CE.

Construction : Through E draw EF parallel to AB and meeting BC in F.

Proof: $\because$ DE ∥ BF and DB ∥ EF, $\therefore$ BDEF is a parallelogram.

$\therefore$ EF = BD = AD ($\because$ D is the mid point of AB).

Then, in $\triangle$s ADE and EFC, AD = EF (proved),

$\angle$AED = corresponding $\angle$ECF ($\because$ ED ∥ BC),

and $\angle$DAE = corresponding $\angle$FEC ($\because$ EF ∥ AB),

$\therefore$ The triangles are congruent. $\therefore$ AE = CE.

Alternative Proof

D is the middle point of AB in $\triangle$ABC, and DE drawn parallel to BC meets AC in E.

To prove that AE = CE.

Construction: Through A drawn PAQ parallel to BC.

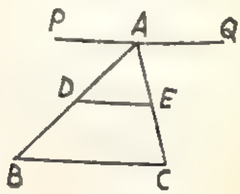


Fig. 43

Proof: $\because$ DE ∥ BC and PAQ ∥ BC, $\therefore$ DE ∥ PAQ.

$\therefore$ PAQ, DE and BC are three parallel st. lines, and AB and AC are two transversals.

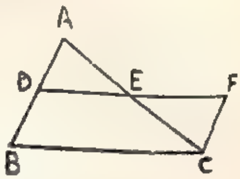
Now, $\because$ AD = BD (Hyp.), $\therefore$ AE = CE [Theorem 23]

Theorem 25

The straight line joining the middle points of two sides of a triangle is parallel to and half of the third side.

Let ABC be a triangle in which D and E, the middle points of AB and AC respectively, are joined.

To prove that DE is parallel to and half of BC.



Construction: Produce DE to F so that EF = DE. Join CF.

Fig. 44

Proof: In $\triangle$s ADE and CEF, AE = CE (Hyp.)

DE = EF (construction) and $\angle$AED = vert. opp. $\angle$CEF,

$\therefore$ the triangles are congruent,

$\therefore$ AD = CF and $\angle$DAE = $\angle$ECF, but these are alternate angles, $\therefore$ AD ∥ CF, i.e., AB ∥ CF.

Again, AD = BD (∵ D is the mid. pt. of AB),

∴ BD = CF. ∴ BD and CF are equal and parallel.

∴ DBCF is a parallelogram.

∴ DF and BC are equal and parallel.

∴ DE ∥ BC, and DE = $\frac{1}{2}$ DF = $\frac{1}{2}$ BC.

Hence, DE is parallel to and half of BC.

Problem 13

To divide a straight line into any number of equal parts.

Let AB be the given st. line to be divided into any number of equal parts (say five).

Construction : At A in AB, draw an angle BAX. From AX cut off 5 equal lengths AP, PQ, QR, RS and ST. Join TB. Through S, R, Q, P draw st. lines parallel to TB, cutting AB in N, M, L, K respectively. Then AB is divided into 5 equal parts in K, L, M, N.

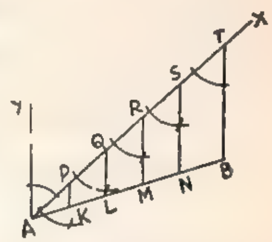


Fig. 45

Proof : Draw AY ∥ BT.

∵ The parallel st. lines AY, PK, QL, RM, SN, TB make equal intercepts on the transversal AX,

∴ They also make equal intercepts on the transversal AB.

∴ AB is divided into 5 equal parts in K, L, M, N.

[Another method]

The st. line AB is to be divided into 5 equal parts.

Construction : At A in AB draw an angle BAX and through B draw BY ∥ XA. From AX cut off five equal parts AP, PQ, QR, RS and ST of any length. Along BY mark off 5 lengths BN, NM, ML, LK and KO each equal to AP. Join PK, QL, RM, SN cutting AB in *p*, *q*, *r*, *s* respectively.

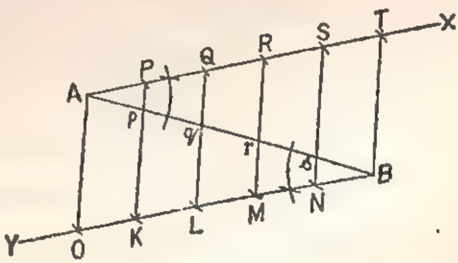


Fig. 46

Then AB is divided into 5 equal parts in p, q, r, s.

Proof : Join AO and TB. $\because$ by construction, AP and OK are equal and parallel,

$\therefore$ AO and PK are equal and parallel.

Similarly PK $\parallel$ QL, QL $\parallel$ RM, RM $\parallel$ SN and SN $\parallel$ TB.

$\therefore$ AO, PK, QL, RM, SN and TB are parallel and divide AT into 5 equal parts.

$\therefore$ They also divide AB into five equal parts in p, q, r, s.

Construction of Quadrilaterals

You have noticed that three independent data are required for the construction of a triangle. But a quadrilateral cannot be drawn if its four sides only are given. To draw a quadrilateral, five independent data are required.

Problem 14

To construct a quadrilateral having given its four sides and an angle.

Let a, b, c, d be the lengths of four sides of a quadrilateral and A be the angle between a and d.

To construct the quadrilateral.

Construction : Take any st. line AX and cut off from it AB equal to a. At A in AB draw $\angle$XAY $= \angle$A, and from AY cut off AD $= d$. With centres B and D and radius b and c respectively draw two arcs cutting each other at C. Join BC and DC. Then ABCD is the quadrilateral required.

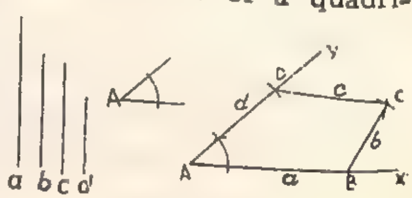


Fig. 47

Proof : By construction AB $= a$, BC $= b$, CD $= c$, AD $= d$ and the angle between AB and AD $=$ the given $\angle$A.

Problem 15

To construct a quadrilateral having given its four sides and a diagonal.

Let a, b, c, d be the given four sides and p be the given diagonal of a quadrilateral, which is to be constructed.

Construction : Take any st. line AX and from it cut off AC equal to p. With centres A and C and radius a and b respectively draw two arcs on one side of AC, cutting each other at B.

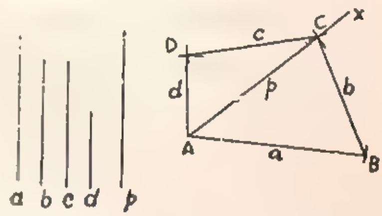


Fig. 48

Again, with centres A and C and radius d and c respectively draw two arcs on the other side of AC, cutting each other at D. Join AB, CB and AD, CD. Then ABCD is the quadrilateral required. For, by construction its sides are equal to the given sides and the diagonal AC$=p$.

Problem 16

To construct a parallelogram having given two adjacent sides and the included angle.

Let p and q be the adjacent sides of a parallelogram and A be the given included angle.

The parallelogram is to be constructed.

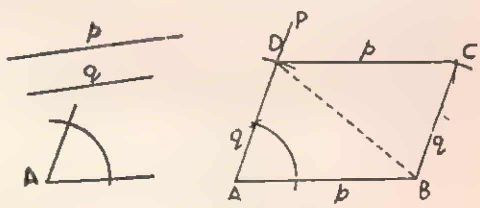


Fig. 49

Construction : Take a st. line AB equal to p. At A in AB make $\angle$BAP$=\angle$A and from AP cut off AD$=q$.

With centres B, D and with radius q and p respectively draw two arcs cutting each other at C. Join BC, DC. Then ABCD is the parallelogram required.

Proof : Join BD. In $\triangle^s$ABD and BCD, $\because$ AB$=p=$CD, AD$=q=$BC, and BD is common,

$\therefore$ the triangles are congruent.

$\therefore$ $\angle$ABD$=\angle$BDC ; but these are alternate angles,

$\therefore$ AB $\parallel$ CD.

$\because$ AB and CD equal and parallel,

$\therefore$ ABCD is a parallelogram, in which AB$=p$, AD$=q$ and the included $\angle$BAD$=$the given $\angle$A.

Problem 17

To construct a square on a given side.

Let AB be the given side, on which a square is to be constructed.

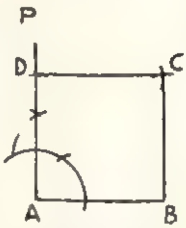


Fig. 50

Construction : At A draw AP perpendicular to AB. From AP cut off AD equal to AB. With centres B and D and radius AB draw two arcs cutting each other at C.

Join BC, DC. Then ABCD is the square required.

Proof : By construction the sides of the quadrilateral ABCD are equal, so ABCD is a parallelogram.

Again one of its angles, *i.e.*, ∠BAD is a right angle.
∴ ABCD is a square.

Areas

1. The *altitude* or *height* of a triangle is the perpendicular drawn from a vertex to the opposite side, considered as *base of the triangle.* [Fig. 51]

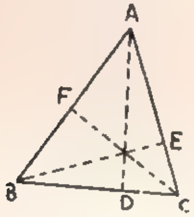


Fig. 51

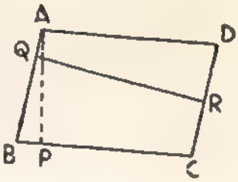


Fig. 52

2. The *altitude* or *height* of a parallelogram is the perpendicular distance between a pair of its parallel sides. [Fig. 52]

3. The angles of a rectangle are right angles. So in the rectangle ABCD, either AB or CD is its altitude.

4. Triangles and parallelograms are said to be *between the same parallels*, when their bases are on one of two parallel st. lines and their opposite vertices or sides are on the other.

5. The triangles and parallelograms between the same parallels are of the same altitude.

6. Triangles and parallelograms having equal altitudes can be placed between two parallel st. lines.

Theorem 26

Parallelograms on the same base and between the same parallels are equal in area.

Let the parallelograms ABCD and ABEF be on the same base AB and between the same parallels AB and DE. *To prove* that pram. ABCD = parm. ABEF.

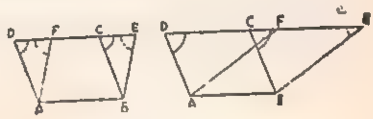


Fig. 53

Proof : In $\triangle$AFD and $\triangle$BCE,

$\angle$ADF = corresponding $\angle$BCE ($\because$ AD $\parallel$ BC),

$\angle$AFD = corresponding $\angle$BEC ($\because$ AF $\parallel$ BE),

and AD = BC (being opposite sides of a parm.)

$\therefore$ the triangles are congruent,

$\therefore$ the quadrilateral ABED $-$ $\triangle$BCE = the quad. ABED $-$ $\triangle$AFD,

$\therefore$ parm. ABCD = parm. ABEF.

[N B. *The area of a parallelogram = base × altitude.*]

Corollaries : (1) Parallelograms on the same base and of equal altitudes are equal in area.

Proof : $\because$ the parms. are of equal altitudes, $\therefore$ they are between the same paralleels.

[Now prove as in Theorem 26]

(2) Parallelograms on equal bases and between the same parallels are equal in area.

[**Hints :** Place the parms. on the same base and then prove as in theorem 26.]

(3) Parallelograms on equal bases and of equal altitudes are equal in area.

[**Hints :** Place the parms. on the same base and on the same side of it. Since their altitudes are equal, they must be between the same parallels. Then prove as in Theorem 26].

Theorem 27

If a triangle and a parallelogram stand on the same base and are between the same parallels, the area of the triangle is half the area of the parallelogram.

Let the $\triangle$ABC and the parm. BCDE stand on the same base BC and between the same parallels BC and EF.

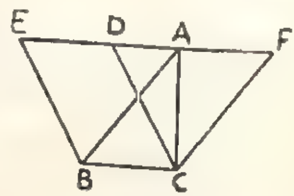


Fig. 54

To Prove that the $\triangle$ABC = half the parm. BCDE.

Construction: Draw CF parallel to BA meeting EF in F.

Proof: $\because$ BC ∥ AF (hyp.) and BA ∥ CF (cons.) ;

$\therefore$ ABCF is a parallelogram of which AC is a diagonal.

$\because$ A diagonal bisects a parallelogram,

$\therefore$ $\triangle$ABC = half the parm. ABCF.

Again, the parms. BCDE and ABCF are on the same base BC and between the same parallels BC and EF ;

$\therefore$ they are equal in area.

$\therefore$ $\triangle$ABC = half the parm. BCDE.

Corollaries: The area of a triangle is half the area of a parallelogram (1) if they are on the same base and of the same altitude, (2) if they are on equal bases and between the same parallels, (3) if they are on equal bases and of equal altitudes.

[N. B. (1) *The area of a triangle* = $\frac{1}{2}$ × *base* × *altitude.*

(2) The area of a quadrilateral = $\frac{1}{2}$ × one diagonal × the sum of the perpendiculars on this diagonal from the opposite vertices.

(3) The area of a trapezium = $\frac{1}{2}$ × altitude × the sum of the parallel sides.

(4) The area of a rhombus = $\frac{1}{2}$ × the product of the diagonals.]

Theorem 28

Triangles on the same base and between the same parallels are equal in area.

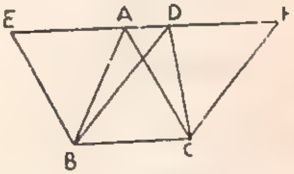


Fig. 55

Let the triangles ABC and DBC be on the same base BC and between the same parallels BC and AD.

To prove that $\triangle^s$ABC, DBC are equal in area.

Construction : Through B, draw BE ∥ CA meeting DA produced at E. Through C, draw CF ∥ BD meeting AD produced at F.

Proof : By construction ACBE and BCFD are parallelograms and they stand on the same base BC and between the same parallels BC and EF.

∴ the parm. ACBE = the parm. BCFD ;

but $\triangle$ABC = $\frac{1}{2}$ the parm. ACBE,

and $\triangle$DBC = $\frac{1}{2}$ the parm. BCFD ; ∴ $\triangle$ABC = $\triangle$DBC.

Corollaries : (1) Triangles on the same base and of the same altitude are equal in area.

(2) Triangles on equal bases and between the same parallels are equal in area.

(3) Triangles on equal bases and of equal altitudes are equal in area.

Theorem 29

Equal triangles on the same base and on the same side of it are between the same parallels.

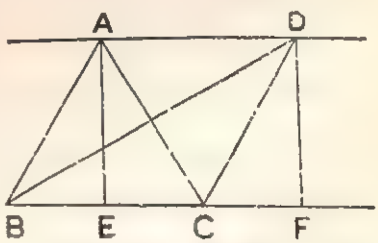


Fig. 56

Let $\triangle^s$ABC and DBC be equal in area and let them be on the same base BC and on the same side of it.

To prove that the triangles are between the same parallels, *i.e.*, to prove that AD and BC are parallel.

Construction : Join AD, and from A and D draw AE and DF perpendicular to BC.

Proof : $\triangle ABC = \frac{1}{2} BC.AE$ [$\because$ AE is its altitude]
and $\triangle DBC = \frac{1}{2} BC.DF$ [$\because$ DF is its altitude];
but $\triangle ABC = \triangle DBC$ (hyp.)
$\therefore$ $\frac{1}{2} BC.AE = \frac{1}{2} BC.DF$, $\therefore$ $AE = DF$.

Again, $\because$ AE, DF are both perpendicular on the same st. line BF, $\therefore$ AE $\parallel$ DF.

$\therefore$ AE, DF are both equal and parallel,

$\therefore$ AD and BC are parallel.

Problem 18

To construct a parallelogram equal in area to a given triangle and having an angle equal to a given angle.

Let ABC be the given triangle and P the given angle. To construct a parallelogram equal in area to $\triangle ABC$ and having an angle equal to $\angle P$.

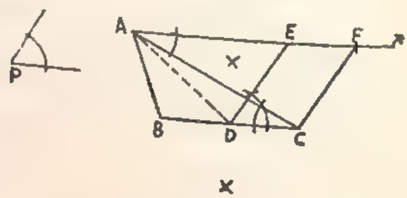


Fig. 57

Construction : Bisect BC at D. At D, draw $\angle CDE = \angle P$. Through A draw AX $\parallel$ BC.

Let AX cut DE at E.

From EX cut off EF equal to DC. Join CF. Then CDEF is the parallelogram required.

Proof : Join AD. $\because$ DC and EF are equal and parallel, $\therefore$ CDEF is a parallelogram.

Now, $\triangle ACD$ and param. CDEF stand on the same base DC and between the same parallels DC and AF,

$\therefore$ the param. $CDEF = 2\triangle ACD$.

Again, $\because$ AD is a median of $\triangle ABC$,

$\therefore$ $\triangle ABD = \triangle ACD$, $\therefore$ $\triangle ABC = 2\triangle ACD$.

$\therefore$ the param. $CDEF = \triangle ABC$ in area, and its $\angle CDE =$ the given $\angle P$. Hence CDEF is the parallelogram required.

[N. B. If a rectangle is to be drawn equal to a triangle, then make the $\angle CDE$ at D a right angle.]

Problem 19

To construct a triangle equal in area to a given quadrilateral.

Let ABCD be the given quadrilateral. To construct a triangle equal in area to quad. ABCD.

Construction : Join AC. Through D, draw DE parallel to AC, cutting BC produced in E. Join AE.

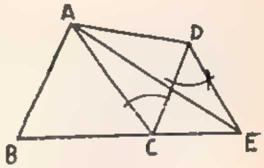


Fig. 58

Then $\triangle$ABE is the triangle required.

Proof : $\triangle$'ACE, ACD are on the same base AC and between the same parallels AC and DE. $\therefore$ $\triangle ACE = \triangle ACD$.

$\therefore$ $\triangle ACE + \triangle ABC = \triangle ACD + \triangle ABC$,

$\therefore$ $\triangle ABE$ = the quadrilateral ABCD.

[N. B. In the similar way *any polygon may* be reduced to an equivalent triangle.

Let ABCDE be the given polygon.

Through B and E draw st. lines parallel to AC and AD respectively, and let them cut CD produced in X and Y respectively.

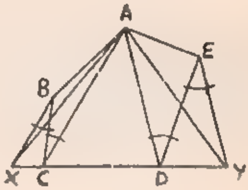


Fig. 59

Join AX, AY. Then $\triangle$AXY is the triangle required.

Proof : $\triangle ACX = \triangle ABC$ ($\because$ they are on the same base AC and between the same parallels AC, BX).

Similarly, $\triangle ADY = \triangle ADE$.

$\therefore$ $\triangle ACX + \triangle ADY = \triangle ABC + \triangle ADE$.

Adding $\triangle$ACD to both sides,

$\triangle ACX + \triangle ACD + \triangle ADY = \triangle ABC + \triangle ACD + \triangle ADE$.

$\therefore$ $\triangle AXY$ = the polygon ABCDE.

Problem 20

To bisect a triangle by a straight line drawn from a given point on one of its sides.

Let P be the given point on the side BC of the $\triangle$ABC.

To bisect $\triangle$ABC by a st. line drawn through P.

Construction : Join AP and bisect BC in D. Draw DQ∥PA, and let DQ meet AB in Q. Join PQ.

Then PQ bisects the △ABC

Proof : Join AD. ∵ AD is a median of △ABC, ∴ △ABD = ½△ABC.

Now, △PDQ and △ADQ are on the same base DQ and between the same parallels QD, AP, ∴ △PDQ = △ADQ.

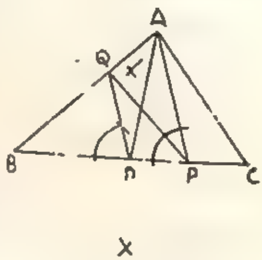


Fig. 60

∴ △PDQ + △BDQ = △ADQ + △BDQ

∴ △BPQ = △ABD = ½△ABC. ∴ PQ bisects the △ABC.

Problem 21

To trisect a triangle by two st. lines drawn from a given point on one of its sides.

Let P be the given point on the side BC of △ABC.

To trisect the △ABC by two st. lines drawn through P.

Construction : Trisect BC at D and E. Join AP. Through D and E draw two st. lines parallel to PA and cutting AB and AC at Q and R respectively. Join PQ, PR. Then PQ and PR trisect the △ABC.

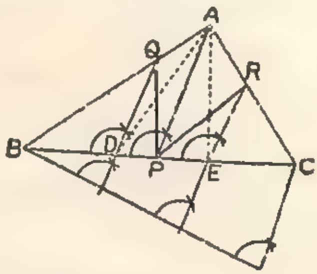


Fig. 61

Proof : Join AD, AE. ∵ △ABD, △ADE and △AEC are on equal bases and of the same altitude, ∴ they are equal in area. ∴ △ABD = △AEC = ⅓ △ABC.

∵ △DPQ and △AQD are on the same base DQ and between the same parallels DQ and PA, ∴ △DPQ = △AQD.

∴ △DPQ + △BDQ = △AQD + △BDQ.

∴ △BPQ = △ABD = ⅓△ABC.

Similarly, △CPR = △AEC = ⅓△ABC.

∴ the remaining portion AQPR = ⅓△ABC.

Hence the st. lines PQ and PR trisect the △ABC.

Problem 22

To bisect a quadrilateral by a straight line drawn from an angular point.

Let ABCD be the given quadrilateral, which is to be bisected by a st. line drawn through the angular point A.

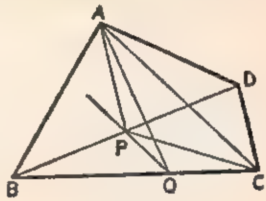


Fig. 62

Construction : Join AC, BD. Bisect BD at P, and draw PQ parallel to AC and meeting BC in Q. Join AQ. Then AQ bisects the quadrilateral ABCD.

Proof : Join AP, CP.

$\because$ AP is a median of $\triangle$ABD, $\therefore$ $\triangle$APD $=\frac{1}{2}\triangle$ABD, Similarly $\triangle$PCD $=\frac{1}{2}\triangle$BCD. $\therefore$ the quadrilateral APCD $=\frac{1}{2}$ quad. ABCD.

Again, $\triangle$APC and AQC are on the same base AC and between the same parallels AC, PQ. $\therefore$ $\triangle$APC $=\triangle$AQC.

Add $\triangle$ADC to both sides Then quad. APCD $=$ quad. AQCD.

$\therefore$ the quad. AQCD $=\frac{1}{2}$ quad. ABCD.

Hence, the st. line AQ bisects the quadrilateral ABCD.

Theorem 30

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

Let ABC be a right-angled triangle in which $\angle$BAC is a right angle.

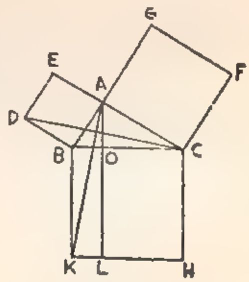


Fig. 63

To prove that $BC^2 = AB^2 + AC^2$.

Construction : On AB, AC and BC draw the squares ABDE, ACFG and BCHK respectively.

From A draw AL parallel to BK, meeting BC in O and KH in L.

Join CD, AK.

Proof : $\because$ the adjacent angles BAC and BAE are right angles, $\therefore$ AC and AE are in the same

st. line. Now, $\angle ABD = \angle CBK$ ($\because$ each is a right angle),

$\therefore$ $\angle ABD + \angle ABC = \angle ABC + \angle CBK$,

$\therefore$ $\angle DBC = \angle ABK$. Then, in $\triangle$s DBC, ABK, $\because$ BD = AB, BC = BK, and the included $\angle DBC$ = the included $\angle ABK$,

$\therefore$ $\triangle DBC \equiv \triangle ABK$. Now, the sq. BE and $\triangle DBC$ are on the same base BD and between the same paralles BD and CE.

$\therefore$ the sq. BF $= 2\triangle DBC$.

Again, the rectangle BKLO and $\triangle ABK$ are on the same base BK and between the same parallels BK and AL.

$\therefore$ the rect. BKLO $= 2\triangle ABK = 2\triangle DBC$.

$\therefore$ the rect. BKLO = the square ABDE.

Similarly, by joining BF and AH, it can be proved that the rect. CHLO = the sq. ACFG.

$\therefore$ the rect. BKLO + the rect. CHLO = the sq. BE + the sq. CG,

$\therefore$ the sq. BH = the sq. BE + the sq. CG, *i.e.*, $BC^2 = AB^2 + AC^2$.

[N. B. This theorem is known as the **Theorem of Phythagoras.**]

Theorem 31

[*Converse of Theorem 30*]

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.

Let ABC be a triangle of which $AC^2 = AB^2 + BC^2$.

To prove that $\angle ABC$ is a right angle.

Construction : Draw EF equal to BC, and on EF draw ED perpendicular to EF and equal to AB. Join DF.

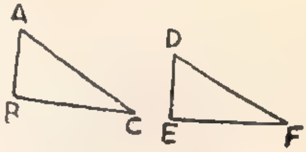


Fig. 64

Proof : $\because$ AB = DE, BC = EF,

$\therefore$ $AB^2 + BC^2 = DE^2 + EF^2 = DF^2$ ($\because$ DEF is a rt. angle),

but $AB^2 + BC^2 = AC^2$ (hyp.), $\therefore$ $AC^2 = DF^2$, $\therefore$ AC = DF.

Now, in $\triangle$s ABC, DEF, $\because$ AB = DE, BC = EF and AC = DF,

$\therefore$ the triangles are congruent.

$\therefore$ $\angle ABC = \angle DEF = 1$ right angle.

Problem 23

To construct a square twice, thrice, four times, etc. a given square.

Let l be the length of a side of the given square. To construct squares equal to twice, thrice, four times, etc. the given square.

Fig. 65

Construction : Take any st. line OX and draw OY⊥OX. From OX and OY cut off OQ and OP respectively each equal to l. Join PQ. From OX cut off OR equal to PQ and join PR. Again, from OX cut off OS equal to PR and join PS. Then the squares drawn on PQ, PR and PS are respectively twice, thrice, four times the given square.

Proof : $\because$ $\angle O$ is a right angle,

$$\therefore \quad PQ^2 = OP^2 + OQ^2 = l^2 + l^2 = 2l^2.$$

$$PR^2 = OP^2 + OR^2 = OP^2 + PQ^2 \; [\because \; OR = PQ]$$
$$= l^2 + 2l^2 = 3l^2.$$

$$PS^2 = OP^2 + OS^2 = OP^2 + PR^2 \; [\because \; OS = PR]$$
$$= l^2 + 3l^2 = 4l^2.$$

In this way a square may be drawn equal to *any times* a given square.

Locus

If a point moves in accordance with some given condition or conditions, the path traced out by it is called its *locus.*

Hence the locus of a point must be a *line*, straight or curved.

Examples : (1) If a point moves without changing its direction, its locus will be a *straight line.*

(2) If a point moves so as to be always equidistant from a fixed point, its locus will be the *circumference of a circle.*

(3) If a point moves so as to be always equidistant from a give st. line, its locus will be a pair of st. lines parallel to the given st. line.

[N. B. A locus is correctly obtained, if it can be shown (1) that every point that satisfies the given condition lies on the locus, and (2) that every point on the locus satisfies the given condition.]

Theorem 32

The locus of a point which is equidistant from two fixed points is the perpendicular bisectors of the straight line joining the two fixed points.

Let A, B be two fixed points.

To prove that the perpendicular bisector of AB is the locus of a point which moves always equidistant from A and B. So it is to be proved that (1) any point equidistant from A and B must lie on that perp. bisector and (2) any point on that perp. bisector is equidistant from A and B.

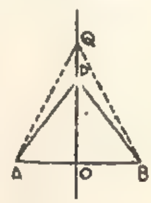


Fig. 66

Construction : Join AB and let O be its middle point. Let P be any position of the moving point equidistant from A, B.

Then PA = PB. Join PA, PB and PO.

Proof : In $\triangle^s$AOP and BOP PA = PB, AO = BO and PO is common to both triangles, ∴ the triangles are congruent. ∴ ∠AOP = ∠BOP, but these are adjacent angles, ∴ PO is perpendicular to AB.

∴ PO is the perpendicular bisector of AB.

Hence P lies on the perpendicular bisector of AB.

(2) Again, suppose Q to be any point on PO, the perp. bisector of AB. Join AQ, BQ.

Now, in $\triangle^s$AOQ, BOQ, ∵ AO = BO, OP is common to both the triangles and the included ∠AOQ = the included ∠BOQ, ∴ the triangles are congruent, ∴ AQ = BQ.

∴ Any point on PO is equisdistant from A, B.

Hence, the perpendicular bisector of AB is the locus of the point which moves equidistant from A and B.

[N. B. (i) Here the st. line PO produced infinitely both ways is the locus.

(ii) Taking any point R outside PO, it can be shown that R is not equisdistant from A and B.]

Theorem 33

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

Let AB and CD be two st. lines intersecting at O.

To prove that the locus of the point equidistant from AB, CD is the pair of bisectors of the angles between AB and CD ; that is, to prove that (1) *any point* equidistant from AB, CD lies on either bisector of the angles between AB, CD and (2) *any point* lying on any of the two bisectors is equidistant from AB and CD.

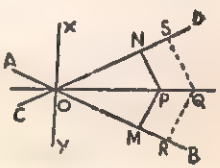


Fig. 67

Proof : (1) Let P be *any* point within ∠BOD so that it is equidistant from AB and CD, *i.e.*, the perps. PM and PN drawn from P on AB, CD respectively are equal. Join OP.

In the right-angled $\triangle^s$ POM and PON, PM = PN (hyp.) and the hypotenuse OP is common, ∴ the triangles are congruent ; ∴ ∠POM = ∠PON, ∴ OP is the bisector of ∠BOD. Hence P lies on the bisector of the ∠BOD.

(2) Let Q be *any* point on OP and let QR and QS be perpendiculars on AB, CD respectively.

Then, in $\triangle^s$ QOR and QOS, ∠QOR = ∠QOS,

∠QRO = ∠QSO [∵ each is a right angle] and OQ is common to both $\triangle^s$, ∴ the $\triangle^s$ are congruent.

∴ QR = QS. ∴ Q is equidistant from AB, CD.

∴ it is proved that the bisector of the ∠BOD is the locus of points equidistant from AB and CD.

If P lies within $\angle AOD$, it can be similarly proved that the bisector XO of $\angle AOD$ is the locus of P.

Hence, the locus of a point equidistant from two intersecting st. lines is the pair of bisectors of the angles between the st. lines.

[N. B. $\because$ PO produced is the bisector of $\angle AOC$, $\therefore$ it will be the locus of P when it lies within $\angle AOC$. Similarly XY is the locus of P when it lies within $\angle AOD$ or $\angle BOC$.]

Theorems regarding concurrent st. lines

If three or more straight lines meet (or intersect) at a point, they are called *concurrent st. lines*. The point where they meet is said to be the *point of concurrence*.

The points that lie on the same straight line are said to be *collinear*.

Theorem 34

The perpendicular bisectors of the sides of a triangle are concurrent.

Let ABC be a triangle and D, E, F be the middle points of AB, BC and AC respectively.

Let DS and ES be drawn perpendicular to AB and BC, meeting at S. Join SF. To show that SF is perp. to AC. Join SA, SB, SC.

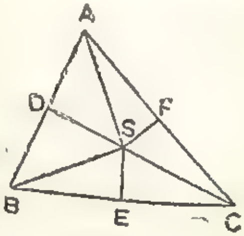


Fig. 68

Proof : $\because$ S lies on DO, the perpendicular bisector of AB,

$\therefore$ AS = BS (Th. 32).

Again, $\because$ S lines on ES, the perp. bisector of BC,

$\therefore$ BS = CS (Th. 32). $\therefore$ AB = BS = CS.

$\therefore$ S is equidistant from A and C.

$\therefore$ S lies on the perpendicular bisector of AC.

$\therefore$ SF is perpendicular to AC.

Hence, the perpendicular bisectors of the sides of a triangle are concurrent.

[N.B. This pt. of concurrence is generally denoted by S and is known as the *circum-centre* of the triangle.]

Theorem 35

The bisectors of the angles of a triangle are concurrent.

Let ABC be a triangle and let BI and CI, the bisectors of ∠ABC and ∠ACB, meet at I. Join AI. To show that AI is the bisector of ∠BAC.

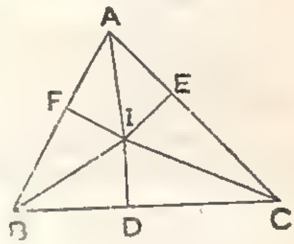


Fig. 69

From I draw ID, IE and IF perp. to BC, AC and AB respectively.

Proof : ∵ the point I lies on the bisector BI of ∠ABC,

∴ ID = IF [Th. 33].

Again, ∵ I lies on the bisector CI of ∠ACB,

∴ ID = IE. ∴ IF = ID = IE.

∴ I is equidistant from AB and AC, ∴ I lies on the bisector of ∠BAC. ∴ AI is the bisector of ∠BAC.

Hence, the bisectors of the angles of a triangle are concurrent.

[N. B. This pt. of concurrence is denoted by I and is known as the *in-centre* of the triangle.]

Corollary : The bisectors of any two exterior angles of a triangle and the bisector of the third interior angle are concurrent.

Theorem 36

The medians of a triangle are concurrent.

Let ABC be a triangle, two of whose medians BE and CF intersect at G.

Join AG and produce it to meet BC in D. To show that AD is the third median of △ABC.

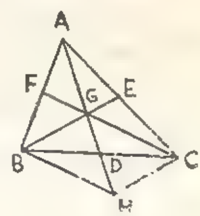


Fig. 70

Construction : From B draw BH ∥ FC and let BH meet AD produced at H.

Join CH.

Proof : In △ABH, F is the middle pt. of AB and FG ∥ BH,

∴ G is the middle pt. of AH.

Again, in $\triangle ACH$, G and E are the mid points AH and AC,

$\therefore$ GE $\parallel$ HC, *i.e.*, BE $\parallel$ HC. $\therefore$ BGCH is a parallelogram.

$\because$ the diagonals of a parm. bisect each other,

$\therefore$ BD = CD. $\therefore$ AD is a median of $\triangle ABC$.

Hence the medians of a triangle are concurrent.

Corollary : The medians of a triangle intersect at a point of trisection of each.

[Draw the fig. 70] It has been shown in Theorem 36 that AG = GH and GD = DH.

$\therefore$ AG = GH = 2GD. $\therefore$ AD = AG + GD = 2GD + GD = 3GD.

$\therefore$ GD = $\frac{1}{3}$AD. Similarly GE = $\frac{1}{3}$BE and GF = $\frac{1}{3}$CF.

$\therefore$ The medians of a triangle intersect at a point of trisection of each.

[N. B. The point of intersection of the medians of a triangle is called the centroid of the triangle and is represented by the letter G.]

(A) *The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.*

Or, The altitudes of a triangle are concurrent.

Let ABC be a triangle and AP, BQ, CR be perpendiculars to BC, CA, AB respectively.

To prove that AP, BQ, CR are concurrent.

Construction : Through A, B and C draw DE, DF and EF parallel to the opposite sides, forming the $\triangle DEF$.

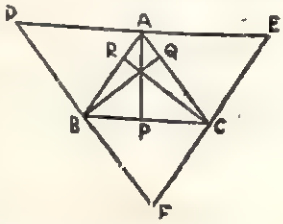


Fig. 71

Proof : By construction, ACBD and ABCE are parms.,

$\therefore$ AD = BC = AE, $\therefore$ A is the mid point of DE.

Similarly B and C are the mid points of DF and DE respectively.

Again, $\because$ AP$\perp$BC and BC $\parallel$ DE, $\therefore$ AP$\perp$DE.

Similarly BQ$\perp$AC and CR$\perp$EF.

$\therefore$ AP, BQ and CR are the perpendicular bisectors of the sides of $\triangle DEF$.

$\therefore$ AP, BQ and CR are concurrent.

[N.B. The point of intersection of these altitudes is called the orthocentre of the triangle and is represented by O.]

For CLASS X

The Circle

Some more definitions are given here excepting those you have already learnt.

Segment : A segment of a circle is a portion of the circle bounded by an arc and the chord of the arc.

The angle subtended by the chord of the segment at any point on the bounding arc is called an **angle in the segment.**

A quadrilateral or a rectilineal figure is said to be **cyclic**, if all its vertices lie on the circumference of the same circle.

The points which lie on the circumference of the same circle are said to be **concyclic.**

Circum-circle : The circle drawn through the vertices of a triangle is called its circum-circle.

Its centre is called the circum-centre and its radius the circum-radius.

A polygon is said to be inscribed in a circle, if the circle passes through all the vertices of the figure.

Then the circle is called its circum-circle.

A rectilineal figure is said to be **circumscribed** about a circle, if all its sides touch the circle.

Here the circle is called the in-circle or inscribed circle.

Theorem 37

One, and only one, circle can be drawn through three points not in the same straight line.

Let A, B, C be three points not in the same st. line.

To prove that one circle, and only one, can pass through A, B and C.

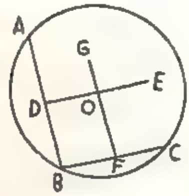


Fig. 72

Construction : Join AB and BC, and let DE, FG be the perpendicular bisectors of AB and BC respectively.

Proof : Since AB and BC are not in the same st. line, DE and FG are not parallel and must therefore intersect at some point.

Suppose they meet at O.

Then, $\because$ DE is the perp. bisector of AB, $\therefore$ every point on DE is equidistant from A and B.

Again, $\because$ FG is the perp. bisector of BC,

$\therefore$ every point on FG is equidistant from B and C.

$\therefore$ O, the point common to DE and FG, is equidistant from A, B and C.

$\therefore$ OA = OB = OC.

$\therefore$ The circle drawn with centre O and with radius OA will pass through A, B and C.

Now, $\because$ two st. lines can intersect at but one point, $\therefore$ O is the only point which is equidistant from A, B and C.

Hence, only one circle can be drawn through A, B and C.

Corollaries : (1) Circles, which have three points (on the circumferences) in common, coincide.

(2) Two circles cannot cut each other in more than two points.

Axiom (Theorem) : In equal circles (or in the same circle), equal chords cut off equal arcs and subtend equal angles at the centres.

Conversely : If two arcs are equal or subtend equal angles at the centres, the chords of the arcs are equal.

Theorem 38

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is perpendicular to the chord.

Conversely, *the perpendicular drawn from the centre to a chord, bisects the chord.*

Let ABC be a circle whose centre is O and let AB be a chord which is not a diameter.

Let OD, drawn from O, bisect AB at D.

To prove that OD is perpendicular to AB.

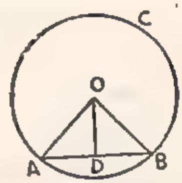


Fig. 73

Join OA and OB.

Proof : In $\triangle^s$ OAD and OBD,

OA = OB (being radii of the circle),

AD = BD (Hyp.), and OD is common.

$\therefore$ the triangles are congruent,

$\therefore$ $\angle$ODA = $\angle$ODB and these being adjacent angles are right angles. $\therefore$ OD is perpendicular to AB.

Conversely : Let O be the centre of the circle ABC and AB be a chord which is not a diameter. Let OD be perpendicular to AB from O.

To prove that AD = BD. Join OA, OB.

Proof : In the right-angled $\triangle^s$ OAD, OBD, the hypotenuse OA = the hypotenuse OB (being radii of the circle), and OD is common to both,

$\therefore$ the triangles are congruent.

$\therefore$ AD = BD, that is, OD bisects AB.

Corollaries : (1) In a circle, the perpendicular bisector of any chord passes through the centre.

(2) A st. line cannot cut a circle at more than two points.

[If possible let the st. line ABC cut a circle at A, B and C. $\because$ AB is a chord, $\therefore$ the centre must lie on the perp. bisector of AB. Also $\because$ AC is a chord, the centre must lie on the perp. bisector of AC. $\therefore$ the pt. of intersection of the two perpendicular bisectors will be the centre of the circle. But ABC being one st. line, the perpendiculars on it must be parallel and will not meet. Also a circle cannot have two centres.]

Theorem 39

Equal chords of a circle are equidistant from the centre.

Conversely, *chords which are equidistant from the centre are equal.*

Let AB and CD be equal chords of a circle whose centre is O.

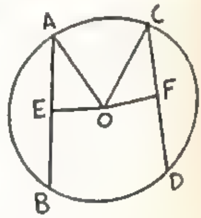


Fig. 74

Let OE and OF be perpendiculars from O to AB and CD respectively.

To prove that OE = OF.

Construction : Join AO and CO.

Proof : $\because$ OE is perp. to the chord AB, $\therefore$ OE bisects AB, $\therefore$ AE $= \frac{1}{2}$AB.

Similarly, CF $= \frac{1}{2}$CD. But AB = CD (hyp.), $\therefore$ AE = CF.

Now, in the right-angled $\triangle^s$ AEO and CFO, the hypotenuse AO = the hypotenuse CO (being radii of the circle) and AE = CF (proved); $\therefore$ the triangles are congruent.

$\therefore$ OE = OF, i e., AB, CD are equidistant from O.

Conversely : O is the centre of the circle, AB and CD are two chords. The perpendiculars OE and OF on AB and CD respectively are equal.

To prove that AB = CD,

Construction : Join OA and OC.

Proof : $\because$ OE$\perp$AB, $\therefore$ AE $= \frac{1}{2}$AB. Similarly, CF $= \frac{1}{2}$CD.

In the right-angled $\triangle^s$ AEO and CFO, the hypot. AO = the hypot. CO, and OE = OF (hypothesis) ;

$\therefore$ the triangles are congruent. $\therefore$ AE = CF.

But AB = 2AE and CD = 2CF, $\therefore$ AB = CD.

[N. B. The above proof will also hold good in the case of equal circles.]

An Additional Theorem 40

Of any two chords of a circle that which is nearer to the centre is greater than one more remote.

Conversely, *the greater of two chords is nearer to the centre than the less.*

Let AB, CD be two chords of a circle whose centre is O and let OE and OF be perpendiculars from O to AB and CD respectively.

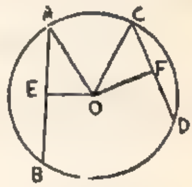


Fig. 75

To prove that (i) if $OE < OF$, then $AB > CD$;

Conversely (ii) if $AB > CD$, then $OE < OF$. Join AO, CO.

Proof : $\because$ $OE \perp AB$, $\therefore$ $AE = \frac{1}{2}AB$.

Similarly $CF = \frac{1}{2}CD$. $\because$ $\angle OEA$ and $\angle OFC$ are right angles,

$\therefore$ $OA^2 = OE^2 + AE^2$ and $OC^2 = OF^2 + CF^2$;

But $OA = OC$ (being radii of the circle),

$\therefore$ $OA^2 = OC^2$. $\therefore$ $OE^2 + AE^2 = OF^2 + CF^2 \cdots(1)$

Now, (i) if $OE < OF$, then $OE^2 < OF^2$,

$\therefore$ from (1) we have $AE^2 > CF^2$,

$\therefore$ $AE > CF$, $\therefore$ $AB > CD$.

Again, (ii) if $AB > CD$, then $AE > CF$, $\therefore$ $AE^2 > CF^2$.

$\therefore$ from (1) we have $OE^2 < OF^2$, $\therefore$ $OE < OF$.

Theorem 41

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Let ABC be a circle whose centre is O and let its arc BKC subtend the $\angle BOC$ at the centre and $\angle BAC$ at any pt. A on the remaining part of the circumference.

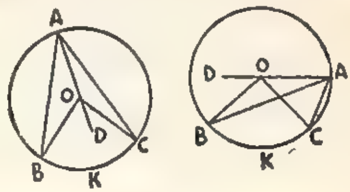


Fig. 76

To prove that $\angle BOC = \ \angle BAC$.

Construction : Join AO and produce it to D.

Proof : In $\triangle AOB$, $OA = OB$ (being radii of a circle).

$\therefore$ $\angle OAB = \angle OBA$.

$\therefore$ The exterior $\angle BOD = \angle OAB + \angle OBA = 2\angle OAB \cdots (1)$

Similarly, $\angle COD = 2\angle OAC \cdots (2)$.

Now, adding (1) and (2) in the first figure, we have

$\angle BOC = 2(\angle OAB + \angle OAC) = 2\angle BAC$.

Again, taking the difference of (2) and (1) in the second figure, we have

$\angle COD - \angle BOD = 2(\angle OAC - \angle OAB)$, *i.e.*, $\angle BOC = 2\angle BAC$.

Corollary : Equal arcs (or the same arc) subtend equal angles at the circumference.

Theorem 42

Angles in the same segment of a circle are equal.

Let BCDAB be a segment of a circle whose centre is O, and let BAC and BDC be angles in the segment.

To prove that $\angle BAC = \angle BDC$.

Construction : Join OB, OC.

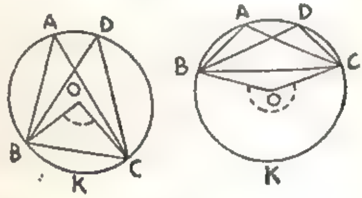


Fig. 77

Proof : $\because$ the arc BKC subtends $\angle BOC$ at the centre and $\angle BAC$ at the circumference, $\therefore$ $\angle BOC = 2\angle BAC$.

Similarly, $\angle BOC = 2\angle BDC$. $\therefore$ $\angle BAC = \angle BDC$.

Theorem 43

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

Let the st. line BC joining the two pts. B and C subtend equal angles BAC and BDC at two points A and D on the same side of BC.

To prove that A, B, C and D lie on a circle.

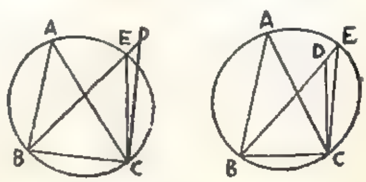


Fig. 78

Proof : Let a circle be drawn through A, B and C. If this circle does not pass

through D, it will cut BD or BD produced at some point. Let it cut BD at E. Join EC.

∵ $\angle$BAC and $\angle$BEC are in the same segment,

∴ $\angle$BAC $=\angle$BEC. But $\angle$BAC $=\angle$BDC (Hyp.),

∴ $\angle$BEC $=\angle$BDC, which is absurd; for CD and CE are not parallel and an exterior angle of a triangle cannot be equal to an interior angle.

∴ The circle drawn through A, B and C must pass through D. Hence A, B, C and D lie on a circle.

Theorem 44

The angle in a semi-circle is a right angle.

Let O be the centre and AB a diameter of the circle ADBC and let ACB be any angle in the semi-circle.

To prove that $\angle$ACB is a right angle.

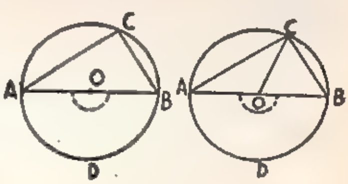


Fig. 79

Proof: $\angle$ACB is at the circumference and $\angle$AOB at the centre on the same arc ADB,

∴ $\angle$ACB $=\frac{1}{2}\angle$AOB; but $\angle$AOB is a straight angle, *i.e.*, 2 right angles,

∴ $\angle$ACB is one right angle.

[Otherwise] Join OC. ∵ OA $=$ OC (being radii of the circle)

∴ $\angle$OAC $=\angle$OCA. Also ∵ OB $=$ OC, ∴ $\angle$OBC $=\angle$OCB,

∴ the whole $\angle$ACB $=\angle$OAC $+\angle$OBC;

But $\angle$ACB $+\angle$OAC $+\angle$OBC $=2$ right angles,

∴ $2\angle$ACB $=2$ rt. angles, ∴ $\angle$ACB is a right angle.

Theorem 45

The angle in a segment greater than a semi-circle is less than a right angle and the angle in a segment less than a semi-circle is greater than a right angle.

Let ABC be a circle whose centre is O, and in the first figure let ACB be a segment greater than a semi-circle. *To prove that* $\angle$ACB in the segment is less than a right angle.

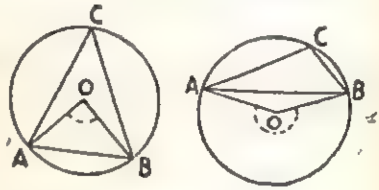


Fig. 80

Cons: Join OA, OB.

Proof: The arc AB subtends $\angle$AOB at the centre and $\angle$ACB at the circumference, $\therefore$ $\angle ACB = \frac{1}{2} \angle AOB$; but $\angle$AOB being an angle of a triangle is less than 2 right angles,

$\therefore$ $\angle$ACB is less than a right angle.

Again, in the second figure let ACB be a segment less than a semi-circle.

To prove that $\angle$ACB is greater than a right angle.

Join OA, OB.

Proof: $\angle$ACB at the circumference $= \frac{1}{2} \angle$AOB at the centre, standing on the same arc AB. But the reflex $\angle$AOB is greater than two right angles,

$\therefore$ $\angle$ACB is less than one right angle.

Theorem 46

The opposite angles of a cyclic quadrilateral are supplementary.

Let ABCD be a quadrilateral inscribed in a circle whose centre is O.

To prove that (*i*) $\angle ABC + \angle ADC =$ 2 right angles,

and (*ii*) $\angle BAD + \angle BCD = 2$ right angles.

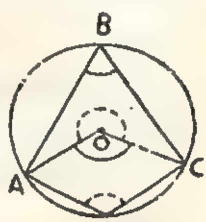


Fig. 81

Cons: Join OA, OC.

Proof: The arc ADC subtends $\angle$AOC at the centre and $\angle$ABC at the circumference, $\therefore$ $\angle ABC = \frac{1}{2} \angle AOC$.

Again, the arc ABC subtends $\angle$AOC at the centre and $\angle$ADC at the circumference,

$\therefore$ $\angle ADC = \frac{1}{2}$ the reflex $\angle$AOC. $\therefore$ $\angle ABC + \angle ADC = \frac{1}{2}(\angle AOC +$ the reflex $\angle AOC) = \frac{1}{2}$ of 4 rt. angles $=$ 2 rt. angles.

Similarly, joining OB and OD, it can be proved that $\angle BAD + \angle BCD = 2$ right angles.

Theorem 47

If a pair of opposite angles of a quadrilateral be supplementary, the quadrilateral is cyclic.

Let ABCD be a quadrilateral in which $\angle B + \angle D = 2$ rt. angles.

To prove that ABCD is a cyclic quadrilateral.

Proof : Let a circle be drawn through A, B and C. If this circle does not pass through D, it will cut AD or AD produced at some point. Let it cut AD at E. Join EC.

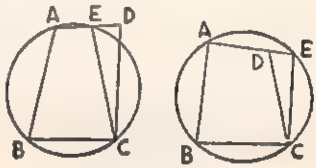


Fig. 82

Now, $\because$ ABCE is a cyclic quadrilateral, $\therefore$ $\angle ABC + \angle AEC = 2$ right angles,

but $\angle ABC + \angle ADC = 2$ right angles (Hyp.) ;

$\therefore$ $\angle AEC = \angle ADC$ which is absurd, because the exterior angle of a triangle cannot be equal to an interior opposite angle.

$\therefore$ The circle drawn through A, B and C must pass through D.

Hence the quadrilateral ABCD is cyclic.

Tangent

A **tangent** to a circle is a straight line which meets the circle at *one point only* and being produced does not cut it. Evidently a tangent has only one point common with the circle and hence no part of a tangent lies within the circle.

A tangent is said to **touch** the circle at the point where it meets the circle and this point is called the **point of contact.**

A secant of a circle is a straight line of indefinite length which cuts the circle at two points.

Two circles are said to **touch** each other if they meet at one point and the common point where they meet is called the **point of contact.**

If of the two circles which touch each other, one is wholly within the other, they are said to touch **internally**; and when one circle is wholly outside the other, they are said to touch **externally**.

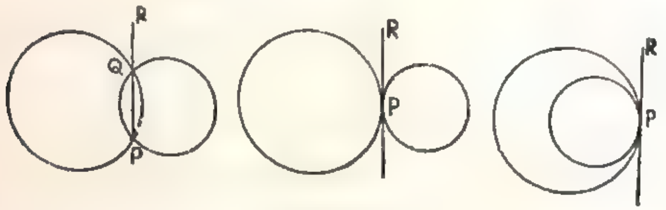


Fig. 83

A straight line that touches two circles is called a **common tangent** to the circles.

Two circles that touch each other have a common tangent at their point of contact.

Conversely, if two circles have a common tangent at a point where they meet, they touch each other at that point.

Theorem 48

The tangent at any point of a circle and the radius through the point are perpendicular to each other.

Let PT be a tangent at the point P to a circle whose centre is O, and let OP be the radius through P.

To prove that PT and OP are perpendicular to each other.

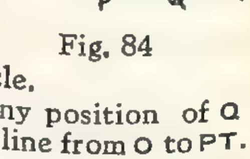


Fig. 84

Proof: Take any point Q in PT.

Since PT is a tangent at P, every point in PT except P must be outside the circle. ∴ Q is a point outside the circle.

∴ the radius OP<OQ, and this is true for any position of Q on PT except P. ∴ OP is the shortest straight line from O to PT.

∴ OP is perpendicular to PT.

Hence PT and OP are perpendicular to each other.

Cor. 1. The perpendicular to a radius at the point where the radius meets the circle is a tangent to the circle.

Cor. 2. One, and only one, tangent can be drawn to a circle at a given point on the circumference.

Cor. 3. The perpendicular to a tangent at the point of contact passes through the centre of the circle.

Theorem 49

Two tangents can be drawn to a circle from an external point.

Let ABC be a circle whose centre is O and let T be a point outside the circle.

To prove that two tangents can be drawn to the circle ABC from T.

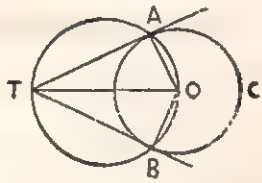


Fig. 85

Construction : Join TO and draw a circle on TO as diameter. Since T is without and O is within the given circle, this circle will cut the circle ABC at two points. Let A and B be the two points of intersection.

Join OA, OB, TA and TB.

Proof : $\because$ $\angle$OAT and $\angle$OBT are angles in a semicircle,

$\therefore$ each of the angles OAT and OBT is a right angle.

$\therefore$ TA and TB are respectively perpendicular to the radii OA and OB.

$\therefore$ TA and TB are two tangents to the circle at A and B respectively.

Hence two tangents can be drawn from T to the circle ABC.

Theorem 50

The two tangents to a circle from an external point are equal and they subtend equal angles at the centre.

Let P be an external point from which tangents PA and PB are drawn to the circle ABC, whose centre is O.

To prove that (*i*) PA = PB and (*ii*) $\angle$POA = $\angle$POB.

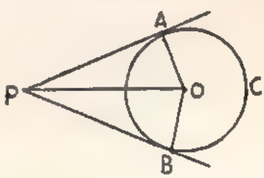


Fig. 86

Construction : Join OP, OA and OB.

Proof : $\because$ PA and PB are tangents to the circle and OA and OB are radii through the points of contact,

$\therefore$ each of the angles OAP and OBP is a right angle.

Now, in the right-angled triangles OAP and OBP, OA = OB (being radii of the circle), the hypotenuse OP is common to both ; $\therefore$ the triangles are congruent.

$\therefore$ PA = PB and $\angle$POA = $\angle$POB.

Corollary : PO bisects the angle between the tangents.

Theorem 51

If two circles touch each other, the point of contact lies in the straight line through the centres.

Let two circles whose centres are A and B touch each other internally or externally at P.

To prove that P lies on the straight line AB, joining the centres A, B.

Cons : Join AP and BP.

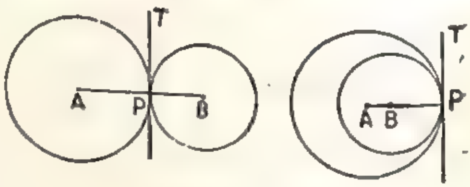


Fig. 87

Proof : ∵ The two circles touch at P, ∴ they have a common tangent at P. Let PT be the common tangent.

∵ AP and BP are two radii through the point of contact P, ∵ AP and BP are perpendicular to PT at P,

∴ AP and BP are in the same straight line.

∴ P lies on the st. line AB joining the centres.

GEOMETRY

SOLUTIONS OF RIDERS & EXERCISES.

1. **Theorem**—If two straight lines intersect, the vertically opposite angles are equal. [C.U. '11, '29]

1. *The bisectors of the vertically opposite angles are in the same straight line.*

Let the straight lines AB and CD intersect at O and let PO and QO bisect the $\angle$AOC and $\angle$BOD respectively.

It is required to prove that PO and QO are in one straight line.

Proof: $\angle$AOP = COP (hyp). $\angle$AOD = vertically opposite $\angle$BOC and $\angle$DOQ = $\angle$BOQ (hyp.)

$\therefore$ $\angle$AOP + $\angle$AOD + $\angle$DOQ

= $\angle$POC + $\angle$BOC + $\angle$BOQ = half of four right angles = 2 right angles.

($\because$ the angles at the point O together = 4 right angles.)

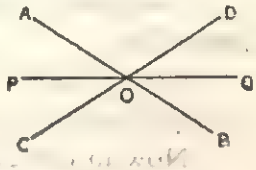


Fig. 1.

$\therefore$ $\angle$POQ is a straight angle.

$\therefore$ PO and QO are in the same straight line.

2. *If the straight lines* AB, CD *intersect at* O, *prove that the bisector of the angle* AOC, *when produced through* O, *also bisects the* $\angle$BOD.

[**Hints:** Let PO bisect $\angle$AOC and be produced to Q. It is required to prove that QO bisects $\angle$BOD. [see the fig. 1]

Proof: $\angle$AOP = vertically opposite $\angle$BOQ

and $\angle$POC = vertically opposite $\angle$DOQ.

Now, $\because$ $\angle$AOP = $\angle$POC, $\therefore$ $\angle$BOQ = $\angle$DOQ.

$\therefore$ OQ bisects $\angle$BOD.

2. **Theorem**—If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are congruent.

3. *The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.* [C. U.]

Of the isosceles triangle ABC, AB = AC. AD bisects the $\angle$BAC meeting the base BC in D. To prove that BD = DC and AD$\perp$BC.

Proof : In the $\triangle$s ABD, ACD, $\because$ AB = AC, AD is common to both triangles and the included $\angle$BAD = the included $\angle$CAD,

$\therefore$ the triangles are equal in all respects ;

$\therefore$ BD = DC and $\angle$ADB = $\angle$ADC, but these are adjacent angles, $\therefore$ each is a right angle. $\therefore$ AD$\perp$BC.

4. *The triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.*

ABC is an equilateral triangle and D, E, F are the middle points of its sides BC, AC and AB respectively. Join the middle points so as to form the triangle DEF.

To prove that the $\triangle$DEF is equilateral.

Proof : F is the middle point of AB, $\therefore$ AF = BF ; similarly, AE = $\frac{1}{2}$AC and BD = $\frac{1}{2}$BC.

But AC = BC, $\therefore$ AE = BD. $\because$ $\triangle$ABC is equilateral, $\therefore$ its angles are all equal.

Now in the $\triangle$s AEF and BFD,

AF = BF, AE = BD and $\angle$FAE = $\angle$DBF,

$\therefore$ $\triangle$s AEF, BFD are equal in all respects, $\therefore$ FE = FD.

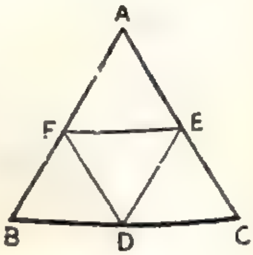


Fig. 2

Similarly, $\triangle$BFD $\equiv$ $\triangle$EDC and FD = DE.

$\therefore$ FE = FD = DE. $\therefore$ $\triangle$DEF is equilateral.

5. ABCDEF *is a regular hexagon. Show that* ACE *is an equilateral triangle.* [C. U. 1911]

ABCDEF is a regular hexagon. To prove that the $\triangle$ACE is an equilateral triangle. Join AC, AE and CE.

Proof : In the triangles ABC and AFE, AB = AF, BC = FE and $\angle$ABC = $\angle$AFE ($\because$ all sides and angles of a regular hexagon are equal.), $\therefore$ the two triangles are congruent. $\therefore$ AC = AE.

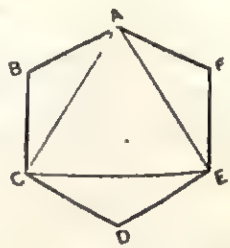


Fig. 3.

Thus, from the $\triangle$s ABC, CDE it can be proved that AC = CE. $\therefore$ AE = AC = CE,

$\therefore$ ACE is an equilateral triangle.

Ex. 6. *If the bisector of the vertical angle of a triangle also bisects the base, then the triangle is isosceles.*

[*C. U.* '37 ; *D. B.* '36]

The bisector AO of the vertical $\angle$A of the $\triangle$ABC also bisects the base BC at O.

To prove that $\triangle$ABC is isosceles.

Produce AO to D so that DO = AO Join DC.

Proof : In the $\triangle^s$ ABO and CDO,

BO = CO (by hypothesis),

AO = DO (by construction) and

$\angle$AOB = vertically opposite $\angle$COD.

$\therefore$ the two triangles are congruent.

$\therefore$ AB = CD and $\angle$CDO = $\angle$BAO = $\angle$CAO,

$\therefore$ AC = CD. But AB = CD, $\therefore$ AB = AC.

$\therefore$ ABC is an isosceles triangle.

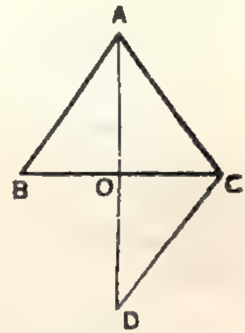


Fig. 4.

3. **Theorem**—If the three sides of one triangle are respectively equal to the three sides of another, then the two triangles are equal in all respects.

Ex. 7. *The straight line joining the vertex of an isosceles triangle to the mid-point of the base, bisects the vertical angle and is perpendicular to the base.*

ABC is an isosceles triangle of which AB = AC and D is the middle point of BC. Join AD. To prove that AD is the bisector of $\angle$BAC and perpendicular to BC.

Proof : In the $\triangle^s$ ABD, ACD, $\because$ AB = AC, BD = DC and AD is common to both triangles, $\therefore$ the two triangles are equal in all respects. $\therefore$ $\angle$BAD = $\angle$CAD and $\angle$ADB = $\angle$ADC, but these are adjacent angles, $\therefore$ each is a right angle.

$\therefore$ AD$\perp$BC. $\therefore$ AD is the bisector of the $\angle$BAC and is perpendicular to BC.

Ex. 8. *The diagonals of a rhombus bisect each other at right angles.*

[*C. U.* 1936]

[A rhombus is a quadrilateral which has all its sides equal, but its angles are not right angles.]

The diagonals AC and BD of the rhombus ABCD intersect each other at O. To prove that AC and BD bisect each other at right angles.

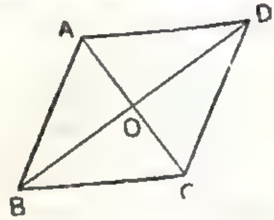


Fig. 5.

Proof : In the $\triangle$s ABC and ADC, AB = AD, BC = DC and AC is common to both triangles. $\therefore$ $\angle$BAC = $\angle$DAC.

Again in $\triangle$s ABO and ADO, AB = AD, the side AO is common and $\angle$BAO = $\angle$DAO, $\therefore$ the two triangles are equal in all respects. $\therefore$ BO = DO and $\angle$AOB = $\angle$AOD, but these are adjacent angles, so each angle is a right angle. $\therefore$ AC bisects BD at right angles.

Similarly it can be proved that BD bisects AC at right angles.

9. *The medians of an equilateral triangle are equal.*

ABC is an equilateral triangle of which AD, BE and CF are medians.

To prove that AD = BE = CF.

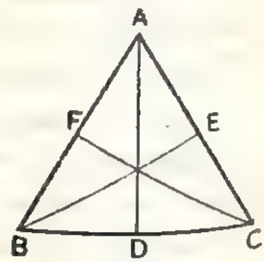


Fig. 6.

Proof : F is the middle point of AB, $\therefore$ BF = $\frac{1}{2}$AB. Similarly, CE = $\frac{1}{2}$AC. $\because$ AB = AC, $\therefore$ BF = CE.

Now, in $\triangle$s BFC and BEC, BF = CE, BC is common to both triangles and the included $\angle$FBC = the included $\angle$BCE ($\because$ all angles of the equilateral triangle ABC are equal), $\therefore$ the two triangles are equal in all respects. $\therefore$ BE = CF. Thus from the $\triangle$s BEA, ADB it can be proved that BE = AD, $\therefore$ AD = BE = CF.

10. *If the opposite sides of a quadrilateral are equal, the opposite angles will be equal.*

ABCD is a quadrilateral of which AB = CD and AD = BC. To prove that $\angle$ABC = $\angle$ADC and $\angle$BAD = $\angle$BCD. Join AC.

Proof : In the $\triangle$s ABC, ADC, $\because$ AB = DC, BC = AD, AC is common to both triangles, $\therefore$ the two triangles are equal in all respects. $\therefore$ $\angle$ABC = $\angle$ADC ; also $\angle$BAC = $\angle$ACD and $\angle$DAC = $\angle$ACB, $\therefore$ $\angle$BAC + $\angle$DAC = $\angle$ACD + $\angle$ACB, that is, $\angle$BAD = $\angle$BCD.

11 *A point* P *is taken within a rhombus* ABCD *such that the distances from the angular points* A *and* C *are equal. Show that* PB *and* PD *are in one and the same straight line.*

[*C. U.* '46]

[N. B. First write down the hypothesis and then the following.]

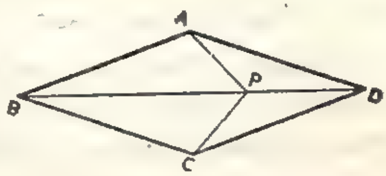


Fig. 7

Join AP, PC BP, and PD.

Proof : In the $\triangle$ APD, PCD, $\because$ AD = DC, AP = PC (hyp.) and PD is common to both triangles, $\therefore$ $\angle$APD = $\angle$CPD.

Similarly, from the $\triangle^s$APB, BPC we have $\angle$APB = $\angle$BPC, $\therefore$ $\angle$APB + $\angle$APD = $\angle$CPB + $\angle$CPD = $\frac{1}{2}$ of 4 right angles = 2 right angles.

$\therefore$ BP and PD are in the same straight line.

12. *If two isosceles triangles stand on the same base and on the same side of it, show that one will fall entirely within the other.*

[*C. U. 1914*]

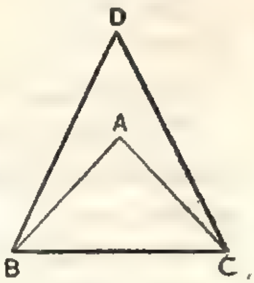


Fig. 8

Two isosceles $\triangle^s$ ABC, DBC stand on the same base BC and on the same side of it. To prove that $\triangle$ABC is entirely within the triangle DBC.

Proof : $\because$ the two isosceles triangles stand on the same base and on the same side of it, $\therefore$ their base angles are unequal to one another. For, if the base angles be equal, the two triangles will coincide. Suppose $\angle$DBC > $\angle$ABC.

$\therefore$ AB must fall within the $\angle$DBC.

Similarly, AC must fall within the $\angle$DCB ($\because$ $\angle$BCD > $\angle$ACB).

$\therefore$ A, the point of intersection of AB and AC, must fall within the $\triangle$DBC.

$\therefore$ $\triangle$ABC will fall entirely within the $\triangle$DBC.

4. **Theorem**—If one side of a triangle be produced, then the exterior angle is greater than either of the interior opposite angles.

Ex. 13. *The sum of the exterior angles formed by producing one side of a triangle both ways is greater than two right angles.*

The side BC of the $\triangle$ABC is produced both ways to D and E. To prove that $\angle$ABD$+\angle$ACE$>$2 right angles.

Proof : Exterior $\angle$ABD$>$ the int. opp. $\angle$ACB. Adding $\angle$ACE to both sides we get $\angle$ABD$+\angle$ACE$>\angle$ACB$+\angle$ACE ; But $\angle$ACB$+\angle$ACE$=$2 right angles.

$\therefore$ $\angle$ABD$+\angle$ACE$>$2 right angles.

Ex. 14. *The base angles of an isosceles triangle are acute.*

[*C. U. 1926*]

ABC is an isosceles triangle of which AB$=$AC.
To prove that $\angle$B and $\angle$C are each acute. Draw AD$\perp$BC.
Proof : Exterior $\angle$ADC$>\angle$ABD ; but $\angle$ADC$=$1 right angle,
$\therefore$ $\angle$ABD is less than one right angle, that is, acute.
$\therefore$ $\angle$ABC is acute and $\angle$ACB being equal to $\angle$ABC is also acute.

Ex. 15. *Three equal straight lines cannot be drawn from a given point to a given straight line.*

AB is a straight line and O is a point outside AB.

To prove that three equal straight lines cannot be drawn on AB from O.

Let OA and OD be two equal straight lines drawn from O to AB. If possible, let another st. line OC drawn to AB be equal to OA and OD.

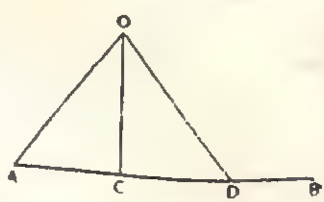


Fig. 9

Now $\because$ AO$=$DO, $\therefore$ $\angle$OAD$=\angle$ODA.
Again, $\because$ OA$=$OC, $\therefore$ $\angle$OAC$=\angle$OCA.
Therefore, $\angle$OCA$=\angle$ODA or $\angle$ODC.

But the $\angle$OCA being the exterior angle of the triangle ODC, cannot be equal to interior opp. $\angle$ODC. $\therefore$ it is impossible to draw three equal straight lines from O to AB.

5. **Theorem**—(i) If one side of a triangle be greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less. Also (ii) Its converse theorem.

Ex. 16. *The hypotenuse is the greatest side of a right-angled triangle.*

$\angle$B of the $\triangle$ABC is a right angle. To prove that the hypotenuse AC of that triangle is the greatest side.

Produce CB to D. Now $\angle$ABD is also a right angle. The exterior $\angle$ABD is greater than each of the interior opposite $\angle$ACB and $\angle$CAB. $\therefore$ $\angle$ABC is the greatest angle of the triangle. $\therefore$ AC being the opposite side of the greatest $\angle$ABC is greater than AB and BC which are opposite sides of the other two angles. $\therefore$ the hypotenuse AC is the greatest side of the right-angled triangle.

Ex. 17. ABCD *is a quadrilateral with* AD *is greatest and* BC *its least side. Prove that the angle at* C *is greater than the angle at* A. [*C. U.* '18, '40]

AD is the greatest and BC is the least side of the quadrilateral ABCD, To prove that $\angle C > \angle A$. Join AC.

Proof : In the $\triangle$ACD, AD>CD ($\because$ AD is the greatest side)

$\therefore$ $\angle ACD > \angle CAD$.

Again, in the $\triangle$ABC, AB>BC

($\because$ BC is the least side),

$\therefore$ $\angle BCA > \angle BAC$,

$\therefore$ $\angle ACD + \angle BCA > \angle CAD + \angle BAC$

$\therefore$ the whole $\angle$BCD>the whole $\angle$BAD.

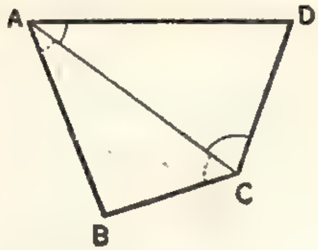


Fig. 10

Ex. 18. *In* $\triangle ABC$, AB *is greater than* AC. AD *bisects the* $\angle A$ *and meets* BC *in* D. *Prove that* $BD > CD$.

[**Hints :** From AB cut off AE equal to AC. Join DE. Now in the $\triangle s$ ADE, ACD, $\because$ $AE = AC$, AD is common to both triangles and $\angle EAD = \angle CAD$ (hyp.),

$\therefore$ $DE = CD$ and $\angle ADE = \angle ADC$.

In the $\triangle ABD$, the ext. $\angle ADC > \angle B$,

$\therefore$ $\angle ADE > \angle B$.

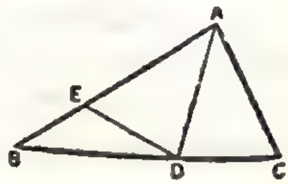


Fig. 11.

Again, the ext. $\angle BED > \angle ADE$. $\therefore$ $\angle BED > \angle EBD$

$\therefore$ $BD > DE$. But $DE = DC$, $\therefore$ $BD > DC$.

6. **Theorem**—Any two sides of a triangle are together greater than the third side.

Ex. 19. *Two sides of a triangle are 2 and 3. Show that the third side is less than 5, but greater than 1.* [*C. U. '25*]

Any two sides of a triangle are together greater than the third side. $\therefore$ here the third side will be less than the sum of the other two sides, that is, it will be less than $(2+3)$ or 5.

Again, as the second side is 3, the sum of the first and third sides will be greater than 3.

But the first side is known to be 2. $\therefore$ the third side will be greater than 1.

Ex. 20. *The difference of any two sides of a triangle is less than the third side.* [*C. U. '34 ; W. B. S. F. '52*]

ABC is a triangle. To prove that the difference of any two of its sides is less than the third side.

Proof : Suppose $AB > AC$. From AB cut off AD equal to AC, then BD is the difference of AB and AC.

Now $AC + BC > AB$, that is, $AC + BC > AD + BD$, but $AC = AD$ (con.), $\therefore$ BC must be greater than BD, *i.e.* $BD < BC$.

Similarly, it can be proved that the difference of any other two sides is also less than the third side.

Ex. 21. *The sum of the four sides of any quadrilateral is greater than the sum of the two diagonals.*

AC and BD are the two diagonals of the quadrilateral ABCD. To prove that $AB+BC+CD+DA>AC+BD$.

Proof : $AB+BC>AC$, $BC+CD>BD$,
$CD+DA>AC$ and $AB+AD>BD$.

$\therefore$ by addition, we get $2(AB+BC+CD+DA)>2\,(AC+BD)$.

$\therefore$ $AB+BC+CD+DA>AC+BD$.

Ex. 22. *Find a point within a quadrilateral such that the sum of its distances from the vertices is the least.* [*C. U.* '44]

Let ABCD be a quadrilateral. To find a point within it such that the sum of its distances from A, B, C and D is the least. Join AC and BD. Let them intersect at O. Then O is the required point.

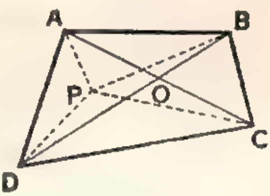


Fig. 12

Proof : Take any point P within the quadrilateral.

Join PA, PB, PC and PD.

Now, $PA+PC>AC$ and
$BP+PD>BD$,

$\therefore$ $AP+PC+BP+PD>AC+BD$.

That is, $AP+PC+BP+PD>OA+OC+OB+OD$.

It is true for any point other than O.

$\therefore$ O is the required point.

Ex. 23. *A is the greatest angle of the* $\triangle ABC$. *Show that it is not possible to construct a triangle with sides equal to* AB, AC, 2BC. [*C. U.* '46]

$\because$ $\angle A$ is the greatest angle of the triangle ABC, $\therefore$ its opposite side BC is the greatest side. $\therefore$ $BA<BC$ and $CA<BC$, $\therefore$ $AB+AC$ together $<2BC$. The sum of any two sides of a triangle is greater than the third side. But here $AB+AC<2BC$, $\therefore$ it is not possible to draw a triangle with sides equal to AB, AC and 2BC.

Ex. 24. *The sum of any two sides of a triangle is greater than twice the median which bisects the third side.*

[*C. U.* '23 ; *D. B.* '32]

Let AO be a median of the triangle ABC.

To prove that $AB+AC>2AO$.

Proof : Produce AO to D so that $DO=AO$. Join DC. In the Δ^s ABO, COD, $\because$ $AO=DO$ (by cons.), $BO=CO$ (hyp.), and $\angle AOB$ = vertically opposite $\angle COD$,

$\therefore$ $\Delta ABO \equiv \Delta COD$. $\therefore$ $AB=CD$.

Now, in ΔACD, $AC+CD>AD$,

$\therefore$ $AB+AC>AD$, but $AD=2AO$,

$\therefore$ $AB+AC>2AO$.

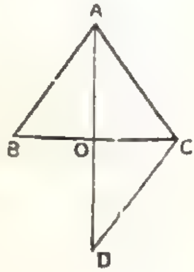


Fig. 13

Ex. 25. *Show that in any triangle the sum of the medians is less than the perimeter.* [*C. U.* '41, '48 *Sup* ; *D. B.* '34]

Let AO, BE and CF be the three medians of ΔABC. To prove that $AO+BE+CF<AB+BC+AC$.

[First prove as in example 24 and then write]

$$\begin{aligned} AB+AC &> 2AO \\ AB+BC &> 2BE \\ AC+BC &> 2CF \end{aligned}$$

$\therefore$ by addition, $2(AB+AC+BC)>2(AO+BE+CF)$,

$\therefore$ $AB+AC+BC>AO+BE+CF$.

$\therefore$ $AO+BE+CF<AB+AC+BC$.

Ex. 26. *If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle.*

[*D. B.* 1927]

Let P be a point within the triangle ABC.

To prove that $BP+PC<AB+AC$.

Proof : Join BP and CP. Produce BP to cut AC at D. Now $AB+AD>BD$.

Again $PD+DC>PC$.

Adding these results we get

$AB+AD+DC+PD>BP+PD+PC$.

$\therefore$ $AB+AD+DC>BP+PC$ (PD is subtracted from both sides).

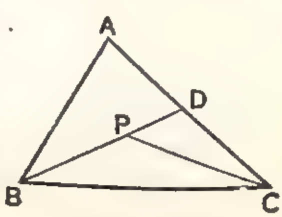


Fig. 14

i. e., $AB+AC>BP+PC$ ($\because AD+DC=AC$), $\therefore BP+PC<AB+AC$.

Ex. 27. *The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.* [*C. U. 1939*]

P is a point within the triangle ABC. Join AP, BP and CP.

To prove that $AP+BP+CP<AB+BC+AC$.

Proof : [Prove as in the example 26 that]

$BP+PC<AB+AC$.

Similarly, $AP+BP<AC+BC$ and $AP+PC<AB+BC$,

$\therefore$ by addition, $2(AP+BP+PC)<2(AB+BC+AC)$,

$\therefore$ $AP+BP+CP<AB+BC+AC$.

Ex. 28. *Prove that the sum of the distances of any point within a triangle from the angular points of the triangle is greater than half the perimeter.* [*C. U. 1927, '39*]

Let P be a point within the $\triangle ABC$. Join AP, BP, CP. To prove that $AP+BP+CP>\frac{1}{2}(AB+BC+AC)$.

Proof : $AP+BP>AB$, $BP+CP>BC$ and $AP+CP>AC$,

$\therefore$ adding $2(AP+BP+CP)>AB+BC+AC$.

$\therefore$ $AP+BP+CP>\frac{1}{2}(AB+BC+AC)$.

Ex. 29. *Show that the sum of the diagonals of any quadrilateral is greater than half the sum of the sides of the quadrilateral.* [*C. U. '43*]

In the quadrilateral ABCD the diagonals AC and BD intersect at the point O.

To prove that $AC+BD>\frac{1}{2}(AB+BC+CD+DA)$.

Proof : $\because$ $AO+BO>AB$,

$BO+CO>BC$,

$CO+DO>CD$

and $AO+DO>AD$

$\therefore$ $2(AO+BO+CO+DO)>AB+BC+CD+DA$,

$\therefore$ $AO+BO+CO+DO>\frac{1}{2}(AB+BC+CD+DA)$,

$\therefore$ $AC+BD>\frac{1}{2}(AB+BD+CD+DA)$.

7. **Theorem.**—The three angles of a triangle are together equal to two right angles.

Ex. 30. (*a*) *Prove that the angles at the base of an isosceles triangle are acute.* [*C. U.* '26]

Suppose ABC is an isosceles triangle of which $AB = AC$. To prove that $\angle ACB$ and $\angle ABC$ are acute angles.

Proof : $\because$ $AB = AC$, $\therefore$ $\angle ABC = \angle ACB$.

Again, $\because$ the sum of the three angles of a triangle is equal to 2 right angles, $\therefore$ $\angle ABC + \angle ACB < 2$ right angles.

$\therefore$ $2\angle ABC < 2$ right angles.

$\therefore$ $\angle ABC$ is less than 1 right angle, *i.e.*, $\angle ABC$ is an acute angle. $\therefore$ $\angle ABC$ and $\angle ACB$ are acute angles.

(*b*) *If the equal sides of an isosceles triangle are produced show that the exterior angles must be obtuse.* [*C. U.* '49 *Sup.*]

[**Hints.** Produce AB and AC to D and E respectively. Prove as before that each of the $\angle ABC$ and $\angle ACB$ is acute.

$\because$ $\angle ABC + \angle DBC = 2$ right angles and $\angle ABC$ is an acute angle,

$\therefore$ $\angle DBC = 2$ right angles $-$ an acute angle $=$ an obtuse angle.
Similarly, $\angle BCE$ is obtuse.

Ex. 31. *The bisectors of any two adjacent angles of a quadrilateral contain an angle equal to half the sum of the remaining angles.*

Let the bisectors of the $\angle A$ and $\angle B$ of the quadrilateral ABCD intersect at O.

To provethat $\angle AOB = \frac{1}{2}(\angle D + \angle C)$.

Proof : In the triangle AOB,

$\angle O + \angle OAB + \angle OBA$

$= 2$ right angles.

In the quadrilateral ABCD,

$\angle BAD + \angle ABC + \angle C + \angle D$

$= 4$ right angles.

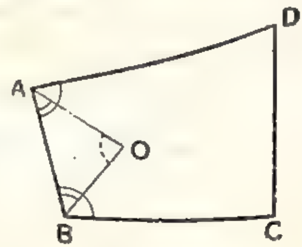


Fig. 15

$\therefore$ $\angle O + \angle OAB + \angle OBA = \frac{1}{2}(\angle BAD + \angle ABC + \angle C + \angle D)$.

But by hypothesis, $\angle OAB = \frac{1}{2}\angle BAD$ and $\angle OBA = \frac{1}{2}\angle ABC$,

$\therefore$ the remaining $\angle O = \frac{1}{2}(\angle C + \angle D)$.

Ex. 32. *The angles* B *and* C *of the* $\triangle ABC$ *are bisected by* BO *and* CO *respectively. Prove that* $\angle BOC = 90° + \frac{\angle A}{2}$.

Proof : In $\triangle BOC$, $\angle BOC + \angle OBC + \angle OCB = 180°$,

$\therefore \quad \angle BOC + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 180° \cdots\cdots(1)$

Again, in $\triangle ABC$, $\angle A + \angle B + \angle C = 180°$,

$\therefore \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90° \cdots\cdots\cdots(2)$

Subtracting (2) from (1) we get $\angle BOC - \frac{1}{2}\angle A = 90°$,

$$\therefore \quad \angle BOC = 90° + \frac{\angle A}{2}.$$

Ex. 33. *The sum of the interior angles of a rectilinear figure is equal to the sum of the exterior angles. Find the number of stdes.* [*C. U. 1944, '48, '49 Supl.*]

The sum of the exterior angles of a polygon = 4 right angles, $\therefore$ Here the sum of the interior angles = 4 right angles. Again all the interior angles of any polygon together with four right angles are equal to twice as many right angles as the polygon has sides. $\therefore$ Here twice as many right angles as the polygon has sides = all the interior angles + 4 right angles = 4 right angles + 4 right angles = 8 right angles. $\therefore$ the required number of sides $= \frac{1}{2} \times 8 = 4$.

Ex. 34. *In a right-angled triangle, the straight line joining the middle point of the hypotenuse to the right angle is half of the hypotenuse.* [*C. U. 1919, P. U. '23 ; D. B. '33*]

In the $\triangle ABC$, $\angle B$ is a right angle. To prove that the line joining B and the middle point of $AC = \frac{1}{2}AC$. At B draw $\angle CBD = \angle C$. Let BD cut AC in D.

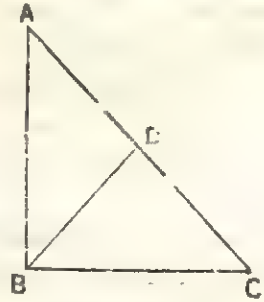


Fig. 16

Proof : $\because \quad \angle C = \angle CBD$,

$\therefore$ BD = DC. $\because$ $\angle ABC = 1$ rt. angle.

$\therefore$ $\angle A + \angle C = 1$ right angle.

$\therefore$ $\angle ABD + \angle DBC = \angle A + \angle C$,

but $\angle DBC = \angle C$ (by construction)

$\therefore$ $\angle ABD = \angle A$, $\therefore$ BD = AD.

$\therefore$ AD = BD = DC.

$\therefore$ D is the mid-point of AC,
and $BD = \frac{1}{2}(AD + DC) = \frac{1}{2}AC$.

Ex. 35. *In a right-angled triangle, if one of the acute angles is double of the other, show that the hypotenuse is double of the shorter side.* [*C. U. 1945*]

Let the $\angle B$ of the $\triangle ABC$ be a right angle and $\angle C = 2\angle A$. To prove that $AC = 2BC$ [vide figure 16]

Proof : $\because$ $\angle B = 1$ right angle,

$\therefore$ $\angle C + \angle A = 1$ right angle $= 90°$

i.e., $2\angle A + \angle A = 90°$, or, $3\angle A = 90°$,

$\therefore$ $\angle A = 30°$ and $\angle C = 60°$.

At B draw $\angle CBD$ equal to $\angle C$. Let BD intersect AC at D. $\angle CBD = \angle C = 60°$. $\therefore$ the remaining $\angle BDC = 60°$.

$\because$ All the angles of the $\triangle BDC$ are equal, $\therefore$ all its sides are equal. Again, $\angle CBD = 60°$ and the whole $\angle B = 1$ right angle, $\therefore$ $\angle ABD = 30° = \angle A$. $\therefore$ $AD = BD$.

$\therefore$ $AC = AD + DC = BD + DC = 2BD = 2BC$.

Ex. 36. A *is the vertex of an isosceles triangle* ABC, *and* BA *is produced to* D *so that* AD *is equal to* AB ; *if* DC *is drawn, show that* $\angle BCD$ *is a right angle.* [*C. U.* '47 ; *D. C.* '32]

$\because$ $AB = AC$, $\therefore$ $\angle ACB = \angle B$.

Again, $\because$ $AD = AB = AC$,

$\therefore$ $\angle ACD = \angle D$.

$\therefore$ $\angle ACB + \angle ACD = \angle B + \angle D$,

that is, in the $\triangle BCD$, the whole

$\angle BCD = \angle B + \angle D$

$= \frac{1}{2}(\angle B + \angle D + \angle BCD)$

$=$ half of 2 right angles

$= 1$ right angle.

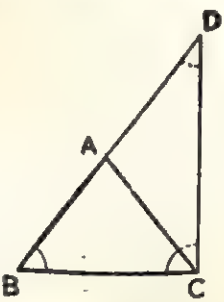


Fig. 17

Ex. 37. *The angle between the bisector of the vertical angle and the perpendicular from the vertex to the base of a triangle is half of the difference of the angles at the base.*

In the $\triangle ABC$, AD is the bisector of the angle $\angle BAC$ and $AO \perp BC$. Let $AB > AC$. To prove that $\angle DAO = \frac{1}{2}(\angle C - \angle B)$.

Proof: $\angle C + \angle CAO = 90^\circ$ and

$\angle B + \angle BAO = 90^\circ$.

$\therefore \quad \angle C + \angle CAO = \angle B + \angle BAO.$

$\therefore \quad \angle C - \angle B = \angle BAO - \angle CAO$

$= \angle BAD + \angle DAO - \angle CAO$

$= \angle CAD + \angle DAO - \angle CAO$

$(\therefore \quad \angle BAD = \angle CAD)$

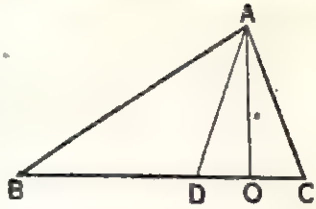


Fig. 18

$= \angle DAO + \angle CAO + \angle DAO - \angle CAO$

$(\because \quad \angle CAD = \angle CAO + \angle DAO) = 2\angle DAO,$

$\therefore \quad \angle DAO = \frac{1}{2}(\angle C - \angle B).$

Ex. 38. *If from the extremities of the base of an isosceles triangle, perpendiculars* BX *and* CY *are drawn to the opposite sides intersecting at* O, *show that the* $\triangle BOC$ *is isosceles.* [*D.B.'26*]

In the $\triangle ABC$, $AB = AC$, $BX \perp AC$ and $CY \perp AB$. BX cuts CY at O. To prove that the $\triangle BOC$ is an isosceles triangle.

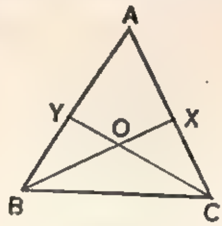


Fig. 19

Proof: In the $\triangle BCX$,

$\angle X = 1$ right angle

$\therefore \quad \angle XCB + \angle XBC = 1$ right angle.

Similarly, in the triangle BYC,

$\angle YBC + \angle YCB = 1$ right angle.

$\therefore \quad \angle XCB + \angle XBC = \angle YBC + \angle YCB \cdots (1)$

But $\because$ $AB = AC$, $\therefore$ $\angle XCB = \angle YBC$.

$\therefore$ from (1) we have $\angle XBC = \angle YCB$. $\therefore$ $OB = OC$.

$\therefore$ OBC is an isosceles triangle.

8. **Theorem**—If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another, then the triangles are congruent.

Ex. 39. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

Suppose O to be any point on the bisector AD of the $\angle BAC$. From O draw OE and OF perpendiculars to AB and AC respectively. To prove that $OE = OF$.

Proof : In the $\triangle OEA$ and $\triangle OFA$, $\angle OAE = \angle OAF$ (by hypothesis), $\angle OEA = \angle OFA$ (right angles) and AO is common to both triangles, $\therefore$ the two triangles are equal in all respects. $\therefore$ $OE = OF$.

Ex. 40. AB *is the hypotenuse of an isosceles right-angled triangle* ABC. *If* AD *bisects the* $\angle A$ *and meets* BC *at* D, *prove that* $AC + CD = AB$. [*B. U.* '23]

In the $\triangle ABC$, $\angle C = 1$ right angle, $AC = BC$ and AD bisects $\angle A$ intersecting BC at D. To prove that $AC + CD = AB$.

Draw $DO \perp AB$.

Proof : In $\triangle^s ACD$ and ADO, $\angle C = \angle O$ (right angles), $\angle CAD = \angle DAO$ and AD is common to both triangles, $\therefore$ $AC = AO$ and $CD = DO$. $\therefore$ $AC + CD = AO + DO$. Again, $\angle DOB = 1$ right angle, $\therefore$ $\angle ODB + \angle OBD = 1$ right angle. But $\angle B = 45°$ ($\because$ $\angle A = \angle B = 45°$, $\angle C$ being a right angle), $\therefore$ $\angle ODB = 45° = \angle B$, $\therefore$ $DO = BO$. $\therefore$ $AC + CD = AO + BO = AB$.

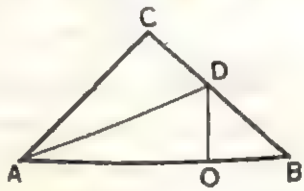


Fig. 20

Ex. 41. *A quadrilateral* $ABCD$ *is such that the diagonal* AC *bisects each of the angles* BAD, BCD. *Prove that* AC *bisects* BD *at right angles.* [*C. U.* '48]

[Hints. Let AC intersect the diagonal BD at O. In the $\triangle^s ABC$ and ADC, $\angle BAC = \angle DAC$, $\angle BCA = \angle DCA$ and AC is common to both triangles, $\therefore$ $AB = AD$. Again, in the $\triangle OAB$ and $\triangle ADO$, $AB = AD$, AO is common to both triangles, and $\angle BAO = \angle DAO$, $\therefore$ the two triangles are congruent. $\therefore$ $BO = DO$ and $\angle AOB = \angle AOD$, but they are adjacent angles, $\therefore$ each is a right angle. $\therefore$ AC bisects BD at right angles.

9. **Theorem**—Two right-angled triangles which have their hypotenuses equal and one side of the one equal to one side of the other, are equal in all respects.

Ex. 42. *The perpendicular from the vertex of an isosceles triangle to the base bisects the base and also the vertical angle.* [*C. U.* 1913]

ABC is a triangle in which $AB = AC$ and $AD \perp BC$.

To prove that AD bisects BC and the $\angle BAC$.

Proof : In the $\triangle$'ABD and ACD, the hypotenuse AB = the hypotenuse AC, AD is common to both triangles and $\angle$ADB = $\angle$ADC (right angles), $\therefore$ the triangles are congruent.

$\therefore$ BD = DC and $\angle$BAD = $\angle$CAD, that is, AD bisects the $\angle$BAC and the base BC.

Ex. 43. *If the perpendiculars from the mid-point of the base of a triangle to the other two sides are equal, show that the triangle is isosceles.* [C.U. '48]

[**Hints :** D is the mid-point of the base BC of the $\triangle$ABC. From D draw the perpendiculars DE and DF on AB and AC respectively. If DE = DF, prove that the $\triangle$ABC is isosceles.

Proof : In the $\triangle$'DBE and DCF, $\angle$E = $\angle$F (right angles), the hypotenuse DB = the hypotenuse DC and DE = DF,

$\therefore$ the two triangles are equal in all respects.

$\therefore$ $\angle$B = $\angle$C, $\therefore$ AB = AC. $\therefore$ $\triangle$ABC is isosceles.

10. **Theorem :** Theorems on parallel straight lines and parallelograms.

Ex. 44. *If the corresponding angles made by a transversal on two parallel straight lines are bisected, the bisectors are parallel.*

Let EG intersect the two parallel straight lines AB and CD at F and G respectively. Let FP and GQ bisect the two corresponding angles EFB and FGD. To prove that FP ∥ GQ.

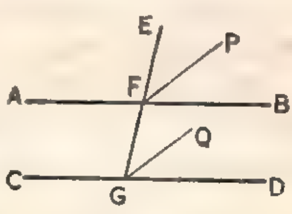


Fig. 21

Proof : $\because$ AB and CD are parallel and EG cuts them, $\therefore$ $\angle$EFB = corresponding $\angle$FGD. Again, $\angle$EFP = $\frac{1}{2}\angle$EFB and $\angle$FGQ = $\frac{1}{2}\angle$FGD, $\therefore$ $\angle$EFP = $\angle$FGQ and these are corresponding angles. $\therefore$ FP ∥ GQ.

Ex. 45. *A straight line cuts two parallel straight lines and the two interior angles on the same side of it are bisected, show that the angle between the bisectors is a right angle.*

Let the transversal EF cut the two parallel straight lines AB and CD at E and F. EG bisects $\angle$BEF and FG bisects $\angle$EFD. To prove that $\angle$EGF is a right angle.

Proof : $\because$ AB $\parallel$ CD, $\therefore$ $\angle$BEF + $\angle$EFD = 2 right angles. Again, $\angle$GEF = $\frac{1}{2}\angle$BEF, and $\angle$GFE = $\frac{1}{2}\angle$EFD,

$\therefore$ $\angle$GEF + $\angle$GFE = $\frac{1}{2}$($\angle$BEF + $\angle$EFD) = half of two right angles = 1 right angle.

$\therefore$ the remaining $\angle$EGF of the $\triangle$GEF is a right angle.

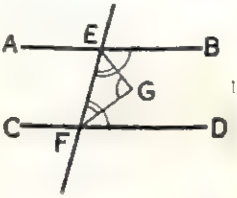


Fig. 22

Ex. 46. *The bisectors of the angles made by a transversal on two parallel straight lines form a rectangle.*

The transversal EF cuts two parallel straight lines AB and CD at E and F.

EG and EH bisect the two interior angles at E. FH and FG bisect the two interior angles at F. To prove that EGFH is a rectangle.

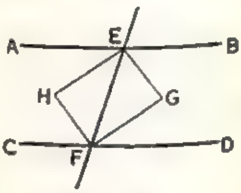


Fig. 23

Proof : $\angle$AEF = alternate $\angle$EFD,

$\therefore$ their halves are also equal,

$\therefore$ $\angle$HEF = $\angle$EFG, but they are alternate angles. $\therefore$ HE $\parallel$ FG.

Similarly, HF $\parallel$ EG, $\therefore$ HEGF is a parallelogram.

Again, $\angle$HEG = $\frac{1}{2}$($\angle$AEF + $\angle$BEF) = half of two right angles = 1 right angle.

Now one angle of the parallelogram HEGF is a right angle, $\therefore$ all its angles are right angles. $\therefore$ HEGF is a rectangle.

Ex. 47. *If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.*

[*D. B. 1926*]

Let CE bisect the exterior $\angle$ACD of the $\triangle$ABC and be parallel to AB. To prove that $\triangle$ABC is an isosceles triangle.

Proof : AB $\parallel$ CE and AC cuts them, $\therefore$ $\angle$BAC = alternate $\angle$ACE. Again, AB $\parallel$ CE and BD cuts them,

$\therefore$ $\angle$ABC = corresponding $\angle$ECD. Now $\because$ $\angle$ACE = $\angle$ECD,

$\therefore$ $\angle$BAC = $\angle$ABC. $\therefore$ AC = BC. $\therefore$ ABC is isosceles.

Ex. 48. *The angles whose arms are parallel are either equal or supplementary.*

The arms AB and BC of the $\angle$ABC are respectively parallel to the arms DE and EF of the $\angle$DEF. To prove that $\angle$B = $\angle$E (in first figure), or $\angle$B and $\angle$E are supplementary (in second figure).

Proof. $\because$ AB $\parallel$ DE, $\therefore$ $\angle$DOC = corresponding $\angle$ABC. Again, $\because$ BC $\parallel$ EF, $\therefore$ $\angle$DOC = corresponding $\angle$DEF.

$\therefore$ $\angle$ABC = $\angle$DEF.

Now, in the second figure,

$\because$ BC $\parallel$ EF, $\therefore$ $\angle$DOB = $\angle$DEF (corresponding angle).

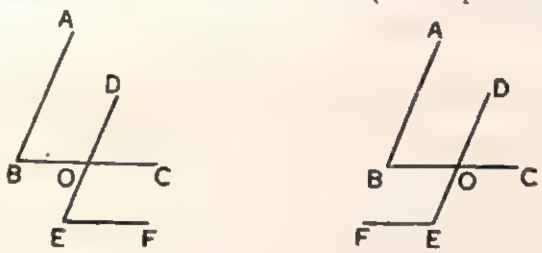


Fig. 24

Again, AB $\parallel$ DE and BC cuts them, $\therefore$ $\angle$B + $\angle$DOB = 2 right angles. $\therefore$ $\angle$B + $\angle$E = 2 right angles, *i.e.*, they are supplementary.

Ex. 49. *Prove that if three sides of one triangle be parallel to three sides of another triangle, the corresponding angles are equal.* [*C. U.* 1932]

In the Δ^s ABC and DEF, AB, BC and CA are respectively parallel to DE, EF and FD. To prove that $\angle$A = $\angle$D, $\angle$B = $\angle$E and $\angle$C = $\angle$F.

Construction : Produce BC so as to cut DE and DF at G and H respectively.

Proof : $\because$ AB $\parallel$ DE and BH cuts them, $\therefore$ $\angle$B = corresponding $\angle$DGH,

Again, $\because$ BH $\parallel$ EF and DE cuts them, $\therefore$ $\angle$DGH = cor. $\angle$E,

$\therefore$ $\angle$B = $\angle$E.

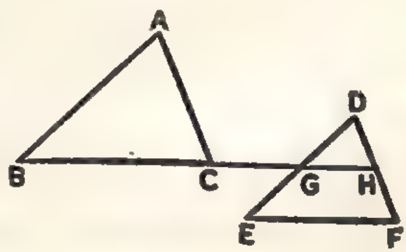


Fig. 25

Again, AC ∥ DF and BH cuts them,

∴ $\angle ACB = \angle DHG$ = corresponding $\angle F$ (∵ BH ∥ EF).

∴ the remaining $\angle A$ of the triangle ABC = the remaining $\angle D$ of the triangle DEF.

Ex. 50. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.* [*C. U. '1911*]

In the quadrilateral ABCD, AB = CD, AD = BC. To prove that ABCD is a parallelogram. Join AC.

Proof : In the $\triangle$ˢ ABC and ADC, AB = CD, BC = AD and AC is common to both triangles, ∴ the two triangles are equal in all respects.

∴ $\angle BAC = \angle ACD$ and $\angle ACB = \angle DAC$,
∵ $\angle BAC$ = alternate $\angle ACD$, ∴ AB ∥ CD and
∵ $\angle ACB$ = alternate $\angle DAC$, ∴ BC ∥ AD.
∴ ABCD is a parallelogram.

Ex. 51. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.* [*C. U. '50 ; D. B. 1936*]

In the quadrilateral ABCD let $\angle A = \angle C$ and $\angle B = \angle D$. To prove that ABCD is a parallelogram.

Proof : ∵ $\angle A = \angle C$ and $\angle D = \angle B$, ∴ $\angle A + \angle D = \angle C + \angle B$ = half of the four angles of the quadrilateral = 2 right angles. ∴ AB ∥ CD. Similarly it can be proved that AD ∥ BC. ∴ ABCD is a parallelogram.

Ex. 52. *Show that two equilateral triangles standing on opposite sides of the same base form a parallelogram.* [*C.U.1913*]

ABC and DBC are two equilateral triangles standing on opposite sides of the same base BC. To prove that ABCD is a parallelogram.

Proof : ∵ $\triangle$ ABC, DBC are equilateral,
∴ each angle of the triangles = 60°.
∴ $\angle ABC = \angle BCD$; but they are alternate angles,
∴ AB ∥ CD. Again $\angle ACB = \angle DBC$, ∴ AC ∥ BD.
∴ ABCD is a parallelogram.

Ex. 53. *If the diagonals of a quadrilateral bisect each other, show that the figure is a parallelogram.*

ABCD is a quadrilateral and its diagonals AC and BD bisect each other at the point O. To prove that ABCD is a parallelogram.

Proof: In the $\triangle^s$AOB and DOC, AO = CO, BO = DO and $\angle$AOB = vertically opposite $\angle$DOC.

$\therefore$ the two triangles are equal in all respects.

$\therefore$ AB = DC and $\angle$OAB = $\angle$OCD, but they are alternate angles, $\therefore$ AB $\parallel$ CD. $\because$ AB $\parallel$ CD and AB = CD,

$\therefore$ AD and BC are both equal and parallel. $\therefore$ ABCD is a parallelogram.

Ex. 54. *If the diagonals of a parallelogram are equal, prove that it is a rectangle.* [*C. U. 1924*]

In the parallelogram ABCD, the diagonal AC = the diagonal BD. To prove that ABCD is a rectangle.

In the $\triangle^s$ABC and BCD, AB = CD, AC = BD and BC is common to both triangles, $\therefore$ the two triangles are congruent. $\therefore$ $\angle$ABC = $\angle$BCD.

But $\angle$ABC + $\angle$BCD = 2 right angles,

$\therefore$ $\angle$ABC = 1 right angle. $\therefore$ ABCD is a rectangle.

Ex. 54. (*a*) *The straight line joining the middle points of any two sides of a triangle is parallel to and half of the third side.*

Let D and E be respectively the middle points of the sides AB and AC of the triangle ABC. To prove that DE $\parallel$ BC and DE $= \frac{1}{2}$ BC. Join DE and produce DE to F so that EF = DE. Join CF.

Proof: In $\triangle^s$ADE and CEF, AE = CE, (by hypothesis), DE = EF (by construction), and $\angle$AED = vertically opposite $\angle$CEF,

$\therefore$ the triangles are congruent.

$\therefore$ CF = AD and $\angle$DAE = $\angle$ECF, but they are alternate angles, $\therefore$ CF $\parallel$ AD, that is, CF $\parallel$ BD.

Again, CF = AD = BD, $\therefore$ BD and CF are equal and parallel.

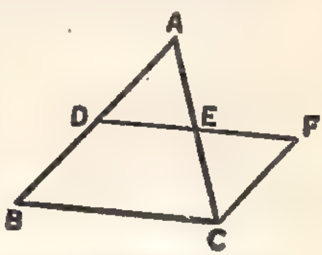


Fig. 26

$\therefore$ DF and BC are also equal and parallel. But DE $= \frac{1}{2}$DF, $\therefore$ DE $= \frac{1}{2}$BC. $\therefore$ DE is parallel to and half of BC.

Ex. 55. *The straight line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides and is half their sum.* [*C. U.* '41]

Let ABCD be a trapezium in which AD ∥ BC and P and Q are middle points of the oblique sides AB and DC. To prove that PQ is parallel to AD and BC and $PQ = \frac{1}{2}(AD + BC)$. Through Q draw RS parallel to BA. Let RS cut BC at R and AD produced at S.

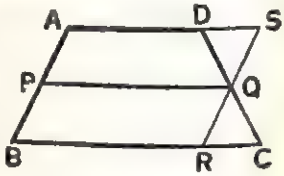


Fig. 27

Proof: In the △ DQS and RQC, $DQ = QC$, ∠SDQ = alternate ∠QCR and ∠DQS = vertically opposite ∠RQC, ∴ $DS = RC$ and $SQ = RQ$. Now, ABRS is a parallelogram, ∴ $AB = RS$. ∴ $\frac{1}{2}AB = \frac{1}{2}RS$, that is, $AP = SQ$.

Now AP and SQ are equal and parallel,

∴ AS and PQ are also equal and parallel.

∴ PQ is parallel to AD and BC.

Again, $PQ = AS = BR$, ∴ $PQ = \frac{1}{2}(BR + AS)$
$= \frac{1}{2}(BR + DS + AD) = \frac{1}{2}(BR + RC + AD) = \frac{1}{2}(BC + AD)$.

Ex. 56. *If the diagonal* AC *of a parallelogram* ABCD *bisects the* ∠A, *show that it bisects the* ∠C *and the parallelogram is a rhombus.* [*C. U. 1926*]

The diagonal AC of the parallelogram ABCD bisects the ∠A. To prove that AC also bisects the ∠C and ABCD is a rhombus.

Proof: ∠BAC = alternate ∠ACD and ∠DAC = alternate ∠ACB; but ∠BAC = ∠DAC (by hypothesis),

∴ ∠ACD = ∠ACB, that is, AC bisects ∠BCD.

Again, ∠BAC = ∠DAC (by hypothesis) = ∠ACB (alternate angle). ∴ $AB = BC$, ∴ all sides of the parallelogram ABCD are equal. ∴ ABCD is a rhombus.

***Ex. 57.** A *and* B *are two points on the same side of an unlimited straight line. Find a point* D *on the straight line such that the distance of the point from* A *and* B *is the least.* [*C. U.*]

PQ is an unlimited straight line and A and B are two points on the same side of PQ. To find a point D on PQ such that the distance of D from A and B is the least.

Construction : From A draw AO perpendicular to PQ, Produce AO to C so that CO = AO. Join CB. CB cuts PQ at D. Then D is the required point.

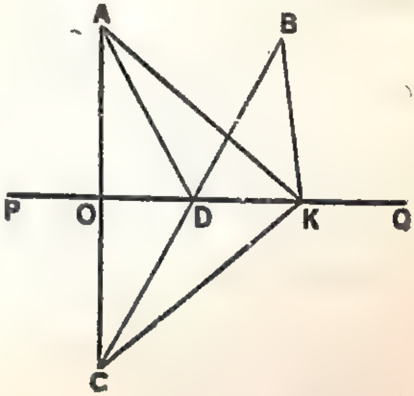


Fig. 28

Proof : Take any point K on PQ. Join AD, BK, AK, CK. PQ is the perpendicular bisector of AC, and D and K are on PQ, $\therefore$ AD = CD and AK = CK. Now, AD + BD = CD + BD = BC.

Again, AK + BK = CK + BK, but in the ΔBKC, BK + CK > BC.

$\therefore$ AD + BD < BK + CK.

$\therefore$ AD + BD < BK + AK, this is true for all positions of K on PQ. $\therefore$ D is the required point.

Ex. 58. *In the side* BC *of a right-angled* ΔABC *right-angled at* C, *find a point* D *such that the perpendicular* DF *drawn from* D *to the hypotenuse shall be equal to* AF. [*C. U.*]

Hints : At A in AB draw the $\angle$BAD = 45°.

Let AD intersect BC or BC produced at D. Then D is the required point.

Proof : Draw DF$\perp$AB. In ΔADF, $\angle$F = 1 right angle,

$\therefore$ $\angle$DAF + $\angle$ADF = 1 right angle ; but $\angle$DAF = 45° ;

$\therefore$ $\angle$ADF = 45° = $\angle$DAF, $\therefore$ DF = AF.

Ex. 59. *The straight lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.*

Let P, Q, R and S be respectively the middle points of the sides AB, AD, DC and BC of the quadrilateral ABCD. Join PQ, QR, RS and SP. To prove that PQRS is a parallelogram.

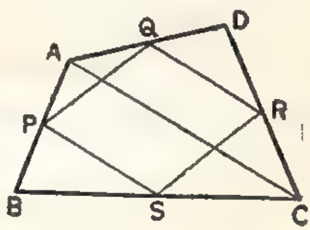


Fig. 29

Proof : Join AC. In ΔADC, QR joins the mid points of AD, CD, $\therefore$ QR is parallel to and half of AC. Similarly, PS is parallel to and

half of AC. $\therefore$ QR and PS are equal and parallel. $\therefore$ PQ and SR are also equal and parallel. $\therefore$ PQRS is a parallelogram.

Ex. 60. *Straight lines joining the middle points of the opposite sides of any quadrilateral bisect each other.*

[*C. U.* '39 ; *D. B.* '35]

Let P, Q, R and S be the middle points of the sides AB, AD, DC and BC of the quadrilateral ABCD. Join PR and QS. To prove that PR and QS bisect each other. [Prove first as in Example 59 that] PQRS is a parallelogram. The diagonals of a param. bisect each other, $\therefore$ PR, QS bisect each other.

11. Theorem (*i*) The medians of a triangle are concurrent. (*ii*) They intersect at a point of trisection.

Ex. 61. *Any two medians of a triangle are together greater than the third.*

Let AD, BE and CF be the medians of the $\triangle ABC$. To prove that the sum of any two medians is greater than the third. Produce AD to H so that $DH = DG$. Join HC.

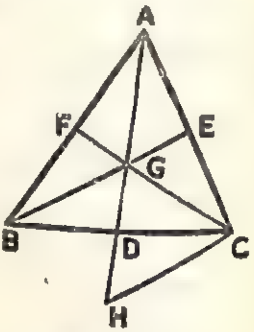


Fig. 30

Proof : $\because$ $BD = DC$, $DG = DH$ and $\angle BDG = \angle HDC$, $\therefore$ $\triangle BGD \equiv \triangle CDH$. $\therefore$ $HC = BG$. In the $\triangle GCH$, $GC + HC > GH$, $\therefore$ $GC + BG > GH$. $\because$ the medians of the triangle intersect at G, $\therefore$ $GD = \frac{1}{2}AG$. $\because$ $AG = GH$. $\therefore$ $BG + GC > AG$. But $AG = \frac{2}{3}AD$, $CG = \frac{2}{3}CF$, $BG = \frac{2}{3}BE$. $\therefore$ $\frac{2}{3}(BE + CF) > \frac{2}{3}AD$, $\therefore$ $BE + CF > AD$. Similarly, it can be proved that $BE + AD > CF$ and $CF + AD > BE$.

Ex. 62. *Straight lines are drawn from a fixed point to a given straight line. Find the locus of their middle points.*

[*C. U. 1938*]

Let PQ be a straight line and O be a point outside PQ.

To find the locus of the middle points of the straight lines which are drawn from O to PQ.

Let two st. limes OA and OB be drawn from O to PQ. Let X and Y be their middle points respectively. Join XY. Then XY is parallel to the base AB of the $\triangle$OAB.

Now, draw any straight line OE from O to PQ, intersecting XY or XY produced at K. Now, from the middle point X of the side OA of the $\triangle$OAE, the line XK is drawn parallel to the base AE, $\therefore$ K is the middle point of OE. It is true for every straight line drawn from O to PQ. $\therefore$ the unlimited straight line XY is the required locus.

Ex. 63. *If two medians of a triangle are equal show that the triangle is isosceles.*

Let the two medians BE and CF of the $\triangle$ABC be equal. To prove that the $\triangle$ABC is isosceles.

Proof : Let the two medians intersect at G. $\because$ the three medians of a triangle intersect at a point of trisection, $\therefore$ EG$=\frac{1}{3}$ BE and FG$=\frac{1}{3}$ CF. But BE$=$CF, $\therefore$ EG$=$FG, $\therefore$ BG$=$CG. Now, in the $\triangle$ˢBGF, CGE, FG$=$GE, BG$=$CG and $\angle$FGB$=\angle$EGC, $\therefore$ The two triangles are equal in all respects $\therefore$ BF$=$CE. $\therefore$ 2BF$=$2CE, that is, AB$=$AC. $\therefore$ $\triangle$ABC is isosceles.

Ex. 64. Trisect a right angle.

$\angle$ABC is a right angle. To divide it into three equal parts.

Construction : With centre B and with any radius draw an arc cutting AB at P and BC at Q. With centres P and Q and with the previous radius draw two arcs cutting the first arc at E and D respectively. Join BD and BE. Now $\angle$ABC is divided into three equal parts, viz, $\angle$ABD, $\angle$DBE and $\angle$EBC.

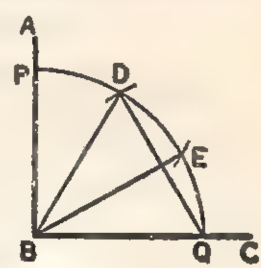


Fig. 31

Proof : Join DQ. $\triangle$BDQ is equilateral, because its sides are equal to the same radius. $\therefore$ $\angle$DBQ$=60°$.

$\therefore$ $\angle$ABD$=90°-60°=30°$. Similarly by joining PE it can be shown that $\angle$ABE$=60°$,

$\therefore$ $\angle DBE = 60° - 30° = 30°$. $\therefore$ the remaining $\angle EBC = 30°$,

$\therefore$ $\angle ABD = \angle DBE = \angle EBC$.

Ex. 65. *Draw the bisector of a given angle without using the vertex in the construction.*

Suppose $\angle AOB$ to be the given angle. Draw any straight line PQ intersecting OA and OB at P and Q. Let the two bisectors of the $\angle OQP$ and $\angle OPQ$ intersect at R and the two bisectors of the $\angle APQ$ and $\angle BQP$ intersect at S. The line SR produced will be the bisector of the $\angle AOB$.

Proof : The two points R and S are equidistant from AO and BO, $\therefore$ they lie on the bisector of the $\angle AOB$. $\therefore$ SR produced will bisect the $\angle AOB$.

Ex. 66. Equilateral $\triangle$'ABD, BCE and CAF are drawn on the sides of a $\triangle ABC$, show that $AE = BF = CD$.

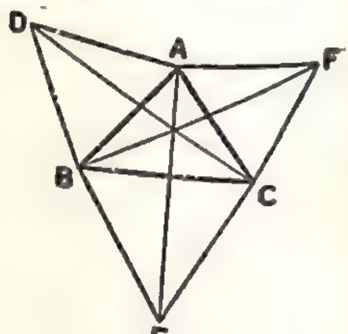


Fig. 32

Proof : $\angle ACF = 60° = \angle BCE$. Adding the $\angle ACB$ to both sides we have $\angle BCF = \angle ACE$. Similarly, $\angle DBC = \angle ABE$. Now, in the $\triangle$'ACE and BCF, $AC = CF$, $CE = BC$ and $\angle ACE = \angle BCF$, $\therefore$ the two triangles are congruent.

$\therefore$ $AE = BF$. Similarly, $\triangle$'ABE, DBC are congruent. $\therefore$ $AE = DC$. $\therefore$ $AE = BF = CD$.

Ex. 67. *The bisectors of the opposite angles of a parallelogram are parallel.*

AF and CG are the bisectors of the $\angle A$ and $\angle C$ of the parallelogram ABCD and BE and DH are the bisectors of the $\angle B$ and $\angle D$. Let them intersect at P, O, Q and R. To prove that AF$\parallel$CG and BE$\parallel$DH.

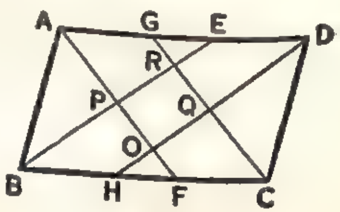


Fig. 33

Proof : $\angle DHC$ = alternate $\angle ADH = \frac{1}{2}$ $\angle ADC = \frac{1}{2}$ $\angle ABC$ ($\because$ the opposite angles of a parallelogram are equal.)

$\therefore \quad \angle DHC = \angle EBC$,

$\therefore \quad DH \parallel BE$. Similarly $AF \parallel CG$.

Ex. 68. *If the two straight lines bisecting two angles of a triangle and terminated at the opposite sides be equal, show that the triangle is isosceles.*

The bisectors BE and CF of the $\angle B$ and $\angle C$ of the $\triangle ABC$ are equal. To prove that the $\triangle ABC$ is isosceles.

Construction : Draw $BD \parallel FC$, so that $BD = FC$. Join DE and DC.

Proof : If AB be not equal to AC, then $\angle B$ and $\angle C$ will not be equal. Suppose $\angle ABC > \angle ACB$.

In the $\triangle ABE$, the exterior $\angle BEC = \angle A + \angle ABE$ $= \angle A + \frac{1}{2}\angle B$. Similarly, $\angle BFC = \angle A + \frac{1}{2}\angle C$.

But $\frac{1}{2} \angle B > \frac{1}{2} \angle C$, $\quad \therefore \quad \angle BEC > \angle BFC$.

$\because \quad BD$ and FC are equal and parallel,

$\therefore \quad BDCF$ is a parallelogram,

$\therefore \quad \angle BFC = \angle BDC, \quad \therefore \quad \angle BEC > \angle BDC \cdots (1)$

Again, $BD = CF$ (opposite sides of the parallelogram) $= BE$.

$\therefore \quad \angle BED = \angle BDE \cdots (2)$.

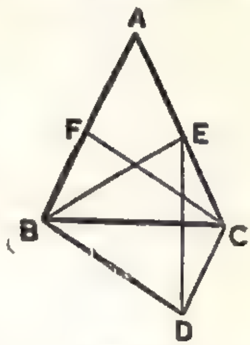


Fig. 34

From (1) and (2) we have

$(\angle BEC - \angle BED) > (\angle BDC - \angle BDE)$,

that is, $\angle CED > \angle CDE$. $\quad \therefore \quad CD > CE$; but $CD = BF$, $\therefore$ $BF > CE$.

Now, in the $\triangle$ˢ BFC and BEC, $FC = BE$, BC is common to both triangles,

but $BF > CE$, $\quad \therefore \quad \angle BCF > \angle CBE$,

$\therefore \angle ACB > \angle ABC$, but this is contrary to the hypothesis. $\therefore$ AB and AC are not unequal, that is, $AB = AC$, $\quad \therefore \quad \triangle ABC$ is isosceles.

Exercise 1

1. The angle between the internal and external bisectors of an angle is a right angle. [D. B. '42]
2. The diagonals of a square are equal.

3. In $\triangle ABC$, $AB = AC$. AB and AC are produced to D and E respectively so that $BD = CE$. Prove that $CD = BE$.

[E. B. S. F. '49]

4. If the exterior angles formed by producing one side of a triangle are equal, then the triangle is isosceles.

[C. U. '24]

5. AB is a chord of a circle whose centre is O; AB is produced to C and D such that $\angle DOA = \angle COB$. Prove that $BC = AD$.

[B. U. '29]

6. If one side of a triangle be produced, then the exterior angle is equal to the sum of the two interior opposite angles.

[C. U. '22]

[**Hints :** In the $\triangle ABC$, BC is produced to D. $\angle ACD + \angle ACB = 2$ right angles. Again $\angle A + \angle B + \angle ACB = 2$ right angles. $\therefore \angle ACD + \angle ACB = \angle A + \angle B + \angle ACB$, $\therefore \angle ACD = \angle A + \angle B$.]

7. If the opposite sides of a quadrilateral are equal its diagonals bisect each other.

8. The straight line which joins the vertices of two isosceles triangles on the same base (1) bisects the vertical angles, (2) bisects the base and (3) is perpendicular to the base.

9. ABC, DBC are two isosceles triangles described on the same base BC, but on opposite sides of it. AD meets BC in E. Prove that $BE = CE$.

[C. U. '28, '33]

[**Hints :** In the $\triangle$s ABD, ACD, $AB = AC$, $BD = DC$, AD is a common side, $\therefore \angle BAD = \angle CAD$. Again in $\triangle$s ABE, ACE, $AB = AC$, AE is common and $\angle BAE = \angle CAE$, $\therefore BE = CE$.]

10. The straight lines joining the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.

11. If a four-sided figure be equilateral, prove that its opposite angles are equal.

[C. U. 1923]

12. Prove that a diagonal of a rhombus bisects each of the angles through which it passes.

[C. U. 1916]

13. Only one perpendicular can be drawn from a given point to a given straight line.

14. The angle subtended by a side of a triangle at any point within it is greater than the angle opposite to that side. [W. B. S. F. '53]

15. Any three sides of a quadrilateral are together greater than the fourth. [C. U. 1913, '33]

16. The greatest side of any triangle makes acute angles with each of the other sides.

17. ABC is a triangle the angles at whose base BC are equal; these angles are bisected by BO and CO, and BO is produced. Prove that the exterior angle at O is equal to either of the base angles of the triangle ABC. [C. U. '22]

[**Hints :** $\because$ $\angle ABC = \angle ACB$, $\therefore$ $\frac{1}{2}\angle ABC = \frac{1}{2}\angle ACB$, *i.e.*, $\angle OBC = \angle OCB$. Again, exterior $\angle DOC = \angle OBC + \angle OCB$ $= 2\angle OCB = \angle ACB = \angle ABC$.]

18. AB, AC are the equal sides of an isosceles triangle; D, E, F are the middle points of AB, BC, CA respectively, prove that $DE = EF$ and $\angle ADE = \angle AFE$. [C. U. 1920]

[**Hints :** $BE = CE$, $DB = FC$ ($\because$ halves of equal sides), $\angle B = \angle C$; $\therefore$ $\triangle DBE \equiv \triangle CEF$ $\therefore$ $DE = EF$ and $\angle BDE = \angle EFC$, $\therefore$ $\angle ADE = \angle AFE$ being supplements of those two angles.]

19. The perpendiculars drawn from the vertices of an equilateral triangle to the opposite sides are equal to one another.

20. If the perpendiculars from two of the angular points of a triangle to the opposite sides are equal, the triangle is isosceles. [D. B. '30]

21. The sum of the interior angles of a quadrilaterial is equal to four right angles.

22. The equal sides BA, CA of an isosceles triangle ABC are produced to the points E and F so that AE is equal to AF, and FB, EC are joined; show that $FB = EC$.

23. If one acute angle at the base of a triangle is double of the other angle at the base and a perpendicular is drawn from the vertex to the base, show that the difference

between the parts into which the base is divided is equal to the smaller side. [B. U.]

[**Hints** : In the $\triangle ABC$, $\angle C = 2\angle B$ and $AD \perp BC$. To prove that, $BD - DC = AC$. From BD cut off DE equal to CD and join AE. Now, $\triangle ACD$ and $\triangle ADE$ are congruent (easy proof). $\therefore$ $CD = DE$, $AC = AE$, and $\angle AED = \angle C = 2\angle B$. But $\angle AED = \angle B + \angle BAE$, $\therefore$ $\angle B = \angle BAE$, $\therefore$ $BE = AE = AC$.]

24. OA and OB are any two perpendicular radii in a circle and AM and BN are perpendiculars drawn from A and B to a diameter of the circle. Prove that $AM = ON$. [B. U.'28]

25. The $\triangle ABC$ has the angles B and C equal. Show that the bisectors of these angles terminated by the opposite sides are equal. [C. U. 1929 ; B. D. '41]

[**Hints** : Suppose the bisectors BD and CE of the $\angle B$ and $\angle C$ cut AC at D and AB at E. In the $\triangle BEC$ and $\triangle BDC$, $\angle EBC = \angle DCB$, $\angle ECB = \angle DBC$ (being half of two equal angles) and BC is a common side. $\therefore$ $BD = CE$.]

26. Two straight lines which are perpendiculars to the same straight line are parallel to each other. [C. U. 1917]

27. In a triangle, the perpendiculars drawn from the extremities of a side to its median are equal.

28. If one angle of a parallelogram is a right angle, all its angles are right angles.

29. The sum of the base angles of a triangle is 108° and their difference is 12°. Find the angles of the triangle.

[**Hints** : $\angle A + \angle B = 108°$, $\angle A - \angle B = 12°$, $\therefore$ (adding) $2\angle A = 120°$, $\therefore$ $\angle A = 60°$, $\therefore$ $\angle B = 108° - 60° = 48°$ and $\angle C = 180° - 108° = 72°$]

30. If one angle of a triangle is equal to the sum of the other two, then the triangle is right-angled. [C. U. 1928]

[**Hints** : In the $\triangle ABC$, $\angle A = \angle B + \angle C$ (Hyp.). But $\angle A + \angle B + \angle C = 2$ right angles. $\therefore$ $\angle A + \angle A = 2$ right angles, that is, $2\angle A = 2$ right angles.

$\therefore$ $\angle A =$ one right angle.]

31. A diagonal of a parallelogram is bisected, and through the point of intersection a straight line is drawn to

be terminated by one pair of opposite sides. Show that the straight line is bisected at the point. [C. U. 1931]

32. In the $\triangle ABC$, BA and CA are produced to D and E so that $AD = AB$ and $AE = AC$. Prove that DE is parallel to BC.

33. Three straight lines meet in a point. Draw another line cutting them so that the segment of it intercepted between the first and second, shall be equal to that intercepted between the second and third.

34. If each side of a triangle is produced both ways, show that the sum of all the six exterior angles is equal to eight right angles. [W. B. S. F. '53]

[**Hints** : If each side of a triangle is produced both ways, there will be two exterior angles at each angular point. At the point A, $\angle A +$ one of the exterior angles $= 2$ right angles. $\therefore$ at the point A, two exterior angles $+ 2\angle A = 4$ right angles. The same is the case also at the points B and C.

$\therefore$ the six exterior angles $+ 2(\angle A + \angle B + \angle C) = 12$ right angles.

But $2(\angle A + \angle B + \angle C) = 4$ right angles, $\therefore$ the six exterior angles are equal to $(12 - 4)$ or 8 right angles.]

35. State what regular polygon has each of its angles equal to $\frac{9}{10}$ of two right angles. [C. U. 1877]

[The sum of the angles of a regular polygon + 4 right angles = twice as many right angles as the polygon has sides.

Here each angle $= \frac{9}{10} \times 2$ right angles $= \frac{9}{5}$ right angles.

Let x be the number of sides, $\therefore$ $2x$ right angles $= (\frac{9}{5}x + 4)$ right angles, $\therefore$ $2x - \frac{9}{5}x = 4$, $\therefore$ $x = 20$.]

36. Find a point equidistant from the sides of a triangle How many such points are possible ?

37. One of the angles of a pentagon is a right angle ; the other four angles are equal to each other. How many degrees are there in each ? [D. B. '27]

[The sum of the angles of the pentagon $= (5 \times 2 - 4)$ right angles $= 6$ right angles ; $\because$ one of its angles $= 1$ right angle,

∴ the sum of the remaining 4 angles = 5 right angles.
∴ each of the remaining angles of the pentagon = $\frac{5}{4}$ right angles = $112\frac{1}{2}$ degrees.]

38. The internal bisectors of the angles of a parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram.

39. An exterior angle of a regular rectilinear figure is double of an interior angle. Find the number of sides.

[C. U. '39 ; Ans. = 3]

40. ABC is a triangle ; straight lines AD, CE bisect the angles at A and C and from B, BE is drawn equal to BC, and BD equal to BA ; show that E, B, D are collinear. [C. U.]

41. D is the middle point of the side BC of the $\triangle$ABC. If BD = CD = AD, prove that $\angle$BAC is a right angle.

[G. U. '48]

42. Find the magnitude of an angle of a regular hexagon.

[C. U. '50 ; Ans. 120°]

43. If the exterior angles formed by producing the sides of a triangle in order are equal, prove that the triangle is equilateral.

[G. U. '55]

[Suppose that in the $\triangle$ABC the sides CA, AB, BC are produced in order to D, E, F, so that $\angle$DAB = $\angle$EBC = $\angle$ACF.

Proof : $\angle$ABC = supplement of $\angle$EBC = supplement of $\angle$ACF = $\angle$ACB. ∴ AB = AC. Similarly, AC = BC,
∴ AB = AC = BC. ∴ $\triangle$ABC is equilateral.]

44. ABC and PQR are two triangles such that AB and AC are respectively equal to PQ and PR. If the medians BD and QS are equal, prove that the triangles are congruent.

[N. U. '49]

45. In $\triangle$ABC the $\angle$C is a right angle and AB is double of BC. Prove that $\angle$B is 60°.

[N. U. '50]

[Solutions]

12. **Theorem**—(*i*) Parallelograms on the same base and between the same parallels (or of the same altitude) are equal in area.

(*ii*) Triangles on the same base and between the same parallels (or of the same altitude) are equal in area.

(*iii*) Equal triangles on the same base and on the same side of it are between the same parallels.

Ex. 1. *Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area.*

ABCD is a parallelogram whose diagonals AC and BD intersect at O.

To prove that $\triangle$' AOB, BOC, COD, AOD are equal in area.

Proof: The two diagonals of a parallelogram bisect each other. $\therefore$ BO = DO and AO = CO. Now $\triangle$' AOB, AOD are on equal bases and of the same height, $\therefore$ $\triangle$AOB = $\triangle$AOD.

Similarly, $\triangle$AOB = $\triangle$BOC and $\triangle$BOC = $\triangle$COD.

$\therefore$ $\triangle$AOB = $\triangle$BOC = $\triangle$COD = $\triangle$AOD.

Ex. 2. (*a*) *If a quadrilateral is bisected by each of its diagonals prove that it is a parallelogram.*

Suppose the quadrilateral ABCD is bisected by the diagonals AC and BD. To prove that ABCD is a parallelogram.

Proof: $\triangle$ABC = $\triangle$DBC (by hypothesis) and they are on the same base and on the same side of it. $\therefore$ AD $\parallel$ BC.

Similarly, AB $\parallel$ DC. $\therefore$ ABCD is a parallelogram.

2. (*b*) *Show that the straight line joining the middle points of any two sides of a triangle is parallel to the third side.*

[W. B. S. F. '53, C. U. 1917]

P and Q are the middle points of the sides AB and AC of the $\triangle$ABC. Join PQ.

To prove that PQ $\parallel$ BC, Join BQ and CP.

Proof: BQ is the median of the $\triangle$ABC,

$\therefore$ $\triangle$BQC = $\frac{1}{2}$ $\triangle$ABC. Again CP is the median of the $\triangle$ABC, $\therefore$ $\triangle$BPC = $\frac{1}{2}$ $\triangle$ABC. $\therefore$ $\triangle$BPC = $\triangle$BQC and they are on the same base BC and on the same side of it. $\therefore$ PQ $\parallel$ BC.

Ex. 3. *Show that the straight line which joins the middle points of the oblique sides of a trapezium, is parallel to each of the parallel sides.* [C. U. 1936]

ABCD is a trapezium of which AD ∥ BC and P and Q are the middle points of the oblique sides AB and CD. Join P and Q. To prove that PQ is parallel to AD and BC. Join AC, PC, BD and BQ.

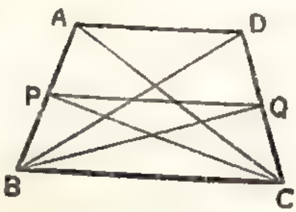


Fig. 35

Proof : Δ^s ABC, BDC stand on the same base BC and between the two parallel st. lines BC and AD, ∴ $\Delta ABC = \Delta BDC$. Again, P is the middle point of AB, ∴ $\Delta BPC = \frac{1}{2}\Delta ABC$.

Q is the middle point of DC, ∴ $\Delta BQC = \frac{1}{2}\Delta BDC$.

∴ $\Delta PBC = \Delta BQC$ and they are on the same base BC and on the same side of it, ∴ PQ ∥ BC.

Again, ∵ AD ∥ BC, ∴ PQ is parallel to BC and AD.

Ex. 4. *A square and a rhombus stand on the same base. Which has the greater area? Give reasons.* [C. U. 1940]

Let the square ABCD and the rhombus BCEF stand on the same base BC.

It is required to find which has the greater area.

Draw FG perp. to BC. The area of the square as well as the rhombus = base × altitude.

∴ the square ABCD = BC.AB and the rhombus BCEF = BC.FG.

Now ∠G is a right angle.

∴ the hypotenuse BF > FG.

But BF = BC = AB, ∴ AB > FG.

∴ BC.AB > BC.FG

∴ The area of the square is greater than the area of the rhombus.

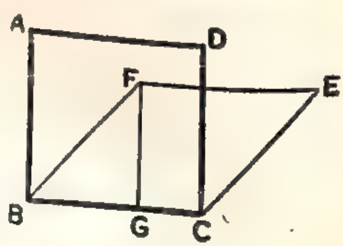


Fig. 36

13. **Theorem**—The area of a triangle is half the area of a rectangle on the same base and having the same altitude.

Ex. 5. *Show that the area of a rhombus is equal to half the rectangle contained by the diagonals.* [*C. U.* '29, '45]

Let the diagonals AC and BD of the rhombus ABCD intersect at O.

To prove that the area of the rhombus $= \frac{1}{2}$ AC.BD.

Proof : The diagonals of a rhombus bisect each other at right angles, $\therefore \triangle ABD = \frac{1}{2}$ BD.AO. Similarly, $\triangle BCD = \frac{1}{2}$ BD.CO.

$\therefore$ the rhombus ABCD $= \triangle ABD + \triangle BCD$

$= \frac{1}{2}$ BD.AO $+ \frac{1}{2}$ BD.CO $= \frac{1}{2}$ BD (AO$+$CO)$= \frac{1}{2}$ BD.AC.

Ex. 6. ABCD *is any parallelogram and* O *is any point within it. Show that the sum of the* $\triangle^s$ AOB, COD *is equal to half the area of the parallelogram.* [*C. U. 1950*]

Let O be a point within the parallelogram ABCD.

To prove that $\triangle AOB + \triangle COD = \frac{1}{2}$ parallelogram ABCD.

Draw PQ parallel to AB or CD through O.

Let it intersect AD and BC at P and Q respectively.

Proof : ABQP and CDPQ are parallelograms by construction.

$\therefore$ $\triangle AOB = \frac{1}{2}$ parallelogram ABQP ($\because$ they stand on the same base AB and between the same parallels AB and PQ.)

Similarly, $\triangle COD = \frac{1}{2}$ parallelogram CDPQ.

$\therefore$ $\triangle AOB + \triangle COD = \frac{1}{2}$ (param. ABQP $+$ param. CDPQ)

$= \frac{1}{2}$ parallelogram ABCD.

Ex. 7. *The triangle formed by joining the middle pt. of one of the oblique sides of a trapezium to the extremities of the opposite side, is half the trapezium.*

Let P be the middle point of the oblique side CD of the trapezium ABCD. Join AP and PB.

To prove that $\triangle APB = \frac{1}{2}$ trapezium ABCD.

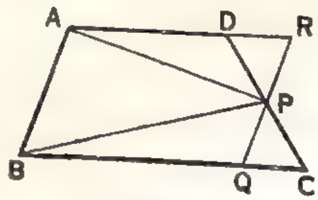


Fig. 37

Through P draw QPR $\parallel$ AB.

Let it intersect BC at Q and AD produced at R.

In the $\triangle^s$ PQC and PRD, PC$=$PD, $\angle$QPC$=\angle$DPR, $\angle$PCQ$=$alternate $\angle$PDR, $\therefore$ the two triangles are congruent.

$\therefore$ trapezium ABCD $=$ parallelogram ABQR.

Now, $\triangle APB$ and parallelogram ABQR stand on the same base AB and between the same parallels AB and RQ,

$\therefore \quad \triangle APB = \frac{1}{2}$ parallelogram $ABQR = \frac{1}{2}$ trapezium ABCD.

Ex. 8. *If the middle points of the sides of a quadrilateral are joined in order, prove that the parallelogram so formed is half the quadrilateral.*

Let P, Q, R, S be the middle points of the sides AB, BC, CD, DA respectively of the quadrilateral ABCD. PQRS is the parallelogram formed by joining these middle points in order.

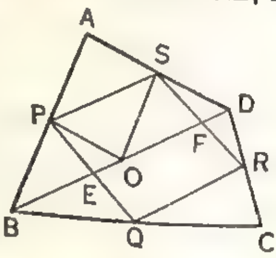


Fig. 38

To prove that the parallelogram $PQRS = \frac{1}{2}$ the quadrilateral ABCD.

Let O be the middle point of BD. Join PO, SO. Let PQ and SR cut BD at E and F.

Proof: $\triangle POS$ is formed by joining the middle points P, O, S of the sides of the $\triangle ABD$. $\therefore \quad \triangle POS = \frac{1}{4} \triangle ABD$.

Again $\because$ PS $\parallel$ EF, $\therefore \quad \triangle POS = \frac{1}{2}$ parallelogram PEFS (being on the same base and between the same parallels).

$\therefore$ parallelogram $PEFS = \frac{1}{2} \triangle ABD$.

Similarly it can be proved that param. $QEFR = \frac{1}{2} \triangle BCD$.

$\therefore$ the whole param. $PQRS = \frac{1}{2}$ of the quadrilateral ABCD.

Ex. 9. *In a parallelogram* ABCD *any point* O *is taken on the diagonal* AC. *If* O *is joined to* B *and* D, *show that the* $\triangle AOB$ *and the* $\triangle AOD$ *are equal in area.* [*C. U. 1942*]

Let the diagonals AC, BD intersect each other at P. The diagonals of a parallelogram bisect each other.

$\therefore \quad \triangle APB = \triangle APD$ [$\because$ their bases BP and PD are equal and they have the same altitude.]

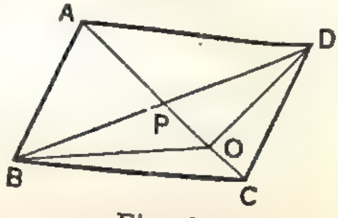


Fig. 39

Again, PO is the median of the $\triangle BOD$, $\therefore \triangle BPO = \triangle POD$.

$\therefore$ the whole $\triangle ABO =$ the whole $\triangle ADO$.

Ex. 10. *Find the locus of the intersection of the diagonals of a parallelogram standing on a given base and having a constant area.*

BC is a fixed st. line. Let ABCD be a parallelogram of constant area standing on BC.

To find the locus of the intersection of the diagonals of all parallelograms that may stand on BC and be equal to the parallelogram ABCD.

Let the diagonals AC, BD intersect at O. Now, $\triangle$BOC is $\frac{1}{4}$ of the whole par^m ABCD. It is true for all par^{ms} standing on BC. $\because$ the areas of all these par^{ms} are equal, $\therefore$ the triangles which are $\frac{1}{4}$ of them are also equal in area. Again, those $\triangle^s$ stand on the same base BC and on the same sides of it, $\therefore$ they are between the same parallels. $\therefore$ the st line (XY) drawn parallel to BC through O is the required locus.

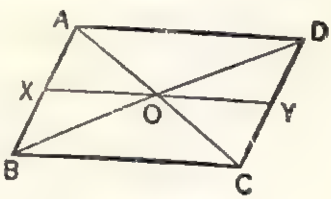


Fig. 40

Ex. 11. *If a diagonal bisects a quadrilateral, prove that it also bisects the other diagonal.* [*B. U. 1920*]

Let the diagonal AC bisect the quadrilateral ABCD.

To prove that AC bisects the diagonal BD. Let AC intersecr BD at O. Draw BP and DR perpendicular to AC.

Proof: BP and DR are altitudes respectively of the $\triangle^s$ ABC, ADC. $\triangle ABC = \frac{1}{2}AC.BP$ and $\triangle ADC = \frac{1}{2}AC.DR$.

$\because$ $\triangle ABC = \triangle ADC$ (by hypothesis),

$\therefore$ $\frac{1}{2}AC.BP = \frac{1}{2}AC.DR$, $\therefore$ $BP = DR$.

Now, in the $\triangle^s$ BOP and DOR, $\angle BOP = \angle DOR$,

$\angle BPO = \angle DRO$ (rt. $\angle s$) and $BP = DR$,

$\therefore$ the two $\triangle^s$ are congruent. $\therefore$ $BO = DO$.

$\therefore$ AC bisects the diagonal BD.

Ex 12. *Two* $\triangle^s$ ABC, DBC *on the same base* BC *and on the same side of it are equal in area. If the* $\triangle$ABC *be isosceles,*

prove that the perimeter of the $\triangle ABC$ *is less than that of the* $\triangle DBC$.

[*B. U. 1926*]

Draw $BE \perp BC$. Produce CA to meet BE at E and produce DA to meet BE at F. $\because$ $\triangle ABC = \triangle DBC$, $\therefore$ $AD \parallel BC$.

$\therefore$ $\angle EAF$ = corresponding $\angle ACB$, $\angle FAB$ = alt. $\angle ABC$.

$\because$ $\angle ABC = \angle ACB$ ($\triangle ABC$ being isosceles),

$\therefore$ $\angle EAF = \angle BAF$.

Again, $\angle EFA$ = corresponding $\angle FBC$ = 1 rt. angle = $\angle AFB$, and AF is common, $\therefore$ $\triangle^s$ AFB, AEF are equal in all respects.

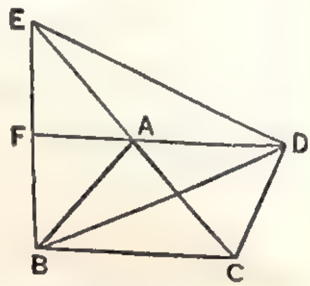


Fig. 41

$\therefore$ FBD is the perpendicular bisector of BE.

$\therefore$ $AE = AB$ and $ED = DB$.

Now $AB + AC = EA + AC = EC$,

and $DB + DC = ED + DC$,

But in the $\triangle DCE$, $ED + DC > CE$,

$\therefore$ $DB + DC > AB + AC$. $\therefore$ $AB + AC + BC < DB + DC + BC$.

Ex. 13. *Bisect a triangle by a straight line drawn through a given point on one of its sides.* [*C. U. 1934, '48 Supp.*]

(See Problem 20)

Ex. 14. *Trisect a triangle by straight lines drawn through a given pt. on one of its sides.* [*C. U. '36, '39, '43*]

(See Problem 21)

Ex. 15. *Bisect a quadrilateral by a straight line drawn through a vertex.* [*C. U. '36, '39*]

(See Problem 22]

Ex. 16. *Bisect a quadrilateral by a straight line drawn through a given point in one of its sides.* [*C. U. 1941, '49*]

Let P be a point on the side AD of the quadrilateral $ABCD$. It is required to bisect the quadrilateral by drawing a st. line through P.

Join PB, PC. From A draw $AY \parallel PB$ and from D draw $DX \parallel PC$. Let AY and DX cut BC produced Y and X respectively. Join PY, PX. Bisect YX at Q. Join PQ. Then PQ bisects the quadrilateral $ABCD$.

Proof : PQ is the median of the $\triangle$PYX, $\therefore$ $\triangle$PYQ = $\triangle$PXQ.

Again $\triangle$APB = $\triangle$BPY, standing on the same base and between the same parallels. Add $\triangle$PBQ to both, then the quadrilateral APQB = $\triangle$PYQ. Similarly, it can be shewn that the quadrilateral PDCQ = $\triangle$PQX. $\therefore$ the quadr. APQB = the quadr. PDCQ, *i.e.*, PQ bisects the given quadrilateral.

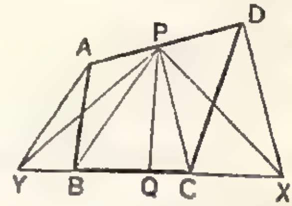


Fig. 43

Ex. 17. *Describe a rhombus equal to a given parallelogram and standing on the same base. When does the construction fail ?* [*C. U. 1935*]

Let ABCD be a par^m of which BC>AB. To draw on BC a rhombus equal in area to the par^m ABCD.

Construction : With centre B and radius BC draw an arc of a circle cutting AD at E. With centre E and radius BC draw an arc cutting AD produced at F. Join CF. Then EBCF is the required rhombus.

Proof : BC = BE = EF, $\therefore$ BC and EF are equal and parallel. $\therefore$ BE and CF are also equal and parallel. $\therefore$ EBCF is a par^m whose all sides are equal, $\therefore$ it is a rhombus Now, rhombus EBCF and par^m ABCD stand on the same base BC and between the same parallels BC, EF.

$\therefore$ they are equal in area.

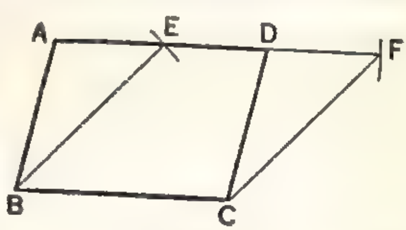


Fig. 44

It is not possible to draw such a rhombus on the smaller side AB or DC of the parallelogram.

Ex. 18. *Construct a parallelogram equal in area to a given triangle and having an angle equal to a given angle.*

[*C. U. 1939, '48 Supp,*]

(Vide Problem 18)

Ex. 19. *Construct a triangle equal in area to a given triangle and having a given base.*

Let ABC be a triangle and b a straight line. To draw a triangle equal to the $\triangle$ABC on a base equal to b.

Cut off BD equal to b from BC or BC produced. Join AD. Draw CE $\parallel$ DA cutting BA or BA produced at E. Join ED. Then EBC is the required triangle.

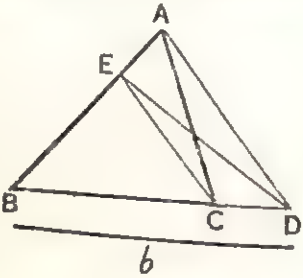


Fig. 45

Proof : $\triangle$ECD $=$ $\triangle$AEC (standing on EC and between the parallels EC and AD). Add the $\triangle$EBC to both. Then $\triangle$BED $=$ $\triangle$ABC, and the base BD of the $\triangle$EBD $= b$ (by construction).

Ex. 20. *Construct a triangle equal in area to a given triangle and having a given altitude.*

To draw a triangle equal in area to the $\triangle$ABC and having its altitude equal to h.

Draw BX$\perp$BC. From it cut off BX equal to h. From X draw XY $\parallel$ BC to cut BA produced at Y. Join CY. From A draw AP $\parallel$ YC to cut BC at P. Join YP. Then $\triangle$BYP is the required triangle.

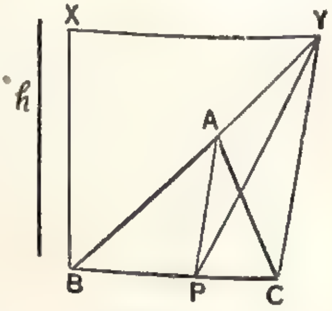


Fig. 46

Proof. The perpendicular drawn from Y to BC will be equal to BX or h.

Again, $\triangle$APY $=$ $\triangle$APC (standing on the same base AP and between the same parallels AP, YC). Adding the $\triangle$ABP to both, the whole $\triangle$ABC $=$ the whole $\triangle$BPY

$\therefore$ $\triangle$BPY is the required triangle.

Ex. 21. *Construct a triangle equal in area to a given triangle having a given vertex and its base in the same straight line as the base of the given triangle.*

Let ABC be the given triangle and P the given point. To draw a triangle equal to the $\triangle$ABC in area and having its base on the same st. line BC and the vertex P.

Draw PQ $\parallel$ CB to cut BA at Q. Join QC and from A draw AD $\parallel$ QC. Let AD cut BC at D. Join BP, DP.

$\triangle$BPD is the required triangle.

Proof : Join QD. $\triangle$AQD $=$ $\triangle$ADC. Adding the $\triangle$ABD to both we have $\triangle$BQD $=$ $\triangle$ABC. Again, $\triangle$BQD $=$ $\triangle$BPD ($\because$ they stand on the base BD and between the parallels BD, PQ), $\therefore$ $\triangle$PBD $=$ $\triangle$ABC. $\therefore$ $\triangle$PBD is the required triangle.

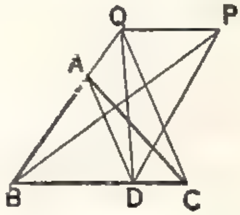


Fig. 47

Ex. 22. *Construct a parallelogram equal in area to a given triangle and having one side of given length and an angle equal to a given angle.*

To draw a par^m equal in area to $\triangle$ABC and having one side $= l$ and one angle $= \angle x$.

Construction. Produce BC to D so that BD $= 2l$. Join AD. Draw CE $\parallel$ DA cutting AB at E. Draw EF $\parallel$ BD. Draw $\angle$DGH $= \angle x$ at the middle point G of BD. Let GH cut EF at H. Draw DF $\parallel$ GH cutting EF at F. Then FDGH will be the required par^m.

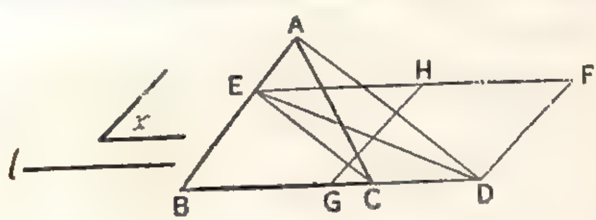


Fig. 48

Proof : Join ED. $\triangle$ECD $=$ $\triangle$AEC (standing on the same base BC and between the same parallels EC, AD).

$\therefore$ $\triangle$BED $=$ $\triangle$ABC. $\because$ HF $\parallel$ GD and GH $\parallel$ DF,

$\therefore$ FDGH is a par^m. Now, the par^m FDGH and $\triangle$BED are between two parallel st. lines and the base GD of the par^m is half the base BD of the triangle.

$\therefore$ the par^m FDGH $=$ $\triangle$BED $=$ $\triangle$ABC, and its $\angle$HGD $= \angle x$ and base GD $= \frac{1}{2}$BD $= l$.

Ex. 23. *Construct a parallelogram equal in area to a given parallelogram, having one side of a given length and an angle equal to a given angle.* [*C. U. 1944*]

To draw a par^m equal in area to the given par^m ABCD having one side $= l$ and one angle $= \angle x$.

Construction. Cut off AE equal to l from AB or AB produced. Draw EF ∥ AD cutting DC at F. Join AF and let it cut BC at G. Through G draw HGO ∥ AE to cut AD at H and EF at O. At A draw $\angle$EAK $= \angle x$. Let AK cut HO at K. Draw EP ∥ AK to cut HO produced at P. Now, AKPE is the required par^m.

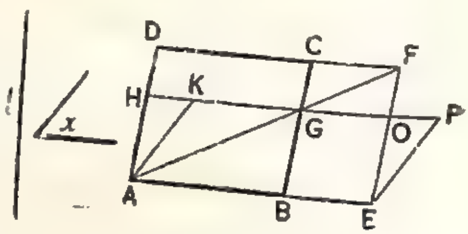


Fig. 49

Proof : By construction, the figures CGOF, ABGH, BEOG, AEFD, AKPE and DCGH are par^{ms}.

$\therefore$ $\triangle$ADF $=$ $\triangle$AFE, $\triangle$CFG $=$ $\triangle$FOG, $\triangle$AGH $=$ $\triangle$ABG.

$\therefore$ $\triangle$ADF $-$ $\triangle$CFG $-$ $\triangle$AGH $=$ $\triangle$AFE $-$ $\triangle$FOG $-$ $\triangle$ABG.

$\therefore$ par^m DCGH $\equiv$ par^m BEOG.

$\therefore$ par^m ABCD $=$ par^m AEOH. Now par^m AEPK $=$ par^m AEOH ($\because$ their bases and altitudes are equal) $=$ par^m ABCD. Again of the par^m AEPK, $\angle$KAE $= \angle x$ and AE $= l$.

Ex. 24. *On a given straight line construct a rectangle equal to a given triangle.* [*C. U. 1946*]

[See the Example 22. Here $\angle x$ will not be given, for each angle of a rectangle is known to be a right angle. Follow the constructions of the example 22. But draw GH$\perp$BD instead of drawing an angle $= \angle x$ at G.]

14. **Theorem**—In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

Ex. 25. *Construct a square equal to the sum of two given squares.*

Let a and b be the respective sides of the two given squares. To draw a square whose area is equal to a^2+b^2,

Construction. Take $AB = a$, At A draw $AC \perp AB$. From it cut off $AC = b$. Join BC. The square on BC will be the required square.

Proof : $\because \angle A = 1$ rt. angle, $\therefore BC^2 = AB^2 + AC^2 = a^2 + b^2$.

[N.B. When you are to draw a square equal to the sum of three given squares, follow the previous construction and then at C draw $CD \perp BC$. Make CD equal to the side of the third square. Join BD. The square on BD will be the required square. If the number of squares be more, follow the above construction gradually in the same way.]

Ex. 26. *Construct a square equal to the difference of two given squares.*

Let a and b be the respective sides of the two given squares and $b > a$. To draw a square whose area $= b^2 - a^2$.

Construction : Take $BA = a$. At A draw $AP \perp BA$ with centre B and radius b draw an arc of a circle to cut AP at C. The square on AC will be the required square.

Proof : $\because \angle A = 1$ rt. angle, $\therefore AB^2 + AC^2 = BC^2$.

$\therefore AC^2 = BC^2 - AB^2 = b^2 - a^2$.

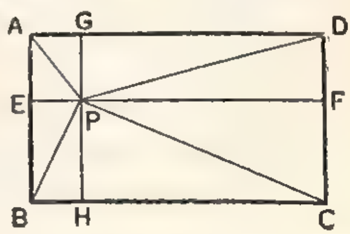
Ex. 27. *Construct a square equal to 3 square inches in area.*

Take $AB = 1$ inch. At A draw AC perpendicular to AB and let $AC = 1$ inch. Join BC. Again, draw $CD \perp BC$ and let $CD = 1$ inch.

The square on BD will be the required square.

Proof : $BD^2 = BC^2 + CD^2 = AB^2 + AC^2 + CD^2 = 1$ sq. in. $+$ 1 sq. in. $+$ 1 sq in. $=$ 3 sq. inches.

Ex. 28. *If any point* P *be joined to* A, B, C, D, *the angular points of a rectangle, show that the squares on* PA *and* PC *are together equal to the squares on* PB *and* PD. [*C. U. 1921*]



Let P be a point within the rectangle ABCD.

To prove that $PA^2 + PC^2 = PB^2 + PD^2$. Through P draw EF and GH parallel respectively to AD and AB. Now, all figures in the diagram are rectangles.

Join PA, PC, PB, PD.

Proof : $PA^2=AG^2+PG^2$ and $PC^2=PH^2+CH^2$.

$\therefore$ $PA^2+PC^2=AG^2+PG^2+PH^2+CH^2=BH^2+PG^2+PH^2+GD^2$ ($\because$ $AG=EP=BH$, and $CH=PF=GD$)$=(BH^2+PH^2)+(PG^2+GD^2)=BP^2+PD^2$.

Ex. 29. *$\triangle ABC$ is right-angled at A ; P, Q are points in AB and AC. Show that $PC^2+QB^2=BC^2+PQ^2$.* [*A.U.* '32]

[**Hints :** Join PQ, PC, BQ. $\because$ $\angle A=1$ rt. angle,

$\therefore$ $PC^2=AC^2+AP^2$, $BQ^2=AB^2+AQ^2$.

$\therefore$ $PC^2+BQ^2=(AC^2+AB^2)+(AP^2+AQ^2)=BC^2+PQ^2$]

Ex. 30. *In a right-angled triangle 4 times the sum of the squares on the medians drawn from the acute angles is equal to 5 times the square on the hypotenuse.* [*D. B.* '30]

Let $\angle A$ of the $\triangle ABC$ be a rt. angle. Let D and E be the middle points of AB and AC respectively.

To prove that $4(BE^2+CD^2)=5\,BC^2$. Join CD and BE.

Proof : $\because$ $\angle A$ is a rt. angle, $\therefore$ $BE^2=AE^2+AB^2$ and $CD^2=AC^2+AD^2$. $\therefore$ $4(BE^2+CD^2)=4AE^2+4AB^2+4AC^2+4AD^2=(2AE)^2+4(AB^2+AC^2)+(2AD)^2$

$=AC^2+4BC^2+AB^2$($\because 2AE=AC, 2AD=AB$ and $BC^2=AB^2+AC^2$)

$=4BC^2+BC^2=5BC^2$.

Ex. 31. *Prove that in an equilateral triangle four times the square on the perpendicular drawn from the vertex on the opposite side is equal to three times the square on any side.*

[*C. U. 1933*]

Let AD be the perpendicular on the base BC of the equilateral $\triangle ABC$. To prove that $4AD^2=3AB^2$.

Proof : $\because$ $\angle D$ is a rt. angle, $\therefore$ $AB^2=AD^2+BD^2$.

$\therefore$ $AD^2=AB^2-BD^2$, $\therefore$ $4AD^2=4AB^2-4BD^2=4AB^2-(2BD)^2$

$=4AB^2-BC^2=4AB^2-AB^2=3AB^2$.

Ex. 32. *ABC is a triangle and P and Q are the middle pts. of the sides AB and AC ; show that if BQ and PC intersect at O, the $\triangle BOC=$ the quadrilateral APOQ.* [*D. B. 1927*]

Proof : CP is the median of the $\triangle ABC$.

$\therefore$ $\triangle BPC=\frac{1}{2}\triangle ABC$. Similarly, $\triangle ABQ=\frac{1}{2}\triangle ABC$.

$\therefore$ $\triangle BPC=\triangle ABQ$. If the common part, *i.e.*, $\triangle BOP$ is taken from these two equal triangles, the remainders are equal. $\therefore$ $\triangle BOC=$ the quadrilateral APOQ,

Ex. 33. *The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.*

Let the diagonals AC and BD of the rhombus ABCD intersect at O.

To prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$.

Proof : $\because$ the diagonals of a rhombus bisect each other at right angles, $\therefore$ $AB^2=AO^2+BO^2$; $BC^2=BO^2+CO^2$; $CD^2=CO^2+DO^2$; $AD^2=AO^2+DO^2$.

$\therefore$ (adding) $AB^2+BC^2+CD^2+DA^2=AO^2+BO^2+BO^2+CO^2+CO^2+DO^2+AO^2+DO^2=2AO^2+2BO^2+2CO^2+2DO^2$ $=4AO^2+4BO^2$ ($\because$ $CO=AO$, $DO=BO$)$=(2AO)^2+(2BO)^2$ $=AC^2+BD^2$.

Ex. 34. *The st. lines bisecting the angles at the base of an isosceles triangle meet the other sides at* D *and* E. *Show that* DE *is parallel to the base.* [*P. U.*]

In the isosceles $\triangle ABC$ let $AB=AC$. Let BE and CD bisect $\angle B$ and $\angle C$ respectively and meet AC, AB at E and D.

To prove that DE $\parallel$ BC. Join DE.

Proof : $\because$ $AB=AC$, $\therefore$ $\angle B=\angle C$ and $\frac{1}{2}\angle B=\frac{1}{2}\angle C$ *i.e.*, $\angle CBE=\angle DCB$.

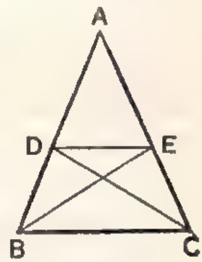


Fig. 51

Now, in the $\triangle^s$ DBC, BEC, $\angle DBC=\angle ECB$, $\angle DCB=\angle EBC$ and BC is common to both.

$\therefore$ $\triangle^s$ DBC, BCE are congruent. $\therefore$ they will stand between two parallel st. lines.

$\therefore$ DE is parallel to BC.

Ex 35. *On the base of a given triangle draw an isosceles triangle of equal area.*

Let ABC be a triangle. To draw an isosceles triangle on the base BC equal to the $\triangle ABC$.

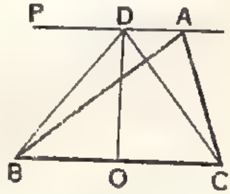


Fig. 52

Construction : Draw OD the perpendicular bisector of BC. Through A draw AP $\parallel$ BC to cut OD at D. Join DB and DC. $\triangle DBC$ is the required triangle.

Proof : $\because$ DO is the perpendicular bisector of BC, $\therefore$ $DB=DC$, $\therefore$ $\triangle DBC$ is an isosceles triangle. Again, $\triangle DBC$ and $\triangle ABC$ stand on the same base BC and between the same parallels BC, PA ; $\therefore$ they are equal in area.

Ex. 36 *Divide a st. line into two parts such that the sum of their squares shall be equal to a given square.*

Let AB be the given st. line and a the side of the given square.

To divide AB into two parts so that the sum of their squares may be equal to a^2.

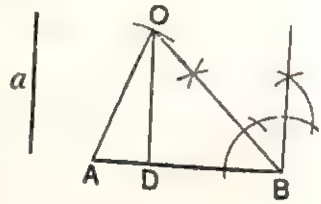


Fig. 53

Construction: At B draw $\angle ABO = 45°$. With centre A and radius a draw an arc of a circle to cut BO at O. Draw $OD \perp AB$. Now AB is divided at D as required.

Proof: $\angle D = 1$ rt. angle, $\angle DBO = 45°$. $\therefore \angle BOD = 45°$. $\therefore DO = DB$. $\because \angle ADO \angle 1$ rt. angle, $\therefore AO^2 = AD^2 + DO^2 = AD^2 + BD^2$. $\therefore AD^2 + BD^2 = a^2$ [$\because AO = a$]

Ex. 37. *Divide a straight line into two parts so that the square on one part may be twice the square on the other.*

Let AB be the given st. line. To divide AB at P so that $AP^2 = 2BP^2$. At A draw a rt. angle and bisect it by AO. Bisect $\angle OAB$ by AC. Now $\angle CAB = 22\frac{1}{2}°$. At B draw $BC \perp AB$ cutting AC at C. At C draw $\angle ACP = \angle CAP$ and let CP cut AB at P.

Now, AB is divided as required at P.

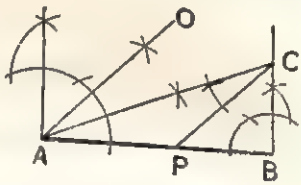


Fig. 54

Proof: In the $\triangle APC$, the ext. $\angle CPB = \angle CAP + \angle ACP = 45°$. $\therefore \angle PCB = 45°$. $\therefore PB = BC$. $\because \angle B = 1$ rt. angle, $\therefore PC^2 = PB^2 + BC^2 = 2PB^2$. $\therefore AP^2 = 2PB^2$ ($\because \angle CAP = \angle ACP$, $\therefore AP = CP$).

Ex. 38. *Divide a st. line into two parts so that the square on one part may be three times the square on the other.*

[*C.U. 1946*]

To divide the st. line AB at P such that $AP^2 = 3BP^2$.

At B draw $\angle ABC = 45°$ and at A draw $\angle BAC = 30°$. Let BC and AC meet each other at C. From C draw $CP \perp AB$. Now, AB is divided as required at P.

Proof: $\angle P = 1$ rt. angle, $\angle B = 45^\circ$, $\therefore$ $\angle PCB = 45^\circ = \angle B$, $\therefore$ $CP = BP$. Again, $\angle A = 30^\circ$, $\therefore$ $\angle ACP = 60^\circ$, $\therefore$ $AC = 2PC$. Now, $AP^2 = AC^2 - PC^2 = (2PC)^2 - PC^2 = 4PC^2 - PC^2 = 3PC^2 = 3PB^2$.

Ex. 39. *Construct an equilateral triangle equal to the sum of two given equilateral triangles.*

Let ABC and DEF be two equilateral triangles.

To draw an equilateral triangle $= \triangle ABC + \triangle DEF$.

Construction: Take $PQ = BC$ and draw $PR \perp PQ$ so that $PR = EF$. Join RQ. Draw the equilateral $\triangle RQS$ on RQ. This will be the required triangle.

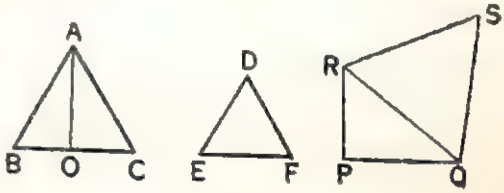


Fig. 55

Proof: Draw $AO \perp BC$. Now, AO is the altitude of the $\triangle ABC$ and it bisects BC. $AO^2 = AB^2 - BO^2 = AB^2 - (\frac{1}{2}AB)^2 = AB^2 - \frac{1}{4}AB^2 = \frac{3}{4}AB^2$ $\therefore$ $AO = \frac{\sqrt{3}}{2}AB$.

So the altitude of an equilateral triangle $= \frac{\sqrt{3}}{2} \times$ base.

$\therefore$ The area of any equilateral triangle $= \frac{1}{2}$ base $\times \frac{\sqrt{3}}{2} \times$ base

$$= \frac{\sqrt{3}}{4} (\text{base})^2$$

Now, $\triangle RQS = \frac{\sqrt{3}}{4} \times (\text{base})^2 = \frac{\sqrt{3}}{4} RQ^2 = \frac{\sqrt{3}}{4}(PQ^2 + RP^2)$

$$= \frac{\sqrt{3}}{4} PQ^2 + \frac{\sqrt{3}}{4} RP^2 = \frac{\sqrt{3}}{4} BC^2 + \frac{\sqrt{3}}{4} EF^2 = \triangle ABC + \triangle DEF.$$

Ex. 40. *Construct an equilateral triangle equal in area to the difference of two given equilateral triangles.*

Let a and b be the sides of two equilateral triangles respectively and let $a > b$. To draw an equilateral triangle equal to the difference of the two given equilateral triangles.

Construction : Take AB $=b$. Draw BC$\perp$AB and with centre A and radius a draw an arc of a circle to cut BC at C. On BC draw an equilateral triangle BCD, which will be the required triangle.

Proof : $\triangle BCD=\frac{\sqrt{3}}{4}BC^2=\frac{\sqrt{3}}{4}(AC^2-AB^2)$

[$\because$ $\angle B=1$ rt. angle.] $=\frac{\sqrt{3}}{4}AC^2-\frac{\sqrt{3}}{4}AB^2=\frac{\sqrt{3}}{4}a^2-\frac{\sqrt{3}}{4}b^2$

$=$the difference of the areas of the two given equilateral triangles.

Ex. 41. *If two sides of a triangle be 15 and 6 inches respectively and the contained angle be 60°, calculate in inches, to the nearest integer, the length of the third side.* [*C.U.* '29]

In the $\triangle ABC$, $BC=15''$, $AC=6''$, and $\angle C=60°$. To find the length of AB.

Draw AD$\perp$BC. Join D and O, the middle point of AC. In $\triangle ADC$, $\angle D=1$ rt. angle and $\angle C=60°$, $\therefore$ $\angle DAC=30°$. $DC=\frac{1}{2}AC=3''$. $\therefore$ $BC=15''-3''=12''$.

Now $AB^2=BD^2+AD^2=BD^2+AC^2-DC^2=(12^2+6^2-3^2)$ sq. inches$=(144+36-9)$ sq. inches$=171$ sq. inches.

$\therefore$ $AB=\sqrt{171}$ inches$=13$ inches (approximately).

Ex. 42. *Bisect a parallelogram by a straight line drawn through a given point.*

Let ABCD be par^m and P any point within or outside it.

To bisect the par^m ABCD by a st. line drawn through P. Join the diagonal BD and take its middle point O. Join PO. Produce PO to cut AD at E and BC at F. Then the st. line EPF bisects the par^m ABCD.

Proof : $\because$ The diagonal BD bisects the par^m,
$\therefore$ $\triangle ABD=\triangle BCD$. Now, in the $\triangle^s$ EOD and BOF,
$\angle EOD=$vert. opp. $\angle BOF$, $\angle EDO=$alt. $\angle OBF$ and $DO=BO$,
$\therefore$ $\triangle EOD\equiv\triangle BOF$. $\therefore$ the quad. $AEOB+\triangle BOF$
$=$the quad. $AEOB+\triangle EOD$,
i.e., quadrilateral $AEFB=\triangle ABD=\frac{1}{2}$par^m ABCD.
$\therefore$ EF bisects the given parallelogram.

Ex. 43. *Describe a triangle equal to the sum of two given triangles.*

To draw a triangle equal to the sum of the two $\triangle^s$ ABC, DEF.

Let $\triangle$ABC > $\triangle$DEF and let AX and DP be their altitudes respectively. Draw a triangle having the altitude AX and equal to the $\triangle$DEF (see Example 20), Let EPQ be that triangle. Produce BC to K so that CK = EP. Join AK. Then $\triangle$ABK is the required triangle.

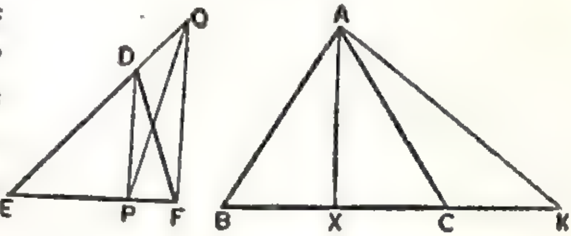


Fig. 56

Proof: In the $\triangle^s$ ACK, EPQ, the base CK = the base EP and AX is their altitude. $\therefore$ $\triangle$ACK = $\triangle$EPQ.

$\therefore$ $\triangle$ABK = $\triangle$ABC + $\triangle$ACK = $\triangle$ABC + $\triangle$EPQ = $\triangle$ABC + $\triangle$DEF.

[N. B. If it is required to draw a triangle equal to the difference of the two given triangles, then from BC cut off BK equal to EP. Join AK. Then ACK will be the required triangle.]

Ex. 44. *Cut off from a given triangle a fourth, fifth or any part required by a straight line drawn from a given point in one of its sides.*

Let P be any point on the side BC of the $\triangle$ABC. It is required to cut off any part (say, *n*th. part) from the $\triangle$ABC by a st. line drawn from P.

Construction: Divide BC into *n* equal parts. Let CE be one of the parts, *i.e.*, *n*th part of BC. Join AP and AE. Draw EF ∥ PA to cut AC at F. Join PF.

Then PF cuts off the *n*th part of the given triangle.

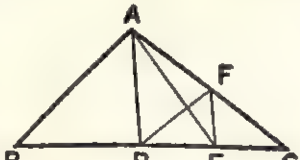


Fig. 57

Proof. EC = *n*th part of the base BC, $\therefore$ $\triangle$AEC = *n*th part of $\triangle$ABC.

Now, $\triangle$EPF = $\triangle$AEF (standing on the same base EF and between the same parallels EF and AP).

$\therefore$ $\triangle$AEC = $\triangle$PFC, $\therefore$ $\triangle$PFC = *n*th part of the $\triangle$ABC.

Ex. 45. *Divide a given straight line into two parts such that the difference between the squares on the two parts may be equal to the square on a given st. line.* [*C. U.* '85]

Let AB be the given st. line and l a side of the given square.

To divide AB into two parts such that the difference of the squares on the parts $= l^2$.

Draw AC⊥AB and let AC $= l$. Join BC. At C draw the ∠BCD equal to the ∠B. Let CD cut AB at D.

Now, AB is divided as required at D.

Proof: ∠B = ∠BCD, ∴ BD = CD. $CD^2 = AD^2 + AC^2$,

∴ $AC^2 = CD^2 - AD^2 = BD^2 - AD^2$,

∴ $l^2 = BD^2 - AD^2$ (∵ l = AC).

Ex. 46. *Describe a right-angled isosceles triangle equal to a given square.*

Let ABCD be a given square. To draw a right-angled isosceles triangle equal to that square.

Produce BA to E so that AE = AB. Join DE and BD. BDE is the required triangle.

Proof: AD is the perpendicular bisector of BE,

∴ ED = BD, ∴ △BED is isosceles. Again, ∵ AE = AD,

∴ ∠E = ∠EDA and ∵ AB = AD, ∴ ∠ABD = ∠ADB

∴ the whole ∠BDE = ∠E + ∠EBD $= \frac{1}{2} \times 2$ rt. angles

= 1 rt. angle.

Again, △BDE = 2△ABD = the square ABCD.

Ex. 47. *Prove that the sum of the perpendiculars from any pt. in the base of an isosceles triangle on the equal sides is equal to the perpendicular from one of the base angles to the opposite side.* [*D. B.* '40]

In the isosceles △ABC let AB = AC and P any point on the base BC. From P draw PQ and PR perpendicular to AB and AC respectively and from B draw BS⊥AC.

To prove that PQ + PR = BS. Join AP.

Proof : The area of a triangle $= \frac{1}{2} \times$ base $\times$ altitude.

$\therefore \quad \triangle APB = \frac{1}{2}AB.PQ = \frac{1}{2}AC.PQ$; $\triangle APC = \frac{1}{2}AC.PR$.

Again, $\triangle ABC = \frac{1}{2}AC.BS$. Now, $\because \triangle APB + \triangle APC = \triangle ABC$,

$\therefore \quad \frac{1}{2}AC.PQ + \frac{1}{2}AC.PR = \frac{1}{2}AC.BS$.

or, $\frac{1}{2}AC(PQ+PR) = \frac{1}{2}AC.BS$, $\quad \therefore \quad PQ+PR=BS$.

Ex. 48. *If any point be taken within an equilateral triangle, the sum of the perpendiculars drawn from it to the sides is equal to the perpendicular from the vertex to the base.*

[**Hints :** $\triangle ABC$ is equilateral. P is any point within it. From P draw perpendiculars on the sides and from A draw a perpendicular on BC. Join AP, BP, CP and prove after the above example 47.]

Exercise 2

1. (*a*) Show that a median of a triangle bisects the triangle. [D. B. '48]

(*b*) If the mid points of the sides of a triangle be joined, it is divided into 4 triangles of equal area.

2. In a quadrilateral ABCD, the diagonal AC bisects the diagonal BD. Prove that AC bisects the quadrilateral. [B. U. '24]

3. In the $\triangle ABC$, R is the middle point of AB and P any pt. in AC. BP is produced to S so that $\triangle RPS$ and $\triangle RCP$ are equal in area. Prove that SC is parallel to AB. [B.U. '32]

4. Find the locus of the vertex of a triangle on a fixed base and of constant area. [See Example 10] [Gau. U. '48]

5. Divide the area of a given square into four parts from which two equal squares can be made up. [C. U. '32]

6. Construct a rhombus equal in area to a given rectangle and having a side equal to a side of the rectangle. (Traces only are required). [C. U. 1933]

7. E is a point on the side of a par^m ABCD. Show that $\triangle EAC + \triangle EBD = \frac{1}{2}$ par^m ABCD. [M. U.]

[**Hints :** $\triangle EAC = \triangle ABE$ (of the same base and altitude).

$\therefore$ $\triangle EAC + \triangle EBD = \triangle ABD = \frac{1}{2}$ par^m ABCD.]

8. Construct a rectangle equal to a given triangle.

9. Construct a triangle equal to a given rectilineal figure. [C. U. 1939]

10. Construct a triangle equal in area to a riven triangle and having one side equal to a given straight line.

11. Show that square on any straight line is four times the square on its half.

12 The sum of the squares on the sides of a rectangle is equal to the sum of the squares on its diagonals.

13. Draw a straight line equal to $\sqrt{2}$ inches.

14. In a right-angled triangle, one of the acute angles is double the other. Show that the square on the greater of the sides containing the right-angle is three times the square on the lesser side.

15. Two triangles of equal area stand on the same base but on opposite sides of it; show that the straight line joining their vertices is bisected by the base. [B. U.]

16. Show that a parallelogram is bisected by any st. line passing through the midde point of one of its diagonals.

17. Shew that a trapezium is bisected by the straight line which joins the middle points of its parallel sides.

18. Draw a $\triangle$ equal to the difference of two given $\triangle$s.

19. Give the construction for drawing a rectangle equal in area to a given rectilineal figure and reducing it to a square of equal area.

[First draw a triangle equal to the given rectilineal figure. Then draw a rectangle equal to that triangle. Then draw a square equal to that rectangle. You will find the method of construction in this book.]

20. Two sides of a triangle are 9 and 12 inches respectively and the angle contained by them is equal to the other two. Find the length of the third side.

(Ans. 15") [C. U. '78]

21. ABC is an isosceles triangle of which B is the vertex; BA, BC are bisected in D, E respectively; AE, CD intersect at F. Show that $\triangle BDE = 3\triangle DEF$. [C. U. 1866]

22. Describe a right-angled isosecles triangle equal in area to a given parallelogram. [D. B. '38]

23. If ABCD is a parallelogram and X, Y are points in DC and AD respectively, show that $\triangle^s$ ABX, BYC are equal in area. [C. U. '47]

24. Draw a parallelogram ABCD with its sides AB, AD equal to two given straight lines and its area equal to that of a given rectangle. [C.U. '49; See Ex. 23]

25. If two triangles have the same altitude, but unequal bases, that which stands on the greater base has the greater area. [C.U. '12]

[Let $\triangle ABC$ and $\triangle ADE$ have the same height h; but the base $BC > DE$. To prove that $\triangle ABC > \triangle ADE$.

Proof: $\triangle ABC = \frac{1}{2}BC.h$, $\triangle ADE = \frac{1}{2}DE.h$; but $BC > DE$. $\therefore$ $\frac{1}{2}BC.h > \frac{1}{2}DE.h$, *i.e.*, $\triangle ABC > \triangle ADE$.]

26. In $\triangle ABC$, P is any pt. on AB. Through P a st. line PQR is drawn equal and parallel to BC, cutting AC at Q. Prove that $\triangle AQR$ and $\triangle BPQ$ are equal in area. [B.U. '22]

[**Hints**: Join AR, PC, BQ, RC. $\because$ BC and PR are equal and parallel, $\therefore$ $PB \parallel RC$. $\triangle PBQ = \triangle PQC$ ($\because$ they stand on the base PQ and between the parallel st. lines PQ and BC).

Similarly, $\triangle APC = \triangle APR$ ($\because AP \parallel RC$)

$\therefore$ $\triangle PQC = \triangle AQR$ (subtracting the common part $\triangle APQ$ from both) $\therefore$ $\triangle BPQ = \triangle PQC = \triangle AQR$.]

27. O is a point within the triangle ABC. If OX, OY, OZ are perpendiculars drawn to BC, CA, AB respectively, show that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$. [C. U.]

[**Hints**: Join AO, BO, CO. $AZ^2 = AO^2 - ZO^2$, $BX^2 = BO^2 - OX^2$, $CY^2 = CO^2 - YO^2$. $\therefore$ $AZ^2 + BX^2 + CY^2 = (AO^2 - YO^2) + (CO^2 - OX^2) + (BO^2 - OZ^2) = AY^2 + CX^2 + BZ^2$.]

28. ABCD is a parallelogram ; E and F are the middle pts. of BC and CD. Prove that $\triangle AEF = \frac{3}{8}$ of the par^m. [B. U.]

[**Hints :** [Draw figure] $\triangle EFC = \frac{1}{4}\triangle BCD = \frac{1}{8}$par^m ABCD ; $\triangle ADF = \frac{1}{2}\triangle ADC = \frac{1}{4}$par^m.ABCD ; $\triangle ABE = \frac{1}{4}$par^m.ABCD.

$\therefore$ $\triangle AEF =$ par^m.ABCD $- \triangle EFC - \triangle ADF - \triangle ABE$

$= \frac{3}{8}$par^m.ABCD.]

29. If the diagonals of a quadrilateral PQRS are at right angles, prove that $PQ^2 + RS^2 = PS^2 + QR^2$. [G. U. '53]

30. ABCD is a parallelogram and the st. line EF is drawn parallel to AC cutting AD and CD at E and F respectively. Prove that $\triangle ABE = \triangle BCF$.

Circle

15. Theorem—(*i*) If a st. line drawn from the centre of a circle bisects a chord, it cuts the chord at right angles. (*ii*) And its converse.

Ex. 1. *The line of centres of two circles bisects their common chord at right angles.* [*C. U.* '50]

Let P and Q be the centres of two circles and AB their common chord.

To prove that PQ bisects the chord AB at right angles. Join AP, AQ, BP, BQ. Join PQ cutting AB at O.

Proof : In the $\triangle^s$ APQ, BPQ, AP = BP, AQ = BQ (being radii) and PQ is common to both,
$\therefore$ $\angle APQ = \angle BPQ$.

Again, in the $\triangle^s$ APO and BPO, AP = PB, PO is common to both and $\angle APO = \angle BPO$.

$\therefore$ AO = BO and $\angle AOP = \angle BOP$, these being adjacent angles, each is a right angle. $\therefore$ PQ bisects AB at right angles.

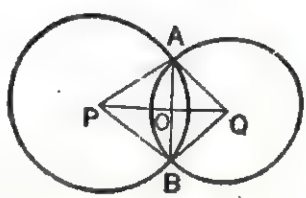


Fig. 58

Ex. 2. *Show that two chords of a circle cannot bisect each other unless both of them pass through the centre.* [*C. U.* '18]

If possible, let the two chords AB and CD bisect each other at X and not pass through the centre.

Join OX. $\because$ OX bisects the chord AB, $\therefore$ $OX \perp AB$; $\therefore$ $\angle OXB = 1$ rt. angle. Similarly, $\angle OXD = 1$ rt. angle.

$\therefore$ $\angle OXB = \angle OXD$, but this is impossible (The whole angle cannot be equal to its part).

It will be possible only when X and O are the same point, *i.e.*, when AB and CD pass through the centre O.

Ex. 3. *Draw a circle of given radius that shall pass through two given points. When is this problem impossible?*

Let A and B be the given points and r the given radius. Join AB. With centres A and B and with radius r draw two arcs of a circle intersecting at C.

Now, the circle drawn with centre C and radius CA will be the required circle.

Proof: $\because$ $CA = CB = r$, $\therefore$ the circle drawn with centre C and with radius r will pass through A and B.

The construction will fail when r is less than $\frac{1}{2}AB$.

Ex. 4. *Prove that a diameter is the greatest chord of a circle.*

Let O be the centre of the circle and PQ one of its chords. Draw $OM \perp PQ$. Join PO. $\because$ $OM \perp PQ$, $\therefore$ $PQ = 2PM$.

$\because$ $\angle M = 1$ rt. angle, $\therefore$ the hypotenuse $PO > PM$.

$\therefore$ $2PO > 2PM$. $\because$ $2PO =$ diameter,

$\therefore$ the diameter $>$ the chord PQ and this is true for any chord. $\therefore$ a diameter is the greatest chord of a circle.

Ex. 5. *Two circles intersect at* A *and* B *and a straight line is drawn parallel to* AB *cutting the circles at* C, D, E, F. *Show that* $CD = EF$.

Let the two circles, whose centres are P and Q, intersect at A and B. Draw $CF \parallel AB$ intersecting the circles at C, D, E and F.

To prove that CD = EF. Let PQ cut AB and CF at R and O respectively.

Proof : PQ the line of centres intersects the common chord AB at R, ∴ $PR \perp AB$.

∵ CF ∥ AB, ∴ $PR \perp CF$.

Now, PO is perpendicular to the chord CF, ∴ CO = FO.

And ∵ QO is perp. to the chord DE, ∴ DO = EO.

∴ CO − DO = FO − EO, ∴ CD = EF.

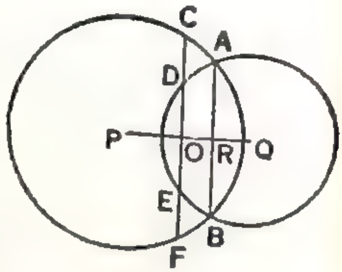


Fig. 59

Ex. 6. *Prove that two different circles cannot cut each other at more than two points.* [*C. U. 1933* ; *W. B. S. F. '52*]

One and only one circle can be drawn through three points not in the same st. line, ∴ if the two circles intersect at more than two points *i.e.*, at three points, then they will completely coincide, *i.e.*, become one circle on passing through the same three points. That is to say that their centres and radii would be the same.

∴ two different circles cannot cut each other at more than two points.

16. Theorem—(*i*) Equal chords of a circle are equidistant from the centre. (*ii*) And its converse.

Ex. 7. *Parallel chords of a circle at the extremities of a diameter are equal.*

Let O be the centre and AB a diameter of the given circle AC and BD are two parallel chords.

To prove that AC = BD. Draw $OE \perp AC$ and $OF \perp BD$.

Proof : In the Δ^s OAE and OBF, ∠E = ∠F (rt. angles), ∠A = alt. ∠B and AO = BO (radii) ∴ the two triangles are congruent, ∴ OE = OF.

∴ AC = BD (being equidistnnt from the centre.)

Ex. 8. AB *and* AC *are two equal chords of a circle, show that the bisector of the* ∠BAC *passes through the centre.* [*C.U. 1926*]

Proof : ∵ The chords AB and AC are equal, ∴ the centre is equidistant from AB and AC. Now, ∵ the bisector

of the $\angle BAC$ is the locus of the point equidistant from two intersecting st. lines AB and AC, $\therefore$ the centre of the circle will lie on the bisector of the angle $\angle BAC$, *i.e.*, the bisector will pass through the centre.

Ex. 9. *Find the locus of the middle points of equal chords of a circle.* [*C. U.* 1913, '21, '33 ; *D. B.* 1935]

Let O be the centre and AB, CD, etc. be several equal chords of a circle. To find the locus of the middle points of these equal chords.

Let P, Q, etc. be the middle points of the chords AB, CD, etc. Join O to each of these middle points. Now OP, OQ, etc. will be perpendicular to AB, CD, etc.

$\because$ the chords are equal, $\therefore$ they are equidistant from the centre.

$\because$ the distance of the middle points of the equal chords from the centre is equal to OP,

$\therefore$ the required locus is a circle drawn with centre O and radius OP.

Ex. 10. *Three equal chords of a circle cannot intersect at a point within it, unless that point is the centre.*

The chords which pass through the centre are called diameters. All diameters are equal and they intersect one another at the centre.

Now, let two equal chords AB and CD, which do not pass through the centre O, intersect at P.

If possible, let EF be another chord equal to AB or CD and passing through P. Draw OK, OL, OR perp. to AB, CD, EF respectively. Join OP.

In the right-angled Δ^s OKP, OLP, OK = OL, OP is common to both, $\therefore$ they are equal in all respects, $\therefore$ $\angle OPK = \angle OPL$.

Similarly, $\angle OPK = \angle OPR$.

$\therefore$ $\angle OPL = \angle OPR$, but this is absurd as the whole angle OPL cannot be equal to its part. $\therefore$ three equal chords of a circle cannot intersect at a point within it, unless that point is the centre.

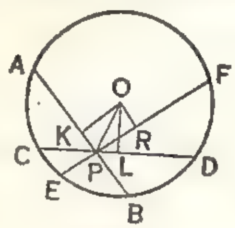


Fig. 60

Ex. 11. *If two equal chords intersect, show that the segments of the one are equal respectively to the segments of the other.* [*C. U. 1935*]

[See the previous diagram.] Let O be the centre and let the equal chords AB and CD intersect at P. To prove that $AP = PD$ and $BP = CP$. Draw $OK \perp AB$ and $OL \perp CD$. Join OP.

Proof : $\because$ $OK \perp AB$, $\therefore$ $AK = \frac{1}{2}AB$.

Similarly, $DL = \frac{1}{2}CD$. Now, $\because$ $AB = CD$, $\therefore$ $AK = DL$.

Again, in the right-angled $\triangle^s$ OKP, OPL, $OK = OL$ ($\because$ the equal chords AB and CD are equidistant from the centre O), OL is their common hypotenuse, $\therefore$ $PK = PL$,

$\therefore$ $AK + PK = DL + PL$, *i.e.*, $AP = DP$.

Again, $AB - AP = CD - DP$, *i.e.*, $BP = CP$.

Ex. 12. PQ *is a fixed chord of a circle, and* AB *is any diameter. Show that the sum of the perpendiculars let fall from* A *and* B *on* PQ *is constant, if* AB *does not intersect* PQ *inside the circle.* [*C. U. 1937, '39 Sup.*]

Let AB be any diameter and PQ be a fixed chord. Let AK and BR be perpendicular to the chord PQ.

To prove that $AK + BR$ is constant. Join BK and draw $OD \perp PQ$ to cut BK at E.

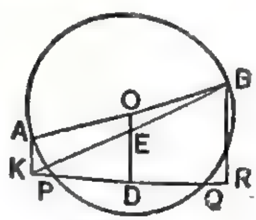


Fig. 61

Proof : AK, OD, BR being perp. to the same straight line are parallel to one another. $\because$ $OD \perp PQ$, $\therefore$ D is the middle point of PQ. $\because$ the centre O and PQ are fixed, $\therefore$ OD is of constant length. In the $\triangle ABK$, OE is drawn parallel to AK from the middle point O of the side AB, $\therefore$ E is the middle point of BK and $AK = 2OE$.

Again, ED is parallel to BR from the middle point E of the side KB of the $\triangle KBR$, $\therefore$ $BR = 2DE$.

$\therefore$ $AK + BR = 2(EO + DE) = 2DO$ = a constant length.

Ex. 13. *Through a given point within a circle draw the least possible chord.* C. U. '35, '42]

Let O be the centre and X be a fixed point within the given circle. Join OX. Draw the chord AB perp. to OX through X. Then AB will be the required least possible chord.

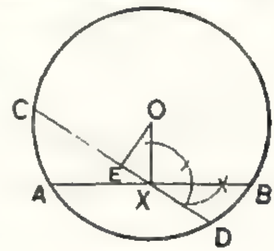


Fig. 62

Proof : Draw any other chord CD through X and from O draw OE⊥CD. In the △OXE, ∠E=1 rt. angle, ∴ the hypotenuse OX>OE,

∴ AB is more remote than CD from the centre O ;

∴ AB<CD. ∴ AB is the least possible chord.

Ex. 14. *In a circle draw a chord equal to twice its distance from the centre.*

Let O be the centre and OA any radius of the circle. Through A draw a chord AB so that ∠OAB=45°. Then AB is the required chord.

Proof : Draw OP⊥AB, then AB=2AP.

∵ ∠OPA=1 rt. angle and ∠A=45°,

∴ ∠AOP=45°=∠A, ∴ AP=PO.

∴ AB=2AP=2PO (PO being the distance of AB from the centre O.)

Ex. 15. *Through a given point in a circle draw two equal chords at right angles to each other.*

Let O be the centre and P be the given point.

Construction : Join OP. At P draw ∠OPA and ∠OPD=45° on both sides of OP. Let AP and DP intersect the circumference at A and B and D and C respectively. AB and CD will be the required chords.

Proof : Draw OM and ON perp. to AB and CD. In the △ˢ OMP, ONP, ∠OMP=∠ONP(right angles), ∠OPM=∠OPN and OP is common to both, ∴ OM=ON.

∴ AB=CD and they are at right angles to each other.

17. Theorem—If the line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

Ex. 16. *L is any point on the arc PM of a circle. The angles LPM and LMP are bisected by straight lines which intersect at O. Find the locus of the pt. O.* [*C. U.* '34, '42]

In the $\triangle OPM$, $\angle O + \angle OPM + \angle OMP = 2$ rt. angles $= 180°$.
$\therefore \quad \angle O + \frac{1}{2}\angle P + \frac{1}{2}\angle M = 180° \cdots (1)$. Again, in the $\triangle LMP$, $\angle L + \angle P + \angle M = 180°$,

$\therefore \quad \frac{1}{2}\angle L + \frac{1}{2}\angle P + \frac{1}{2}\angle M = 90° \cdots (2)$

Subtracting (2) from (1),
$\angle O - \frac{1}{2}\angle L = 90°$, $\quad \therefore \quad \angle O = 90° + \frac{1}{2}\angle L$.
$\angle L$ is always constant wherever L may be on the arc PM. $\therefore$ $\angle O$ is also constant. $\therefore$ the arc POM of the segment of the circle on the chord PM containing an angle $90° + \frac{1}{2}\angle L$ is the required locus of the point O.

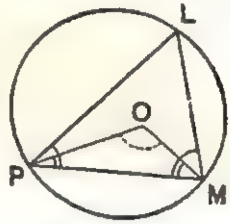


Fig. 63

18. Theorem—The angle at the centre of a circle is double of the angle at the circumference standing on the same arc.

Ex. 17. *The chords AB and CD of a circle, whose centre is O, intersect at P. Show that $\angle AOC + \angle BOD = 2\angle APC$.*
[*C. U.* '38, *W. B. S. F.* '53]

Proof : Join BC. $\angle AOC$ at the centre $= 2\angle ABC$ at the circumference standing on the same arc AC.

Similarly, $\angle BOD = 2\angle BCD$.
$\therefore \angle AOC + \angle BOD = 2(\angle ABC + \angle BCD)$
$= 2(\angle PBC + \angle BCP) = 2\angle APC$
($\because$ in the $\triangle BCP$, the ext. $\angle P$ $= \angle PCB + \angle PBC$).

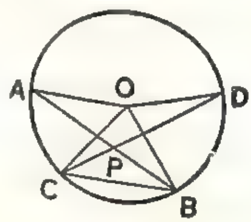


Fig. 64

Ex. 18. *Parallel chords of a circle intercept equal arcs.*

Let AB and CD be two parallel chords. To prove that the arc AC = the arc BD. Join BC.

Proof : $\angle$ABC = alt. $\angle$BCD and they are angles at the circumference.

$\therefore$ they stand on equal arcs. $\therefore$ the arc AC = the arc BD.

Ex. 19. *If two chords of a circle intersect inside the circle, the angle between them is equal to the angle at the circumference subtended by an arc equal to the sum of the intercepted arcs.*

Let the chords AB and CD intersect at E. To prove that $\angle$AEC = the angle at the circumference subtended by an arc equal to the sum of the arcs AC and BD.

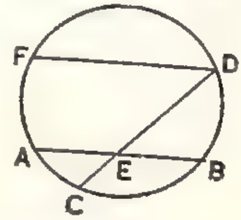


Fig. 65

Proof : Draw DF $\parallel$ AB. $\because$ AB $\parallel$ FD, $\therefore$ arc AF = arc BD, $\therefore$ arc AC + arc BD = arc AC + arc AF = arc FC.

Now, $\angle$AEC = corresponding $\angle$D = the angle at the circumference subtended by the arc FC.

[N. B. If AB and CD intersect at E outside the circle, prove that $\angle$AEC = the angle at the circumference subtended by an arc equal to the difference of the two arcs AC and BD.]

19. Theorem—In equal circles (or in the same circle) (*i*) if arcs subtend equal angles either at the centres or at the circumferences they are equal. (*ii*) Its converse.

Ex. 20. *Prove that the bisectors of the angles in the same segment of a circle pass through a common point.* [*C.U.'14,'51*]

Let APB be any angle in the segment APB of a circle and let its bisector PQ cut the circle at Q.

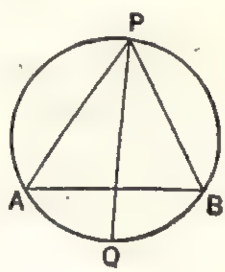


Fig. 66

$\because$ $\angle$APQ = $\angle$BPQ, $\therefore$ arc AQ = arc BQ. $\therefore$ Q is the middle point of the conjugate arc AB. $\therefore$ the bisector of $\angle$P will pass through the middle point of the arc AQB. It is true for any position of the $\angle$P in that segment of the circle. Again, the arc AB being fixed, its middle point Q is a fixed point. $\therefore$ the bisector of any

angle in the segment APB of the circle will pass through the fixed point Q.

Ex. 21. *Two circles cut at* P *and* Q *and any chord* APB *is drawn through* P *and terminated by the circles. Show that* $\angle$AQB *is constant.*

Join AQ and BQ.

Proof : $\because$ the arcs PQ of both circles are fixed. $\therefore$ the $\angle$A and the $\angle$B at the circumference on them are constant.

$\therefore$ $\angle A + \angle B =$ a constant.

$\because$ the sum of the three $\angle s$ of a triangle $= 180°$ (constant),

$\therefore$ the remaining $\angle Q$ of the $\triangle ABQ$ will also be constant.

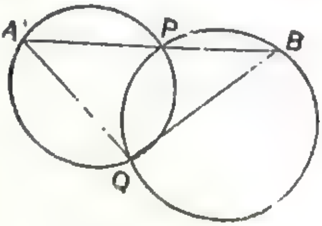


Fig. 67

Ex. 22. *A triangle* ABC *is inscribed in a circle and the bisectors of the angles meet the circumference at* X, Y, Z. *Show that the angles of the* $\triangle$XYZ *are respectively* $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$. [*C. U.* '39 *Sup.*]

Let ABC be a triangle inscribed in a circle and let the bisectors AX, BY, CZ of the $\angle A$, $\angle B$, $\angle C$ intersect the circle at X, Y, Z. Join XY, YZ, ZX.

To prove that $\angle X = 90° - \frac{1}{2}\angle A$,

$\angle Y = 90° - \frac{1}{2}\angle B$, $\angle Z = 90° - \frac{1}{2}\angle C$.

Proof : $\angle AXY = \angle ABY$ standing on the same arc AY $= \frac{1}{2}\angle B$. Standing on the same arc AZ, $\angle AXZ = \angle ACZ = \frac{1}{2}\angle C$.

$\therefore$ The whole $\angle X = \frac{1}{2}\angle B + \frac{1}{2}\angle C$.

In the $\triangle ABC$, $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90°$.

$\therefore$ $\frac{1}{2}\angle B + \frac{1}{2}\angle C = 90° - \frac{1}{2}\angle A$.

$\therefore$ $\angle X = 90° - \frac{1}{2}\angle A$.

Similarly, it can be proved that $\angle Y = 90° - \frac{1}{2}\angle B$ and $\angle Z = 90° - \frac{1}{2}\angle C$.

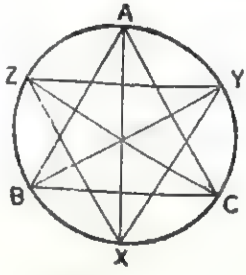


Fig. 68

Ex. 23. *Triangle* ABC *is inscribed in a circle, and the bisectors of the angles meet the circumference at* X, Y, Z. *Prove that* AX *is perpendicular to* YZ. [*B. U. 1920*]

(Draw the previous diagram). Let AX cut YZ at P.

$\because \quad \angle AXY + \angle Y = \angle AXY + \angle BYX + \angle BYZ$

$= \angle ABY + \angle BAX + \angle BCZ = \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 1$ rt. angle,

i. e., in the $\triangle PYX$, $\angle YXP + \angle PYX = 1$ rt. angle,

$\therefore \quad \angle P = 1$ rt. angle. $\quad \therefore \quad AX \perp YZ$.

Ex. 24. AB *is a fixed chord of a circle and* APB *is a triangle inscribed in it. Perpendiculars are drawn from* A *and* B *to* PB *and* PA *respectively intersecting at* O. *Find the locus of* O.

Let AB be a fixed chord of a circle and APB be a triangle inscribed in it. Let AC and BD be prep. to PB and PA respectively cutting each other at O.

To find the locus of the point O.

$\angle D$ and $\angle C$ of the quadrilateral PCOD are each a rt. angle.

$\therefore \quad \angle P + \angle DOC = 2$ rt. angles $= 180°$.

$\therefore \quad \angle DOC = 180° - \angle P$,

But $\angle P$ being an angle in the fixed segment of a circle is always equal or constant.

$\therefore \quad \angle AOB = \angle DOC = 180° - \angle P =$ a constant angle.

$\therefore$ the arc AOB of the circle passing through A, O, B is the locus of the point O.

Ex. 25. ABC *is an equilateral triangle inscribed in a circle. If* P *be any point on the arc* BC *opposite to* A, *show that* AP = BP + CP. [*C. U. 1939*]

Let ABC be an equilateral triangle inscribed in a circle. Let P be any point on the arc BC. To prove that AP = PB + PC.

From AP cut off AO equal to PC. Join BO, BP and CP.

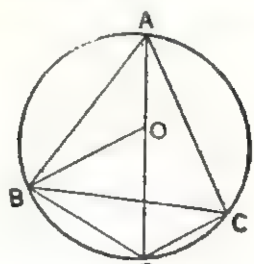


Fig. 69

Proof: In the $\triangle^s$ ABO, BPC, AO = PC, AB = BC and $\angle BAO = \angle BCP$ (angles at the circumference standing on the same arc),

$\therefore$ BO = BP. $\therefore$ $\angle BOP = \angle BPO = \angle ACB$ (angles at the circumference standing on the same arc.) $= 60°$. $\therefore$ $\angle OBP = 60° = \angle BOP$

$\therefore \quad BP = PO. \quad \therefore AP = BP + PC.$

Ex. 26 *If a circle can be described about a parallelogram, the parallelogram must be a rectangle.* [C. U. 1915, 20]

The parallelogram being cyclic, the sum of its two opposite angles is two right angles. Again, the opposite angles of a parallelogram are equal. $\therefore$ each angle of a cyclic parallelogram is a right angle.

$\therefore$ the cyclic parallelogram must be a rectangle.

20. Theorem—(*i*) The opposite angles of a cyclic quadrilateral are supplementary. (ii) Its converse theorem.

Ex. 27, ABC *is an isosceles triangle and* XY *is drawn parallel to the base* BC *cutting the sides in* X *and* Y ; *show that* B, C, X, Y *are concyclic.* [C. U. '48. Sup. ; A. U. 1931]

$\because$ XY $\parallel$ BC, $\therefore$ $\angle$AXY = corresponding $\angle$B = $\angle$C
($\angle$B = $\angle$C in the isosceles $\triangle$).

$\because$ $\angle$AXY + $\angle$BXY = 2 rt. angles, $\therefore$ $\angle$BXY + $\angle$C = 2 rt. angles, $\therefore$ BXYC is a cyclic quadrilateral.

$\therefore$ B, C, X. Y are concyclic.

Ex. 28. *A triangle is inscribed in a circle ; show that the angles in the three segments exterior to the triangle are together equal to four right-angles.*

Let $\triangle$ABC be inscribed in a circle. To prove that the sum of the three angles in the segments APB, AQC and BRC exterior to the triangle = 4 rt. angles.

Join AP, BP, AQ, CQ, BR, CR.

Proof : $\because$ APBC is a cyclic quadrilateral

$\therefore$ $\angle$P + $\angle$ACB = 2 rt. angles.

Similarly, $\angle$Q + $\angle$ABC = 2rt. angles and $\angle$R + $\angle$BAC = 2 rt. angles.

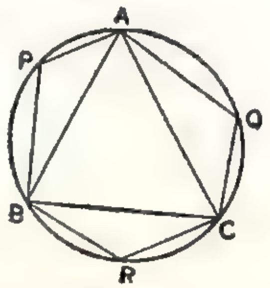


Fig. 70

Adding the results we have $\angle$P + $\angle$Q + $\angle$R + $\angle$ABC + $\angle$BCA + $\angle$CAB = 6 rt. angles.

But $\angle$ABC + $\angle$BCA + $\angle$CAB of the $\triangle$ABC = 2 rt. angles.

$\therefore$ $\angle$P + $\angle$Q + $\angle$R = 6 rt. angles − 2 rt. angles = 4 rt. angles

Ex. 29. *Prove that the internal bisector of any angle of a cyclic quadrilateral and the external bisector of the opposite angle intersect on the circle.* [*C. U. 1924 ; D. B. '36*]

Let ABCD be a cyclic quadrilateral. To prove that the internal bisector of $\angle$B and the external bisector of the opposite $\angle$D will intersect at a point on the circle. Let the internal bisector BE of $\angle$B intersect the circumference at E. Join ED and produce CD to F.

Proof: $\because$ EBCD is a cyclic quadrilateral,

$\therefore$ $\angle$EDF = interior opposite $\angle$EBC.

Again, $\angle$EDA = $\angle$ABE (angles at the O^{ce} on the same arc AE).

$\therefore$ $\angle$ABE = $\angle$CBE (by hypothesis),

$\therefore$ $\angle$ADE = $\angle$EDF, $\therefore$ ED is the bisector of the $\angle$ADF, *i.e.*, ED is the external bisector of $\angle$ADC.

$\therefore$ the internal bisector of $\angle$B and the external bisector of the opposite $\angle$D intersect at E on the circle.

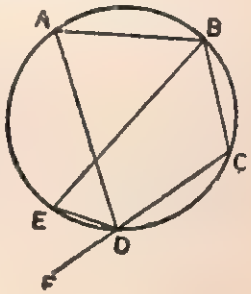


Fig. 71

Ex. 30. *Prove that the internal bisectors of the angles of any quadrilateral form a cyclic quadrilateral.* [*C. U. 1925*]

Let the internal bisectors of the angles of the quadrilateral ABCD form the quadrilateral OPQR.

To prove that OPQR is a cyclic quadrilateral.

Proof: In the $\triangle$OBC, $\angle O + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 2$ rt. angles.

In the $\triangle$QAD, $\angle Q + \frac{1}{2}\angle A + \frac{1}{2}\angle D = 2$ rt. angles. $\therefore$ $\angle O + \angle Q + \frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 4$ rt. angles.

But $\angle A + \angle B + \angle C + \angle D = 4$ rt. angles. $\therefore \frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 2$ rt. angles.

$\therefore$ $\angle O + \angle Q = 2$ rt. angles.

$\therefore$ OPQR is a cyclic quadrilateral.

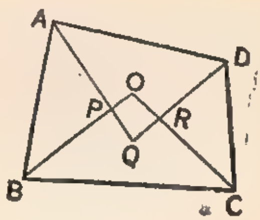


Fig. 72

Ex. 31. *Find the locus of the middle points of all chords of a circle which pass through a fixed point.*

Let O be the centre of the circle and P be any fixed pt. within it. So OP is a fixed st. line. To find the locus of the middle points of all chords which pass through P. Draw any chord AB through P and let R be its middle point. Join OP and OR. $\because$ OR is the line joining the centre and the middle point of the chord, $\therefore$ OR$\perp$AB.

$\therefore$ OP subtends a rt. angle at R and it is true for every chord passing through P. $\therefore$ if a circle be drawn with OP as diameter, it will pass through each similar rt. angles, *i.e.*, through the middle pts. of the chords.

$\therefore$ This circle is the required locus.

If P be outside the circle, the arc of that circle which is within the given circle will be the required locus.

Ex. 32. *Through each of the points of intersection of two circles straight lines are drawn, cutting one circle at* A *and* B *and the other at* C *and* D. *Prove that* AB *is parallel to* CD. [*C.U. 1911*]

Let two circles intersect at P and R.

Let the lines AC and BD drawn through P and R intersect the two circles at A, C and B, D respectively. Prove that AB ∥ CD.

Proof: Join AB, PR, CD.

$\because$ ABRP is a cyclic quadrilateral,

$\therefore$ $\angle$A+$\angle$PRB=2 rt. angles.

Again, the external $\angle$PRB of the cyclic quadrilateral PRDC=the interior opposite $\angle$C.

$\therefore$ $\angle$A+$\angle$C=2 rt. angles.

$\therefore$ AB ∥ CD.

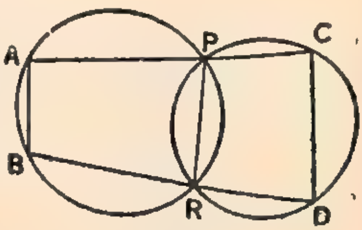


Fig. 73

Ex. 33. *Find a point within a triangle so that the angles subtended at it by the sides of the triangle may be equal.*

To find a pt. within the $\triangle$ABC so that the angles subtended at it by AB, AC and BC are equal.

Draw two equilateral triangles ABE and ACD on AB and AC.

Draw two circumcircles of the $\triangle$s ABE, ACD. Let them intersect at O within the $\triangle$ABC. Then O is the required point.

Proof : Join AO, BO, CO.

$\because$ ADCO is a cyclic quadrilateral,

$\therefore$ $\angle AOC + \angle D = 180°$.

But $\angle D = 60°$ (being an angle of an equilateral triangle) ;

$\therefore$ $\angle AOC = 120°$.

Similarly $\angle AOB = 120°$.

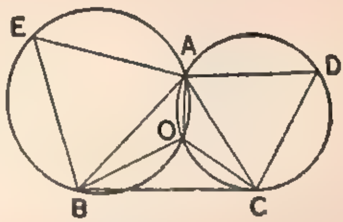


Fig. 74

$\because$ the sum of the three angles at $O = 4$ rt. angles $= 360°$, $\therefore$ the remaining $\angle BOC = 120°$.

$\therefore$ AB, BC and CA subtend equal angles at O.

Ex. 34. *ABCD is a quadrilateral in which a pair of opposite angles are supplementary. If AC bisects the $\angle BAD$, show that $BC = CD$.* [*D. B. 1930*]

Let ABCD be a quadrilateral in which $\angle A + \angle C = 2$ rt. angles and the diagonal AC bisects $\angle BAD$.

To prove that $BC = CD$.

Proof : $\because$ $\angle A + \angle C = 2$ rt. angles,

$\therefore$ the quadrilateral is cyclic.

$\because$ $\angle BAC = \angle DAC$, $\therefore$ arc $BC =$ arc CD.

Again, $\because$ arc $BC =$ arc CD, $\therefore$ chord $BC =$ chord CD.

21. Theorem—The angle in a semi-circle is a right angle.

Ex. 35. *A variable st. line passes through a fixed pt. Find the locus of the foot of the perpendicular drawn to it from another fixed point.*

Let P and Q be two fixed points and let PR be any st. line drawn through P. From Q draw $QO \perp PR$. To find the locus of O. Join QP. P and Q being fixed pts., QP is a fixed st. line and QP subtends a rt. angle at O. It is true for all positions of PR. $\therefore$ the circle drawn on QP as diameter is the required locus.

Ex. 36. *Circles described on any two sides of a triangle as diameters intersect on the third side.*

To prove that circles described on any two sides of the $\triangle ABC$ as diameters intersect on the third side.

Let the circle drawn on AB as diameter intersect BC or BC produced at D. Join AD. $\angle ADB$ being in a semicircle is a right angle. $\therefore$ $\angle ADC = 1$ rt. angle.

$\therefore$ the circle drawn on AC as diameter will pass through the right-angular point D. $\therefore$ the two circles drawn on AB and AC as diameters will intersect each other at D on the third side BC.

Ex. 37. *The internal and external bisectors of the vertical angle of a triangle meet the circum-circle in* P *and* Q. *Show that* PQ *is a diameter of the circle.*

Let the $\triangle ABC$ be inscribed in a circle and let AP and AQ be respectively the internal and external bisectors of the $\angle A$. To prove that PQ is a diameter.

Proof: $\because$ AP and AQ are respectively the internal and external bisectors of the $\angle A$, $\therefore$ $\angle PAQ = 1$ rt. angle and $\therefore$ it must be an angle in a semi-çircle. $\therefore$ PQ is a diameter.

Ex. 38. *If the bisectors of any two opposite angles of a cyclic quadrilateral cut the circle, the straight line joining the points of intersection is a diameter of the circle.*

Let ABCD be a cyclic quadrilateral. Let the bisector BP of the $\angle B$ and the bisector DQ of the $\angle D$ intersect the circle at P and Q. To prove that PQ is a diameter.

Join BQ and PQ.

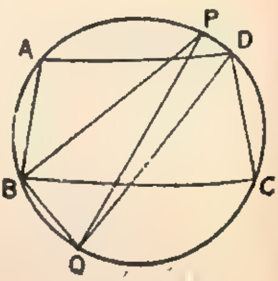


Fig. 75

Proof: $\angle QBC = \angle QDC$ (angles at the circumference on the same arc) $= \frac{1}{2}\angle ADC$ and $\angle PBC = \frac{1}{2}\angle ABC$ (hyp.) $\therefore$ the whole $\angle PBQ = \frac{1}{2}(\angle ADC + \angle ABC)$ $= \frac{1}{2} \times 2$ rt. angles ($\because$ ABCD is a cyclic quadrilateral) $= 1$ rt. angle. $\angle PBQ$ being one rt. angle, it is an angle in a semicircle. $\therefore$ PQ is a diameter of the circle.

Ex. 39. *If two circles cut one another at* A *and* B *and from* A *diameters* AP *and* AQ *are drawn, one in each circle, show that* P, B *and* Q *are collinear.*

Let the two circles cut one another at A and B. AP and AQ are two diameters of the two circles. To prove that P, B and Q are in one st. line. Join PB, QB and AB.

Proof : $\because$ AP is a diameter, $\therefore$ $\angle ABP = 1$ rt. angle.
Similarly, $\angle ABQ = 1$ right angle.
$\therefore$ $\angle ABP + \angle ABQ = 2$ rt. angles,
$\therefore$ PB and QB are in the same st. line.
$\therefore$ P, B and Q are collinear.

Ex. 40. *Two equal circles intersect at* A *and* B, *and through* A *any straight line* PAQ *is drawn terminated by the circumferences. Show that* $BP = BQ$.

Let the two equal circles intersect at A and B and let a st. line PAQ cut the circles at P and Q.

To prove that $BP = BQ$.

Proof : Join PB and BQ. $\because$ the circles are equal,
$\therefore$ the two arcs (AB) cut off by them are equal,
$\therefore$ $\angle P = \angle Q$, $\therefore$ $BP = BQ$.

Ex. 41. *The sum of the arcs, intercepted by any two chords of a circle perpendicular to each other, is equal to half the circumference.*

Let the two chords AB and CD be perpendicular to each other and intersect at P.

To prove that the arc AC + the arc BD = half circumference.

Through D draw DE parallel to AB to cut the circle at E,

Proof : $\because$ the chords DE and AB are parallel,
$\therefore$ the arc DB = the arc AE.
$\therefore$ the arc AC + the arc BD = the arc CAE.
Again, $\angle D$ = corresponding $\angle APC = 1$ right angle.
$\therefore$ $\angle D$ is in a semi-circle.
$\therefore$ the arc CAE = half the circumference, *i.e.*,
the arc AC + the arc BD = half the circumference.

22. Theorem—The tangent at any point of a circle is perpendicular to the radius through the point.

Ex. 42. *All chords of a circle which touch a concentric circle are equal and are bisected at the point of contact.*

Let O be the centre of two concentric circles ABC and PQR and let any two chords AB and CD of the circle ABC touch the circle PQR at P and Q.

To prove that AB = CD and AB and CD are bisected at P and Q respectively.

Proof: Join OP and OQ. ∵ OP is a radius drawn through the point of contact of the tangent AB, ∴ OP ⊥ AB.

Similarly, OQ ⊥ CD. Again, OP = OQ (radii of the same circle),

∴ AB and CD are equidistant from the centre O, ∴ AB = CD.

Again, OP and OQ drawn from the centre O are perp. to chords AB and CD,

∴ AB and CD are bisected at P and Q.

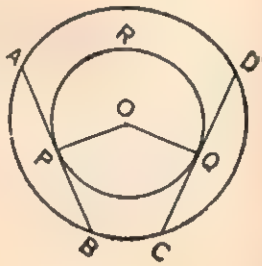


Fig. 76

Ex. 43. *Find the locus of a point from which the tangents drawn to a given circle are of given length.*

[*C. U.* '22, '39 ; *G. U.* '49]

Let l be the given length and O be the centre of the given circle of which the radius is r.

Let T be a point so that the tangents drawn from T to the circle are equal to l. To find the locus of the point T.

Construction: Take any radius OP and draw PT ⊥ OP at P. Let PT be equal to l. The circle drawn with centre O and radius OT is the required locus.

Proof: Join OT. ∵ PT is perp. to the radius OP at P,

∴ PT is a tangent to the circle.

Now, $OT^2 = OP^2 + PT^2$

(∵ $\angle P = 1$ rt. angle) $= r^2 + l^2$,

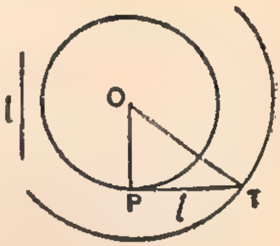


Fig. 77

$\therefore$ OT $= \sqrt{r^2+l^2}$ = a constant quantity.

$\therefore$ the point T is always equidistant from the centre O.
$\therefore$ the circle drawn with centre O and with radius OT (*i. e.*, $\sqrt{r^2+l^2}$ is the required locus.

Ex. 44. *The angle contained by the segment* ABC *of a circle is half of a right angle. Prove that the tangent at* A *and* C *are at right angles to one another.* [*A. U. 1934*]

Let the $\angle$ABC in the segment ABC of the circle be half of a right angle. Let O be the centre and let the tangents AP and CP at A and C intersect at P.

To prove that $\angle$P = 1 right angle.

Proof: Join AO and CO. The $\angle$AOC at the centre $= 2\angle$B at the circumference standing on the same arc AC = 1 rt. angle.

$\because$ OA and OC are radii passing through the points of contact, $\therefore$ $\angle$OAP and $\angle$OCP are each a rt. angle.

Now, in the quadrilateral OAPC each of the angles $\angle$O, $\angle$A and $\angle$C is a right angle, $\therefore$ the fourth $\angle$P is also a right angle.

Ex. 45. *In the greater of two concentric circles chords* AB *and* AC *are drawn to touch the inner circle at* P *and* Q. *Show that* PQ $= \frac{1}{2}$BC.

Let O be the centre of the two concentric circles. Let the chords AB and AC of the greater circle touch the smaller circle at P and Q.

To prove that PQ $= \frac{1}{2}$BC.

Join BC, PQ, OP, OQ.

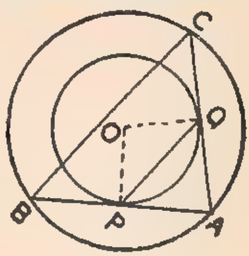


Fig. 78

Proof: OP being a radius through the point of contact, OP$\perp$AB. Similarly, OQ$\perp$AC, $\therefore$ P and Q are the middle points of the chords AB and CD.

$\therefore$ PQ is half the base BC of the ΔABC.

Ex. 46. *AB is a diameter of a circle, and AC a tangent at A equal to AB ; CB is joined cutting the circle in D ; show that CB is bisected in D and* $AD = \frac{1}{2}BC$. [*C. U*]

Let AB be a diameter of the given circle. AC is a tangent to the circle and $AC = AB$. Join CB cutting the circle at D.

To prove that CB is bisected at D and $AD = \frac{1}{2}BC$. Join AD.

Proof : $\angle ADB = 1$ rt. angle being in the semi-circle. $\therefore$ $\angle ADC = 1$ right angle.

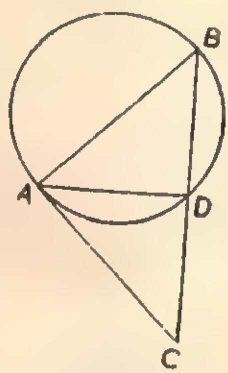


Fig. 79

In the $\triangle^s$ ADC and ADB, $\angle ADC = \angle ADB$ (rt. angles), $AC = AB$ and AD is common to both.

$\therefore$ $\triangle ADC \equiv \triangle ADB$. $\therefore$ $CD = BD$, *i.e.*, CB is bisected at D. Again, $\because$ AB is a diameter from the point of contact, $\therefore$ $\angle BAC = 1$ rt. angle. AD is the st. line which joins the right angle to the middle point D of the hypotenuse BC, $\therefore$ $AD = \frac{1}{2}BC$.

Ex. 47. *The tangents to a circle at the extremities of a chord include an angle double of that between the chord and the diameter through either point of contact.*

Let O be the centre of a circle and PQ be a chord. AP and AQ are two tangents at P and Q. POR is a diameter passing through P.

To prove that $\angle A = 2\angle OPQ$. Join OQ.

Proof : OP and OQ are radii from the points of contact,

$\therefore$ $\angle OPA = \angle OQA = 1$ rt. angle.

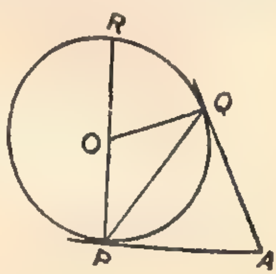


Fig. 80

$\therefore$ $\angle OPQ + \angle APQ + \angle OQP + \angle AQP = 2$ rt. angles $= \angle A + \angle APQ + \angle AQP$. $\therefore$ $\angle A = \angle OPQ + \angle OQP$. $\because$ $OP = OQ$,

$\therefore$ $\angle OPQ = \angle OQP$, $\therefore$ $\angle A = \angle OPQ + \angle OQP = 2\angle OPQ$.

Ex. 48. *If the circumference of a circle is divided into three equal arcs, the tangents drawn to the circle at the points of section form an equilateral triangle.* [*C. U. 29*]

Let the circumference of the circle whose centre is O be divided into three equal parts at the points D, E and F.

Draw tangents AB, BC and CA at the points D, E and F respectively so as to form the $\triangle$ABC.
To prove that ABC is an equilateral triangle.

Proof : $\because$ the arcs DE, EF and FD are equal, $\therefore$ the angles at the centre standing on them are all equal, *i.e.*, $\angle$DOE $=\angle$EOF $=\angle$FOD $=\frac{1}{3}$ of 4 rt. angles $=120^\circ$ ($\because$ the angles at the pt. O $=360^\circ$.)

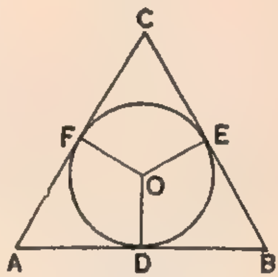


Fig. 81

Now, in the quadrilateral ADOF, $\angle$D $=\angle$F $=1$ right angle ($\because$ OD, OF are radii drawn from the pts. of contact.),

$\therefore$ $\angle$A $+\angle$DOF $=2$ rt. angles $=180^\circ$.

But $\angle$DOF $=120^\circ$, $\therefore$ $\angle$A $=60^\circ$. Similarly, it can be proved that $\angle$B $=60^\circ$ and $\angle$C $=60^\circ$.

$\therefore$ $\angle$A $=\angle$B $=\angle$C. $\therefore$ AB $=$ BC $=$ CA.

$\therefore$ $\triangle$ABC is an equilateral triangle.

23. **Theorem**—If two tangents are drawn to a circle from an external point, they are equal and subtend equal angles at the centre.

Ex. 49. *Two circles touch externally at* A *and a straight line touches the circles at* B *and* C. *Prove that* $\angle$BAC *is a right angle.* [*C.U. 1913*]

Let two circles touch each other externally at A. Let their common tangent BC touch them at B and C. Join AB and AC. To prove that $\angle$BAC $=1$ rt. angle. Let AP be the common tangent to the two circles at A to cut BC at P.

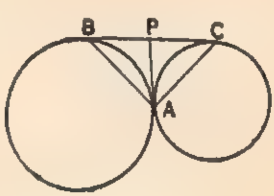


Fig. 82

Proof : $\because$ PA and PB are two tangents from the same point P to the same circle, $\therefore$ PB $=$ PA.

$\therefore$ $\angle PBA = \angle PAB$. Similarly, $PA = PC$ and $\angle PCA = \angle PAC$.

$\therefore$ in the $\triangle ABC$, the whole $\angle BAC = \angle ABC + \angle ACB$ $= \frac{1}{2}$ of 2 rt. angles $=$ 1 rt. angle.

Ex. 50. *Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.* [*D. B. 1929 ; B. U.*]

Let AP and BR be two parallel tangents touching the circle at A and B. Let a third tangent PR touch the circle at T and cut AP and BR at P and R. Join OP and OR.

To prove that $\angle POR = 1$ rt. angle.

Join OA, OT and OB.

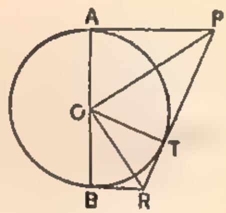


Fig. 83

Proof : In the $\triangle$s POA and POT, $\angle A = \angle T$ (rt. angles), $OA = OT$ and the hypotenuse OP is common,

$\therefore$ $\angle APO = \angle TPO$,

i.e., $\angle OPT = \frac{1}{2} \angle APT$. Similarly, $\angle ORT = \frac{1}{2} \angle BRT$.

$\therefore$ $\angle OPR + \angle ORP = \frac{1}{2}(\angle APR + \angle BRP) = \frac{1}{2}$ of 2 rt. angles $=$ 1 rt. angle ($\because$ $AP \parallel BR$, $\therefore$ $\angle APR + \angle BRP = 2$ right angles).

$\therefore$ the remaining $\angle POR$ of the $\triangle OPR = 1$ right angle.

Ex. 51. OA, OB *are two fixed tangents to a circle.* PQ *is any other tangent cutting* OA *and* OB *at* P *and* Q. *Prove that* PQ *subtends a constant angle at the centre of the circle.* [*C. U. 1923*]

Let OA and OB be two fixed tangents to the circle. Let the tangent PQ touch the circle at T and cut OA and OB at P and Q. Join the centre C to P and Q. To prove that $\angle PCQ$ is constant. Join CB, CA and CT.

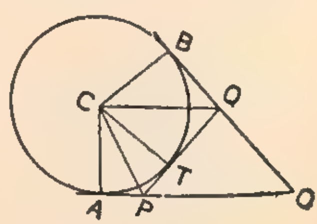


Fig. 84

Proof : $\because$ PA and PT are two tangents drawn from P,

$\therefore$ $\angle ACP = \angle TCP$,

i.e., $\angle PCT = \frac{1}{2} \angle ACT$.

Similarly, $\angle QCT = \frac{1}{2} \angle BCT$.

$\therefore$ the whole $\angle PCQ = \frac{1}{2}(\angle ACT + \angle BCT) = \frac{1}{2} \angle ACB$.

Now, $\because$ OA, OB are two fixed tangents,

$\therefore$ A and B are two fixed points. $\therefore$ $\angle ACB$ is constant.

$\therefore$ $\angle PCQ$(*i.e.*, $\frac{1}{2}\angle ACB$) is constant.

Ex. 52. *In any quadrilateral circumscribed about a circle the sum of one pair of opposite sides is equal to the sum of the other pair.* [*C. U. 1931*]

Let the sides of the quadrilateral ABCD touch the given circle at P, Q, R, S. To prove that $AB+CD=AD+BC$.

Proof : $\because$ AP and AS are two tangents to the circle from the point A, $\therefore$ $AP=AS$.

Similarly, $BP=BQ$, $CR=CQ$, $DR=DS$,

$\therefore$ $AP+BP+CR+DR=AS+BQ+CQ+DS$

i.e., $AB+CD=AD+BC$.

Ex. 53. *Prove that rhombus is the only parallelogram that can be circumscribed about a circle.*

[**Hints :** $\because$ ABCD is a par^m, $\therefore$ $AD=BC$ and $AB=CD$. Again, $AB+DC=AD+BC$ (vide example No. 52).

$\therefore$ $2AB=2BC$. $\therefore$ $AB=BC$. $\therefore$ all sides of the par^m are equal, $\therefore$ it is a rhombus.]

Ex. 54. *If a quadrilateral is described about a circle the angles subtended at the centre by any two opposite sides are supplementary.* [*B. U. 1935*]

[**Hints :** Draw the figure as in example 52. Suppose, O to be the centre of the circle. Join the pts. A, P, B, Q, C, R, D, S to the point O. $\because$ AP and AS are tangents, $\therefore$ $\angle AOS=\angle AOP$. Similarly, $\angle DOS=\angle DOR$,

$\angle COQ=\angle COR$, $\angle BOQ=\angle BOP$. $\therefore$ $\angle AOS+\angle DOS+\angle COQ+\angle BOQ=\angle AOP+\angle BOP+\angle DOR+\angle COR$,

i.e. $\angle AOD+\angle BOC=\angle AOB+\angle COD=\frac{1}{2}$ of 4 rt. angles $=2$ rt. angles.

Ex. 55. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, prove that the quadrilateral can be circumscibed about a circle.*

Let $AB+CD=AD+BC$ in the quadrilateral ABCD. To prove that a circle can be drawn within the quadrilateral

ABCD touching its sides. Let AO and BO bisect the $\angle A$ and $\angle B$, intersecting each other at the pt. O. From O draw OP, OQ, OS perp. to AB, BC, AD respectively.

$\because$ AO is the bisector of $\angle A$, $\therefore$ OP$=$OS.

Similarly, OP$=$OQ, $\therefore$ OP$=$OQ$=$OS.

Now, the circle drawn with centre O and radius OP will touch the sides AB, BC and AD at P, Q and S, for the radii OP, OQ, OS are perps. to the sides at those points.

If the circle do not touch the side CD, suppose DR to be a tangent to this circle and let it cut BC at R.

$\because$ the circle touches the sides of the quadrilateral ABRD, $\therefore$ AB$+$DR$=$AD$+$BR$\cdots$(1) and AB$+$DC$=$AD$+$BC (hyp.)$\cdots$(2).

Now, subtracting (1) from (2) we have DC$-$DR$=$BC$-$BR $=$RC, *i. e.*, in the $\triangle$RDC the difference of two sides is equal to the third side ; but this is impossible. $\therefore$ the circle will touch the side DC also.

Ex. 56. *Draw a circle of given radius to touch two intersecting straight lines.*

Let AX and AY be two intersecting straight lines and r be the given radius. Bisect $\angle A$ by AO. Draw AR$\perp$AX so that AR$=r$. From R draw RO par^l to AX cutting AO at O. Draw OP$\perp$AX and OQ$\perp$AY. Now, the circle drawn with centre O and with radius OP will be the required circle.

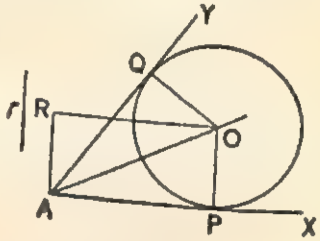


Fig. 85

Proof : $\because$ AO bisects $\angle A$, $\therefore$ OP$=$OQ and they are perp. to AX and AY. $\therefore$ The circle drawn with centre O and with radius OP will pass through P and Q and touch AX and AY respectively at those points.

Now, $\because$ APOR is a par^m, $\therefore$ OP$=$AR$=r$.

[N. B. There may be four such circles. Draw the other three circles.]

24. Theorem—If two circles touch, the point of contact lies on the straight line through their centres.

Ex. 57. *Two circles touch externally, and through the point of contact a straight line is drawn terminated by the circumferences, show that the tangents at its extremities are parallel.*

Let the two circles whose centres are A and B touch externally at P. A st. line RPQ is drawn through P to cut the circles at R and Q. At R and Q, RS and QT are drawn tangents to the two circles. To prove that RS ∥ QT. Join AR, AP, BP, BQ.

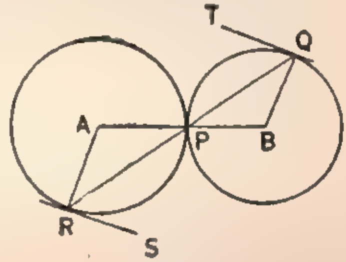


Fig. 86

Proof : The centres A and B and the point of contact P are in the same st. line. ∵ AR = AP, ∴ ∠ARP = ∠APR = vertically opp. ∠BPQ = ∠BQP (∵ BP = BQ). Again, RS is a tangent and AR is a radius passing through the point of contact,

∴ ∠ARS = 1 rt. angle. Similarly, ∠BQT = 1 rt. angle.

∴ ∠ARS = ∠BQT, but ∠ARP = ∠BQP,

∴ ∠PRS = ∠PQT, but they are alternate angles,

∴ RS ∥ QT.

Ex. 58. D, E, F *are the middle points of the sides of a triangle and* P *is the foot of the perpendicular let fall from one vertex on the opposite side. Show that* P, D, E, F *are concyclic.* [*C. U.* '43 ; *D. B.* '27, '29, '37]

Proof : Join EF, FD and EP. ∵ E is the middle point of the hypotenuse AC of the right-angled △ APC, ∴ EP = $\frac{1}{2}$AC = EC.

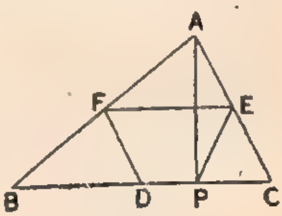


Fig. 87

∴ ∠EPC = ∠ECP.

Now, EF joining the middle pts. of AB and AC is par^{l} to DC.

Similarly, FD ∥ EC.

∴ DCEF is a par^{m}.

∴ ∠EPC = ∠C = ∠DFE.

∵ ∠EPD + ∠EPC = 2 rt. angles,

∴ ∠EPD + ∠EFD = 2rt. angles. ∴ P,D,E,F are concyclic.

Ex. 59. *Of all triangles standing on the same base and having the same vertical angle, show that the isosceles is that which has the greatest area.* [*C. U. 1941*]

Let ABC be an isosceles triangle and DBC any other triangle on the base BC and let the vertical $\angle$A and $\angle$D of the triangles be equal

To prove that $\triangle$ABC$>\triangle$DBC. Draw AP and DQ perp. to BC from A and D.

Proof. $\because$ the angles at A and D on the base BC are equal, $\therefore$ A, B, C, D are concyclic. AP is drawn perp. on the base from the vertex of an isosceles triangle, $\therefore$ it bisects the base BC. $\therefore$ AP passes through the centre. Now draw AR$\perp$AP, then AR is a tangent to the circle. $\therefore$ every pt. on AR except A is outside the circle. $\therefore$ if a perp. is drawn from any pt. on the O^{ce} on BC, it is less than AP. $\therefore$ AP$>$DQ. $\therefore$ the altitude AP of the $\triangle$ABC is greater than the altitude DQ of the $\triangle$DBC.

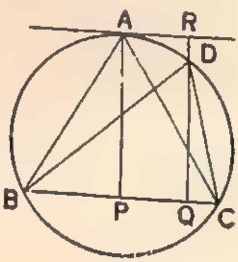


Fig. 88

$\because$ the area of a triangle $=\frac{1}{2}\times$base $\times$altitude,
$\therefore$ $\triangle$ABC$>\triangle$DBC.

Ex. 60. AB, *a diameter of a circle, is produced to meet the tangent at* C *in* D. *Show that* $\angle$BDC$+2\angle$BCD *is a right angle.*

Let O be the middle point of AB. Then O is the centre of the circle. Join OC and BC.

Proof. $\because$ OB$=$OC, $\therefore$ $\angle$OCB$=\angle$OBC.
In the $\triangle$BCD, the ext. $\angle$OBC$=\angle$BCD$+\angle$BDC.
$\therefore$ $\angle$OCB$=\angle$BCD$+\angle$BDC. Add $\angle$BCD to both sides. Then $\angle$OCB$+\angle$BCD$=2\angle$BCD$+\angle$BDC, but CD is a tangent and CO is a radius drawn from the pt. of contact,
$\therefore$ $\angle$OCB$+\angle$BCD, *i.e.*, the whole $\angle$OCD is a rt. angle.
$\therefore$ $\angle$BDC$+2\angle$BCD$=1$ rt. angle.

Ex. 61. *Show how to draw a tangent to a given circle parallel to a given straight line. How many such tangents are possible?* [*C. U.* '39]

Let O be the centre of the circle and AB any given st. line. To draw a tangent to the circle par^l to AB.

Construction: From O draw OM perp. to AB. Let OM or OM produced intersect the circle at P. At P draw PQ⊥OP. Then PQ is the required tangent.

Proof: ∵ PQ is perp. to the radius OP at P, ∴ PQ is a tangent.

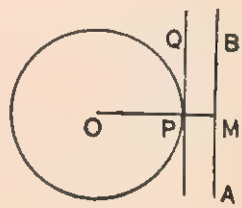


Fig. 89

Again, ∵ PQ and AB are perp. to the same st. line OM, ∴ PQ ∥ AB.

PO produced can intersect the circle at another pt. The tangent at that pt. is also par^l to AB.

∴ Two such tangents can be drawn.

Ex. 62. *Show that all chords parallel to the tangent at any pt. of a circle are bisected by the radius through the point.* [*C. U. 1918*]

Let O be the centre of the circle. Let PT be a tangent at the pt. P and CD be any chord par^l to the tangent PT.

To prove that the chord CD is bisected by the radius OP drawn from the pt. of contact.

Proof. The radius OP drawn from the pt. of contact is perp. to the tangent PT. ∵ CD ∥ PT, ∴ OP is perp. to CD.

∵ OP is perp. to the chord CD from the centre,

∴ OP bisects CD. This is true for every chord par^l to the tangent PT.

Ex. 63. *Find the locus of the centres of circles which touch two concentric circles.* [*D. B. 1934*]

Let O be the centre of two concentric circles ABQ and CDR and let KRQ be any circle touching the circles ABQ and

CDR at Q and R. Let P be the centre of the circle KRQ. To find the locus of the pt. P. Join OR, PR, PO.

$\because$ the circles CDR and KRQ touch each other at R,

$\therefore$ ORP is the same st. line.

Again, $\because$ the circles ABQ and KRQ touch each other at Q,

$\therefore$ OPQ is the same st. line. $\therefore$ ORPQ is a st. line.

Now, $RQ = OQ - OR$,

$\therefore$ $PR = \frac{1}{2}RQ = \frac{1}{2}(OQ - OR)$.

$\because$ the two circles ABQ and CDR are fixed, $\therefore$ their radii QO and OR are constant.

$\therefore$ $PR =$ a constant length.

$\therefore$ $OP = OR + PR =$ a constant length.

$\therefore$ the circle drawn with centre O and with radius OP is the required locus of the pt. P.

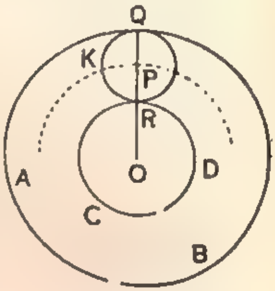


Fig. 90

Ex. 64. *Through a given point draw a chord in a circle equal to a given st. line. When does the construction fail ?*

Let O be the centre and P be any fixed pt. (within or outside) the circle. Let l be the given st. line.

To draw a chord through P equal to l.

Construction : With centre A, any point on the $\odot^{ce}$, and with radius l draw an arc of a circle cutting the given circle at B. Join AB. Then $AB = l$.

Draw $OQ \perp AB$. With centre O and with radius OQ draw a circle. Draw a tangent PT to that circle from P so that is cuts the given circle at C and D. Then CD is the required chord.

Proof : $\because$ AB and CD are tangents to the inner circle,

$\therefore$ $OQ \perp AB$ and $OT \perp CD$. $\because$ $OQ = OT$ (radii),

$\therefore$ the chord $CD =$ the chord $AB = l$.

If the given st. line be greater than the diameter of the circle, it is not possible to draw such a chord.

Again, if P be within the circle, it is not possible to draw such a chord when the given length is less than the chord through P perp. to OP.

Ex. 65. *A straight rod of given length slides between two straight rulers placed at right angles to one another. Find the locus of its middle point.*

Suppose AB and AC are two straight rulers placed at right angles and PQ a straight rod of given length sliding between them, its extremities being always on AB and CD.

To find the locus of the middle point O of the rod PQ.

Join AO. $\because$ in every position of PQ, PAQ is a right-angled triangle,

$\therefore$ AO$=\frac{1}{2}$PQ$=$a constant length, *i.e.*, O is always equidistant from A. $\therefore$ the arc of the circle, drawn with centre A and radius $\frac{1}{2}$PQ, that lies between AB and AC is the required locus of O.

Ex. 66. A *and* B *are the centres of two fixed circles which touch internally. If* P *is the centre of any circle which touches the larger circle internally and the smaller externally prove that* AP$+$BP is constant. [*D. B. 1935*]

Suppose A to be the centre of the greater circle which is touched internally by a smaller circle whose centre is B. Let P be the centre of the third circle which touches the circle (A) internally at C and the circle (B) externally at D.

To prove that AP$+$BP is constant. Let the radii of the three circles be r_1, r_2, and r_3 respectively.

Proof : $\because$ the second and third circles touch each other at D, $\therefore$ B, D, P are in the same st. line, *i.e.*, BDP is a st. line.

$\therefore$ BP$=r_2+r_3$. Again, $\because$ the first and the third circles touch each other at C, $\therefore$ APC is the same st. line.

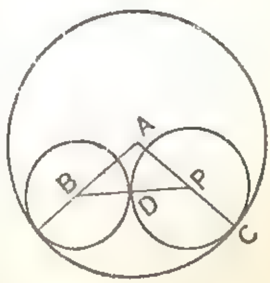


Fig. 91

$\therefore$ AP$=$AC$-$PC$=r_1-r_3$. Now, AP$+$BP$=r_1-r_3+r_2+r_3=r_1+r_2$.

$\because$ the first and the second circles are fixed, $\therefore$ r_1 and r_2 are fixed.

$\therefore$ AP$+$BP is constant.

Exercise 3

1. If a st. line cuts two concentric circles the parts of it intercepted between the circumferences are equal.

2. If two chords of a circle cut one another and make equal angles with the straight line joining the centre to the point of intersection, they are equal.

3. The join of the middle points of two parallel chords of a circle passes through the centre.

4. Given the base and the vertical angle of a triangle, find the locus of its vertex. [C. U. '11]

5. Two circles intersect at A and B and each passes through the centre of the other. Through A any st. line PAQ is drawn terminated by the circumferences. Prove that ΔPBQ is equilateral.

[**Hints :** Suppose O and R to be the centres of the two circles. $\because$ each circle passes through the centre of the other, $\therefore$ their radii are equal. Join AO, BO, AR, BR, RO.

Now, $AO = BO = RO = AR = BR$ (being equal radii).

$\therefore$ ΔAOR is equilateral. $\therefore$ $\angle AOR = 60^\circ$.

Similarly, $\angle BOR = 60^\circ$, $\therefore$ $\angle AOB = 120^\circ$.

Again, $\angle P$ at the circumference $= \frac{1}{2}$ $\angle AOB$ at the centre (both standing on the same arc) $= 60^\circ$. Similarly, it can be proved that $\angle Q = 60^\circ$. $\therefore$ the remaining $\angle PBQ = 60^\circ$. $\therefore$ ΔPBQ is equilateral.]

6. Two circles touch externally at P. Show that their common tangent is bisected by the tangent at P.

7. If one side of a cyclic quadrilateral is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral. [D. B. '26]

8. In a given circle draw a chord equal to a given st. line and parallel to another.

9. Find the locus of intersection of two st. lines which are at right angles and pass through two fixed points. [C.U. '17]

10. Find the locus of the centres of circles of given radius which touch a given circle.

11. Similar segments of circles on equal chords are equal to one another.

[N. B. The segments of a circle in which all angles are equal are similar segments of a circle.]

12. A circle is described on the hypotenuse of a right-angled triangle as diameter. Prove that the circle passes through the opposite angular point. [C. U. '27]

13. Find the locus of the vertex of a triangle having given the base and the sum of the base angles.

14. A circle is drawn on one of the equal sides of an isosceles triangle as diameter. Show that it passes through the middle pt. of the base.

15. Two circles that touch externally at a pt. have a common tangent PQ. Show that the line of centres of the circles is a tangent to the circle drawn on PQ as diameter.

16. Equilateral triangles are described on the three sides of a triangle externally and circles are circumscribed about these equilateral triangles. Prove that they intersect in a common point. [C. U. '25. '26]

17. AB is a fixed chord of a given circle and P is any pt. on the circumference. Show that the bisector of the $\angle$APB passes through one or other of two fixed points. [C. U. '23]

18. Circles are drawn with a fixed point as centre and tangents are drawn to them from another fixed point. Find the locus of the points of contact of these tangents.

19. ABCD is a parallelogram. The circle which passes through A and B cuts AD and BC at E and F. Prove that E, F, C and D are concyclic. [B. U. '26]

20. If in two circles equal chords subtend equal or supplementary angles at the circumference, the circles are equal.

21. Prove that, if two circles touch, the distance between their centres is equal to the sum or difference of their radii. [C. U. '19]

22. (*a*) If a circle can be described about a parallelogram, the parallelogram must be a rectangle.

(*b*) The rectangle circumscribed about a circle must be a square.

23. AC and AB are respectively a diameter and a chord of a circle. Tangents at A and B meet at D. Show that $\angle ADB = 2\angle BAC$.

24. If a parallelogram can be inscribed in a circle, the pt. of intersection of its diagonals must be at the centre of the circle. [Cf. D. B. '48]

[Let ABCD be a cyclic par^m whose diagonals AC and BD intersect at O. The par^m being cyclic is a rectangle.

$\therefore$ $\angle BAD = 1$ rt. angle. $\therefore$ it is an angle in the semi-circle. $\therefore$ BD is a diameter of a circle. Again, O is the middle point of BD, $\therefore$ O is the centre of the circle.]

25. Two circles cut at A and B and any straight line PAQ is drawn through A and terminated by the circles. Show that $\angle PBQ$ is constant.

[**Hints :** $\angle P$ and $\angle Q$ at the $\odot^{ce}$ standing on the fixed arc AB are constant, $\therefore$ $\angle B$ is constant.]

26. Find the locus of points such that the pairs of tangents drawn from them to a given circle contain a constant angle.

27. A quadrilateral is inscribed in a circle. Show that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles. [C. U. 1887]

28. Describe a circle to touch two parallel straight lines and a transversal. [C. U. 1935]

29. From a pt. A, on the circumference of a circle, AD is drawn perpendicular on a chord BC ; if AE be the diameter through A, prove that $\angle$ BAD, EAC are equal. [C.U. '48, '49]

30. In an acute-angled triangle show that the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the triangle formed by joining the feet of these perpendiculars.

31. The sides AB and DC of a cyclic quadrilateral ABCD are produced to meet in E. Prove that the triangle EBC, EAD are equiangular.

[Hints : In the cyclic quadrilateral the exterior $\angle$EBC =interior opposite $\angle$ADE. Similarly $\angle$ECB=$\angle$EAD, and $\angle$E is common.]

32. Prove that non-parallel sides of a cyclic trapezium are equal. [U. U. '52]

33. Prove that if two tangents are drawn to a circle from an external point, the line joining the point to the centre of the circle cuts the chord of contact at right angles.
[E B. S. B. '51]

[SOLUTION OF EXAMPLES ABOUT DIFFERENT KINDS OF CENTRES]

Ex. 1. *If* O *is the orthocentre of the triangle* ABC, *show that the angles* BOC, BAC *are supplementary.* [*C. U.* '36]

In the $\triangle$ABC, BE and CF are perp. respectively to AC and AB and they intersect each other at O.

Then O is the orthocentre.

To prove that $\angle$BOC+$\angle$BAC=2 rt. angles.

Proof : In the quadrilateral AEOF, $\angle$E and $\angle$F are each a right angle ; $\therefore$ $\angle$FAE+$\angle$FOE=2 rt. angles ; but $\angle$FOE =vert. opp. $\angle$BOC. $\therefore$ $\angle$BAC+$\angle$BOC=2 rt. angles.

Ex. 2. *If* O *be the orthocentre of a triangle* ABC *and if* AO *be produced to meet the circumcircle at* P, *show that* OP *is bisected at right angles by* BC.

[*C. U. 1944*]

O is the orthocentre of the $\triangle ABC$. Join BO and produce it to meet AC at E. Join AO and produce it to meet BC at D and the circumcircle at P.

To prove that BC bisects OP at rt. angles.

Proof: O being the orthocentre, $AD \perp BC$ and $BE \perp AC$.

$\therefore$ DOEC is a cyclsc quadrilateral.

$\therefore$ its exterior $\angle BOD =$ int. opp. $\angle C = \angle P$ (being in the same segment of the circle); Now, in the $\triangle^s$ BPD and BDO, $\angle P = \angle BOD$, $\angle BDP = \angle BDO$ (right angles) and BD is common to both. $\therefore$ PD $=$ DO and the angles at D are rt. angles. $\therefore$ BC bisects OP at right angles.

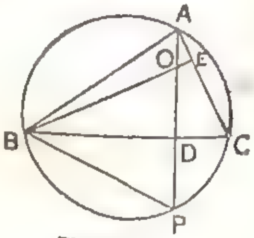


Fig. 92

Ex. 3. *Prove that in an equilateral triangle* I, S, O *and* G *are coincident.*

[N. B. The in-centre of a triangle is denoted by I, its circumcentre by S, its orthocentre by O and its centroid *i.e.*, the point of intersection of its three medians by G.]

Let ABC be an equilateral triangle. To prove that its incentre, circumcentre, orthocentre and centroid are coincident, *i.e.*, the same point.

In any triangle the pt. of intersection of the bisectors of its angles is its incentre, the pt. of intersection of the perp. bisectors of its sides is the circumcentre, the pt. of intersection of the perpendiculars drawn from the vertices to the opposite sides is the orthocentre and the pt. of intersection of the three medians is called the centroid.

Suppose in the $\triangle ABC$, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$ and the perpendiculars meet at O. Then O is its orthocentre. $\triangle ABD$ and $\triangle CAD$ are congruent, ($\because$ $\angle ADB = \angle ADC = 1$ rt. angle, $AB = AC$, AD is common), $\therefore$ AD is the bisector of $\angle BAC$.

Similarly, BE and CF are bisectors of $\angle ABC$ and $\angle ACB$ respectively. $\therefore$ O is the incentre of the triangle. Again, $\triangle^s$ ABD, ACD being congruent, $BD = CD$. $\therefore$ AD is a median. Similarly, BE and CF are other two medians. $\therefore$ O is the centroid of the triangle. Again, the medians being perpendiculars to the sides, O is the circumcentre of the triangle.

$\therefore$ in the equilateral triangle I, S, O and G are coincident.

Ex. 4. *ABC is a triangle, O is its orthocentre, and AK a diameter of the circumcircle. Show that BOCK is a parallelogram.*

Let BD and CE be perps. to AC and BA. Let them meet at O. Then O is the orthocentre. Let AK be the diameter of the circumcircle. Join BK, CK. To prove that BOCK is a par^m.

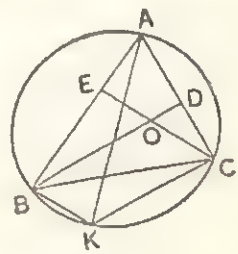


Fig. 93

Proof : $\angle ABK$ being in a semi-circle is a right angle.

Now, $\because$ KB and CE are both perp. to AB, $\therefore$ $BK \parallel CE$.

Similarly, $BD \parallel KC$. $\therefore$ BOCK is a parallelogram.

Ex. 5. *If on the sides of any triangle three equilateral triangles be drawn, and circles be inscribed in each of these triangles the st. lines joining the centres of these circles will form an equilateral triangle.*

Three equilateral $\triangle^s$ ABD, FBC, EAC are drawn on the sides of the $\triangle ABC$. Let three circles be inscribed in those three triangles. Let their centres be P, R and Q.

To prove that $\triangle PRQ$ is an equilateral triangle.

Construction : Draw three circumcircles of the $\triangle^s$ ABD, FBC and EAC intersecting one another at O. Join AO, BO and CO.

Proof: The quadrilateral AOBD being cyclic,

$\angle D + \angle AOB = 180^\circ$; but $\angle D = 60^\circ$, $\therefore$ $\angle AOB = 120^\circ$.

Again, P, R and Q are also centres of the circumcircles, $\therefore$ PQ, the line of centres, bisects their common chord AO at right angles at M.

$\therefore$ $\angle PMO = 1$ rt. angle.

Similarly, $\angle PNO = 1$ rt. angle.

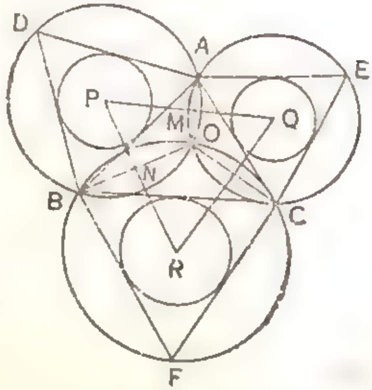


Fig. 94

$\therefore$ in the quadrilateral PMON, $\angle P + \angle MON = 180^\circ$, but $\angle MON = 120^\circ$ (proved), $\therefore$ $\angle P = 60^\circ$. Similarly, it can be proved that $\angle Q = 60^\circ$, $\angle R = 60^\circ$. $\therefore$ PR = RQ = QP.

$\therefore$ PRQ is an equilateral triangle.

Ex. 6. *Given the base and the vertical angle of a triangle, find the locus of its orthocentre.*

Let BE and CF be perp. to AC and AB and let them intersect at O, so that O is the orthocentre.

The base BC and the vertical $\angle A$ of the triangle are fixed. To find the locus of the point O.

$\because$ $\angle F$ and $\angle E$ of the quadrilateral AFOE are each a right angle, $\therefore$ $\angle FOE + \angle A = 2$ rt. angles, but $\angle A$ is constant.

$\therefore$ $\angle FOE$ is constant. $\therefore$ $\angle BOC = \angle FOE =$ a constant.

$\because$ $\angle BOC$ subtended by the base BC at the othocentre O is constant,

$\therefore$ the arc BOC of the circle passing through B, O, C is the locus of the pt. O.

Ex. 7. *Given the base and the vertical angle of a triangle, find the locus of its incentre.* [*C. U.* '*19*]

Suppose the base BC and the vertical $\angle A$ of the $\triangle ABC$ to be fixed. The bisectors BI and CI of the $\angle B$ and $\angle C$

intersect each other at I. To find the locus of the incentre I.

In the $\triangle BIC$, $\angle BIC + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 180°$······(1) and in the $\triangle ABC$, $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90°$...(2).

Subtracting (2) from (1) we have $\angle BIC - \frac{1}{2}\angle A = 90°$.

$\therefore$ $\angle BIC = 90° + \frac{1}{2}\angle A$ = a constant quantity ($\because$ $\angle A$ is constant), $\therefore$ the fixed base BC subtends a constant $\angle BIC$ at the pt. I, $\therefore$ the arc BIC of the circle passing through B, I, C is the locus of the incentre I.

Ex. 8. *Given the base and the vertical angle of a triangle, find the locus of its centroid.*

The base BC and the vertical $\angle A$ of the $\triangle ABC$ are given. The medians BE and CF intersect at G. To find the locus of the centroid G.

Draw GP and GK par[l] respectively to AB and AC to meet BC at P and K. $\angle GPK$ = corresponding $\angle ABC$ and $\angle GKP$ = corresponding $\angle ACB$.

$\therefore$ $\angle PGK = \angle A$ = a constant angle.

Again, $\because$ the medians of a triangle intersect at a point of trisection,

$\therefore$ $FG = \frac{1}{3}CF$.

$\because$ $GP \parallel BF$, $\therefore$ $BP = \frac{1}{3}BC$.

Similarly, $KC = \frac{1}{3}CB$, $\therefore$ $PK = \frac{1}{3}BC$.

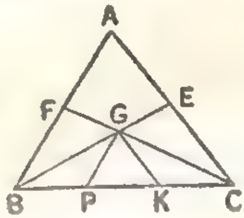


Fig. 95

Now, $\because$ B and C are fixed points, $\therefore$ PK is always of constant length and it subtends a constant angle at G.

$\therefore$ the arc PGK of the segment of the circle containing $\angle G$ or $\angle A$ on the st. line PK is the required locus.

CONSTRUCTION OF TRIANGLES

[N. B. Construction of triangles involving their areas, has already been dealt with. In questions in which "The traces of construction only are required"—you are to give all traces of construction only, but no statement of construction and proof. If there be no such restriction, you are to give both the statement of construction and proof.]

Ex. 1. *Construct a right-angled triangle, having given the hypotenuse and the sum of the remaining sides.* [C. U. '22]

Let h be the hypotenuse and AP the sum of the remaining sides. At P draw $\angle APB = 45°$. With centre A and with radius h draw an arc of a circle cutting PB at B. Draw $BC \perp AP$, and join AB. Then ABC is the required triangle.

Proof : $\because$ $\angle C = 90°$ and $\angle P = 45°$, $\therefore$ $\angle PBC = 45°$.
$\therefore$ $BC = PC$. $\therefore$ $AC + BC = AC + PC$ and $AB = h$.

Ex. 2. *Construct a right-angled triangle having given the hypotenuse and the difference of the other two sides.*

Let h be the hypotenuse and AP the difference of the other two sides. Produce AP and at P draw $\angle CPB = 45°$. With centre A and radius h draw an arc to cut PB at B. Draw $BC \perp AP$. Now, ABC is the required triangle.

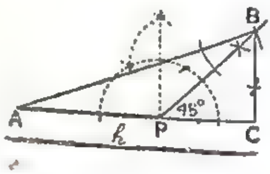


Fig. 96

Proof : $\because$ $\angle C = 90°$ and $\angle CPB = 45°$, $\therefore$ $\angle PBC = 45°$. $\therefore$ $BC = PC$,
$\therefore$ $AC - BC = AC - PC = AP$, and the hypotenuse $AB = h$.

Ex. 3. *Construct an isosceles right-angled triangle, having given the sum of the hypotenuse and one side.*

Suppose, AX to be the sum of the hypotenuse and one side. Draw $\angle XAB = 45°$ at A and $\angle AXB = 22\frac{1}{2}°$ at X.

Let AB and XB intersect each other at B. At B draw $\angle XBC = \angle AXB = 22\frac{1}{2}°$. Let BC cut AX at C. ABC is the required triangle.

Proof : $\angle BCA = \angle X + \angle XBC = 45°$ and $\angle A = 45°$.

$\therefore$ $\angle ABC = 90°$ and $AB = BC$.

$\because$ $\angle X = \angle XBC$, $\therefore$ $XC = BC$, $\therefore$ $AC + BC = AX$.

Ex. 4. *Construct a right-angled triangle having given the hypotenuse and the perpendicular from the right angle on it.*

[**Hints** : AB is the given hypotenuse and h the perpendicular drawn from the rt. angle on AB. Draw a semi-circle on AB as diameter. At A draw $AX \perp AB$ and make $AX = h$. From X draw $XC \parallel AB$ to cut the semi-circle at C and D. Now, ACB and ADB are the required triangles.

Proof : Draw $CM \perp AB$. Then AMCX is a rectangle, $\therefore$ $CM = AX = h$ and $\angle ACB = 1$ rt. angle, being in a semi-circle.]

Ex. 5. *Draw an isosceles triangle having given the base and the vertical angle.*

Let AB be the given base and $\angle PXY$ the given vertical angle.

Construction : Produce PX to R and bisect the $\angle YXR$ by QX. At A and B draw $\angle BAC$ and $\angle ABC$ equal to $\angle RXQ$. Let AC and BC intersect at C. Then ABC is the required triangle.

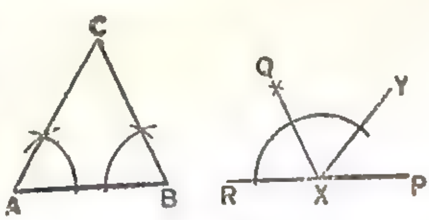


Fig. 97

Proof : $\because$ $\angle A = \angle B$,

$\therefore$ $AC = BC$,

$\therefore$ ΔABC is isosceles. $\angle A + \angle B + \angle C = 180°$ and $\angle YXR + \angle YXP = 180°$. But $\angle A + \angle B = \angle YXR$, $\therefore \angle C = \angle PXY$.

Ex. 6. *Construct an isosceles triangle having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base.* [*C. U.* '42]

Let AB the base and h the sum of one of the equal sides and the perpendicular from the vertex to the base.

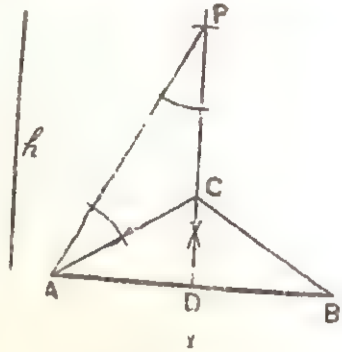


Fig. 98

Construction: Draw DP, the perp. bisector of AB and let DP $=h$. Join AP and at A draw $\angle$PAC $=\angle$P. Let AC cut PD at C. Join BC. Then ABC is the required triangle.

Proof: $\because$ PD is the perp. bisector of AB, $\therefore$ AC $=$ BC, $\therefore$ $\triangle$ABC is isosceles.

$\because$ $\angle$P $=\angle$PAC, $\therefore$ AC $=$ PC.

$\therefore$ the side AC $+$ the perp. CD $=$ PC $+$ CD $=$ PD $=h$.

Ex. 7. *Construct an equilateral triangle having given the length of the perpendicular from one of the vertices to the opposite side.*

[Hints: Draw any equilateral triangle APQ. Draw AR$\perp$PQ. From AR cut off AD $=$ the given length of the perp. Through D draw BC $\parallel$ PQ to cut AP at B and AQ at C. ABC is the required triangle.]

Ex. 8. *Construct a triangle having given the base angles equal to two given angles and the perpendicular from the vertex on the base equal to a given line.* [C. U. '37 ; G. U. '49]

Let $\angle$X and $\angle$Y be the base angles and h the given perpendicular from the vertex on the base.

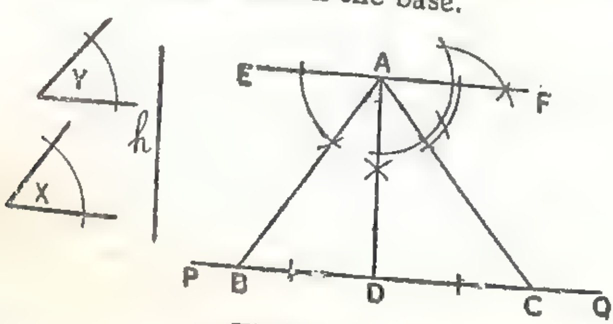


Fig. 99

Construction: Take any st. line PQ and at any pt. D on it draw DA$\perp$PQ. Let DA $=h$ and draw EAF $\parallel$ PQ.

At A on EF draw $\angle EAB = \angle X$ and $\angle FAC = \angle Y$. Let AB and AC cut PQ at B and C. Then ABC is the required triangle.

Proof : $\because$ EF $\parallel$ PQ. $\therefore$ $\angle ABC$ = alternate $\angle EAB = \angle X$ and $\angle ACB$ = alt. $\angle CAF = \angle Y$, and the perp. AD $= h$.

Ex. 9. *Construct a triangle having given the base, the median which bisects the base and the altitude.*

[Hints : Draw DP perp. bisector of the base BC and let DP = the given perp. h. Draw EPQ $\parallel$ BC. Draw an arc of a circle with centre D, middle pt. of BC, and with radius m, the given median. Let the arc cut EQ at A. Join AB and AC. Then ABC is the required triangle.]

Ex. 10. *Construct a triangle having given two angles and a side opposite to one of them.* [*C. U.* '38, '40 ; *D.B.* '39]

Let $\angle A$ and $\angle B$ be the two given angles and a be the given side opposite to $\angle A$. To draw the triangle.

Construction : Cut off BC $= a$ from any st. line BD. At C draw $\angle DCE = \angle B$ and $\angle ECA = \angle A$. At B draw $\angle CBA = \angle B$. Let BA cut CA at A. Then ABC is the required triangle.

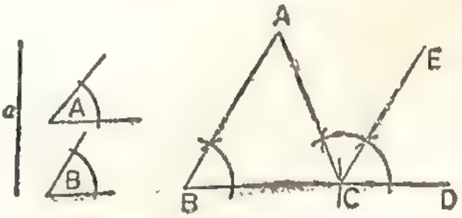


Fig. 100

Proof : The exterior $\angle ACD = \angle ABC + \angle BAC$, but $\angle ABC = \angle B = \angle ECD$, $\therefore$ $\angle BAC = \angle ACE = \angle A$ and BC $= a$.

Ex. 11. *Construct a triangle having given the base, one of the angles at the base and the sum of the remaining sides.*

[*D. B.* '48 ; *C. U.* '20]

Let a be the base, K the sum of the remaining sides and $\angle B$ an angle at the base.

Construction : Take the base BC $= a$. At B draw $\angle CBD = \angle B$. Let BD = K. Join DC and at C draw $\angle DCA = \angle CDA$. Let CA cut BD at A. Then ABC is the required triangle.

Proof: $\because$ $\angle ACD = \angle ADC$, $\therefore$ $AC = AD$.
$\therefore$ $AB + AC = BD = K$.

Ex. 12. *Construct a triangle having given the base, the difference of the other sides and an angle opposite to one of them.*

[**Hints:** Let BC be the base. Draw $\angle CBA$ = the given angle. From BA cut off BD equal to the difference of the other two given sides. Join CD. Draw $\angle DCA = \angle CDA$. Let CA intersect BD at A. Then ABC is the required triangle. $\because \angle ADC = \angle ACD$, $\therefore$ $AD = AC$, $\therefore$ $AB - AC = AB - AD = BD$.]

[N. B. If the given angle be obtuse, then from AB produced cut off BD equal to the difference of the sides.]

Ex. 13. *Construct a triangle having given the perimeter and the angles at the base.* [*C. U.* '38, '45, '51]

Let XY be the perimeter and $\angle P$ and $\angle Q$ be the angles at the base.

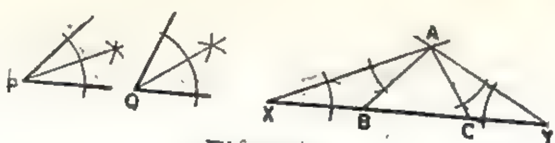


Fig. 101

Construction: Bisect $\angle P$ and $\angle Q$. At X draw $\angle YXA = \frac{1}{2}\angle P$ and at Y draw $\angle XYA = \frac{1}{2}\angle Q$. Let XA and YA meet at A. At A draw $\angle XAB = \angle X$ and $\angle YAC = \angle Y$. Let AB and AC cut XY at B and C respectively. Then ABC is the required triangle.

Proof: $\angle ABC = \angle X + \angle BAX = 2\angle X = \angle P$,
and $\angle ACB = \angle Y + \angle CAY = 2\angle Y = \angle Q$.
Again, $\because$ $\angle X = \angle BAX$, $\therefore$ $AB = BX$, and $\because$ $\angle Y = \angle CAY$,
$\therefore$ $AC = CY$. $\therefore$ $AB + BC + AC = XB + BC + CY = XY$.

Ex. 14. *Construct a triangle having given the middle points of its sides.*

Let D, E and F be the middle points of the sides of a triangle. Join DE, EF, FD. Through D draw AB ∥ EF, through E draw AC ∥ DF and through F draw CFB ∥ DE, ABC is a triangle formed by the st. lines AB, AC and BC. ABC is the required triangle.

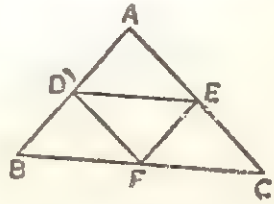


Fig. 102

Proof: By construction, ADFE and DBFE are two parallelograms, ∴ EF = AD and EF = BD,

∴ AD = BD, ∴ D is the middle pt. of AB. Similarly, it can be proved that E and F are the middle pts. of AC and BC respectively.

Ex. 15. *Construct an isosceles triangle having given the altitude and the vertical angle.*

[**Hints:** Let PQ be any st. line. Draw AD perp. to PQ and equal to the given altitude h. Draw $\angle$DAB and $\angle$DAC each equal to half the given vertical angle on either side of the pt. A. Let AB and AC meet PQ at B and C. ABC is the required triangle.]

Ex. 16. *Construct a triangle having given one of the base angles, the altitude and the sum of the other two sides.*

[**Hints:** Let $\angle$B be the given base angle, h altitude and s the sum of the other two sides. Draw a st. line BX and at B draw $\angle$XBY = $\angle$B. Let BY = s. Draw BP perp. to BX and equal to h. Draw PA ∥ BX to cut BY at A. Draw a circle with centre A and with radius AY to cut BX at C and D. Join AC and AD. $\triangle$s ABC, ABD are the required triangles.]

Ex. 17. *Construct an isosceles triangle, the altitude and the base angle being given.*

[**Hints:** Let $\angle$B be the base angle and h the altitude. Draw any st. line BP and at B draw $\angle$PBQ = $\angle$B.

Draw BD perp. to BP and equal to h. Draw DA ∥ BP to cut BQ at A. Draw an arc of a circle with centre A and radius AB to cut BP at C. Join AC. $\triangle$ABC is the required triangle.

Ex. 18. *Construct an isosceles triangle having given the base and the difference of one of the remaining sides and the altitude.*

Let BC be the base and d the difference of one of the remaining sides and the altitude. To draw the isosceles triangle. Draw PDQ, the perp. bisector of BC, to cut BC at D. Let DQ $= d$. Join BQ and draw $\angle$QBA $= \angle$Q.

Let BA cut PQ at A. Join AC.

Then ABC is the required triangle.

Proof : AD is the altitude of $\triangle$ABC

$\because$ $\angle$ABQ $= \angle$AQB, $\therefore$ AB $=$ AQ.

$\therefore$ AB $-$ AD $=$ AQ $-$ AD $=$ DQ $= d$.

Again, AD is the perp. bisector of BC,

$\therefore$ AB $=$ AC, $\therefore$ ABC is the isoscles triangle.

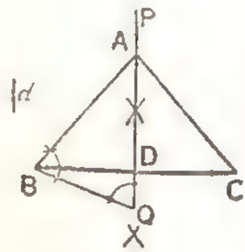


Fig. 103

Ex. 19. *Construct a triangle having given the base, the difference of the other two sides and the difference of the angles opposite to those two sides (or, the difference of the base angles).* [*C. U.* '39, '41 ; *D. B.* '41]

Let BC be the base, d the difference of the other two sides and $\angle$X the difference of the base angles. To draw the triangle. Bisect $\angle$X. At B draw $\angle$CBD $= \frac{1}{2}\angle x$. Draw an arc of a circle with centre C and with radius d to cut BD at D. Join CD and produce it to A. At B draw $\angle$DBA $= \angle$ADB.

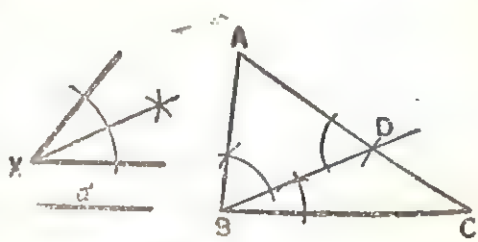


Fig. 104

Let BA cut DA at A. Then ABC is the required triangle.

Proof : $\because$ $\angle$ABD $= \angle$ADB, $\therefore$ AB $=$ AD.

$\therefore$ AC $-$ AB $=$ AC $-$ AD $=$ CD $= d$.

Again, in the $\triangle$BDC, the exterior $\angle$ADB $= \angle$DBC $+ \angle$C,

$\therefore$ $\angle$ABD $= \angle$DBC $+ \angle$C.

Adding $\angle DBC$ to both sides, the whole $\angle ABC = 2\angle DBC + \angle C$.

$\therefore \angle ABC - \angle C = 2\angle DBC = \angle X$ ($\because \angle DBC = \frac{1}{2} \angle x$).

Ex. 20. *Construct a triangle having given the base, the base angles and the sum of the remaining sides.*

Given the base BC, the sum of the remaining sides s, and the difference of the base angles $\angle X$. To draw the triangle.

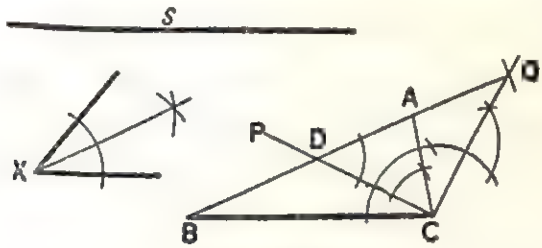


Fig. 105

Construction : At C in BC draw $\angle BCP = \frac{1}{2}\angle X$. Draw $CQ \perp PC$. With centre B and with radius s draw an arc to cut CQ at Q. Join BQ cutting CP at D.

Draw $\angle DCA = \angle QDC$ so that CA cuts DQ at A.

Then ABC is the required triangle.

Proof : $\angle DCA + \angle ACQ = 1$ rt. angle ($\because CQ \perp PC$),

$\therefore \angle CDQ + \angle DQC = 1$ rt. angle.

$\therefore \angle DCA + \angle ACQ = \angle CDQ + \angle DQC$, but $\angle DCA = \angle CDQ$,

$\therefore \angle ACQ = \angle AQC$, $\therefore AC = AQ$, $\therefore AB + AC = BQ = s$.

Again, $\angle ACD = \angle ADC = \angle B + \angle BCD$.

$\therefore \angle ACD + \angle BCD = 2\angle BCD + \angle B$,

i.e., $\angle ACB = 2\angle BCD + \angle B$. $\therefore \angle ACB - \angle B = 2\angle BCD = \angle X$.

Ex. 21. *Construct a triangle having given two sides and the median which bisects the third side.*

Suppose p and q to be the lengths of two sides and AO to be the median bisecting the third side. To draw the triangle.

Construction : Produce AO to D so that OD = AO.

With centres A and D and radii p and q respectively, draw two arcs of a circle. Let them intersect at B. Join BO and produce it to C so that CO = BO. Join AB and AC.

ABC is the required triangle.

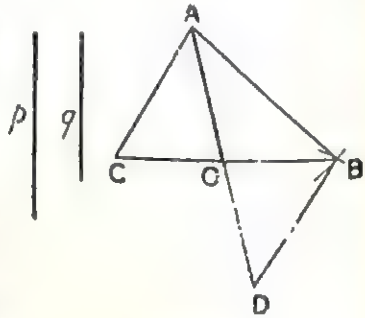


Fig. 106

Proof : Join BD. In the $\triangle^s$ AOC, BOD, $\because$ AO = DO, CO = BO and $\angle$AOC = vert. opp. $\angle$BOD,

$\therefore$ the triangles are congruent.

$\therefore$ AC = BD = q. AB = p (Cons.)

and $\because$ BO = CO, $\therefore$ AO is a median.

Ex. 22. *Construct a triangle having given the lengths of one side and the medians which bisect the other two sides.*
[D. B. '49 ; C. U. '42 ; G. U. '48]

Let BC be the base and p and q be the medians bisecting the other two sides. To draw the triangle.

Construction : Divide p and q into three equal parts. With centres B and C and with radii $\frac{2}{3}p$ and $\frac{2}{3}q$ respectively, draw two arcs cutting each other at G.

Let D be the middle pt. of BC. Join DG and produce it to A so that AG = 2DG. Join AC and AB. Then ABC is the required triangle. Join BG and CG.

Let BG and CG produced cut AC and AB at E and F.

Proof : Produce GD to O so that DO = GD. Join OC. $\triangle$BDG $\equiv$ $\triangle$ODC ($\because$ BD = DC, GD = DO and $\angle$BDG = $\angle$CDO),

$\therefore$ $\angle$GBD = $\angle$OCD, $\therefore$ CO $\parallel$ BE.

Now $\because$ GE is drawn parallel to OC from G, the middle pt. of AO, $\therefore$ E is the middle pt. of AC and GE = $\frac{1}{2}$ CO = $\frac{1}{2}$BG.

$\therefore$ BE is a median and BE = BG + GE = $\frac{2}{3}p + \frac{1}{3}p = p$.

Similarly, it can be proved that CF is a median and CF = q.

Ex. 23. *Construct a triangle having given the three medians.*
[C. U. 1940 ; W. B. S. F. '33]

Given the lengths p, q, r, of the three medians. To draw the triangle.

Construction : Divide p, q, r into three equal parts. Take $GH = \frac{2}{3}p$.

With centres G and H and radii $\frac{2}{3}r$ and $\frac{2}{3}q$ respectively, draw two arcs of circles intersecting each other at C.

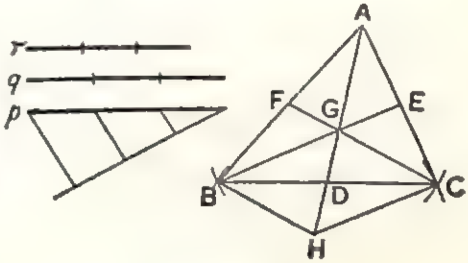


Fig. 107

Join GC and HC. Draw the par^m GCHB.

Produce HG to A so that AG=GH. Join AB and AC. Then ABC is the required triangle.

Proof : Produce BG and CG to cut AC and AB at E and F respectively.

$\because$ BGCH is a par^m, $\therefore$ GD=DH and BD=DC.

$\therefore$ AD is a median and AD=AG+GD=GH+$\frac{1}{2}$ GH $=\frac{2}{3}p+\frac{1}{3}p=p$. GE is drawn par^l to HC from the middle pt. G of the side AH of the $\triangle$AHC, $\therefore$ E is the middle pt. of AC and GE=$\frac{1}{2}$ CH=$\frac{1}{3}q$.

$\therefore$ BE is a median and BE=BG+GE=CH+$\frac{1}{2}$ CH=q.

Similarly, CF is the third median and CF=r.

Ex. 24. *Construct a triangle having given the base, the vertical angle and (i) one side, or (ii) the altitude or (iii) the median which bisects the base.*

Let BC be the base, $\angle$P the vertical angle and (i) a, one side or (ii) h the altitude or (iii) m the median which bisects the third side. To draw the triangle.

(i) **Construction :** On BC draw a segment BAC of a circle containing $\angle$P. With centre B and radius a draw an arc of a circle to cut the segment of the circle at A. Join AB and AC. ABC will be the required triangle, for by construction, $\angle$A=$\angle$P, AB=a.

(ii) **Construction :** Draw the segment of the circle as before, cut off DE=h from the perp. bisector DE of BC.

Through E draw RET ∥ BC to cut the segment of the circle at R and T. Complete the Δ^s RBC and TBC. They are the required triangles.

Proof: By construction $\angle BRC = \angle P$

and $\angle BTC = \angle P$ and RT ∥ BC,

∴ if perpendiculars are drawn from R and T on BC they (*i. e.* the altitudes of the two triangles) will be equal to DE, *i.e.*, h.

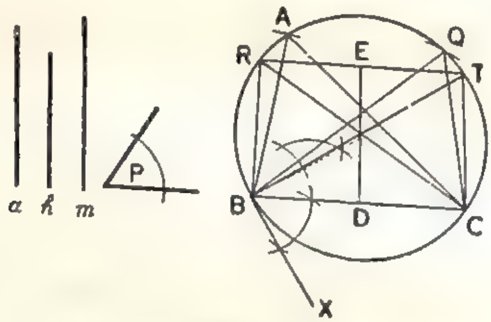


Fig. 108

(iii) **Construction**: [Draw the segment of the circle as before.] With the centre D, the middle pt. of BC, and with radius m, draw an arc of a circle to cut the segment of the circle at Q. Join BQ and CQ. Now, BQC is the required triangle, for $\angle Q = \angle P$ and the median $DQ = m$.

Ex. 25. *Construct a triangle having given the base, the vertical angle, and (1) the point at which the base is cut by the bisector of the vertical angle, or, (2) the point where the perpendicular from the vertex falls on the base.*

Let BC be the base, $\angle X$ the vertical angle and (1) D be the pt. at which the base is cut by the bisector of the vertical angle, or (2) K be the point where the perpendicular from the vertex falls on the base BC.

To draw the triangle.

(1) On BC draw the segment BAC of a circle containing an angle equal to $\angle X$ and complete the circle. Take P, the

middle pt. of the arc BPC. Join PD and produce it to meet the circumference at A.

Join AB and AC. Then ABC is the required triangle.

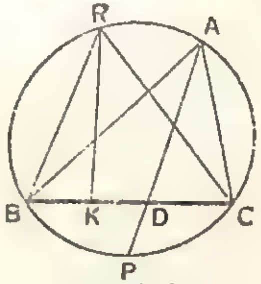


Fig. 109

Proof : $\angle A = \angle X$ (by construction). $\because$ arc BP = arc CP,

$\therefore$ $\angle BAP = \angle CAP$,

$\therefore$ the bisector AP of the vertical angle intersects the base BC at D.

(2) [Draw the segment of the circle as before. Draw KR$\perp$BC to cut the circle at R. Join BR and CR. Then RBC is the required triangle].

Ex. 26. *Construct a triangle having given the base, the vertical angle and the sum of the remaining sides.*

Given the base BC, the vertical $\angle x$ and the sum of the remaining sides s. To draw the triangle.

Construction. On the base BC draw a segment BAC of a circle containing $\angle x$ and a segment BDC of a circle containing $\frac{1}{2}\angle x$. With centre B and radius s draw an arc of a circle to cut the arc BDC at D. Join BD cutting the arc BAC at A. Join AC. Then ABC is the required triangle.

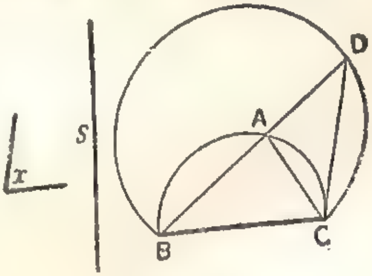


Fig. 110

Proof : $\angle BAC = \angle x$.
Again, the exterior $\angle BAC = \angle ADC + \angle ACD$.
But $\angle ADC = \frac{1}{2}\angle X = \frac{1}{2}\angle BAC$. $\therefore$ $\angle ACD = \frac{1}{2}\angle BAC$.
$\therefore$ $\angle ADC = \angle ACD$, $\therefore$ AC = AD, $\therefore$ AB + AC = AB + AD = s.

Alternative method. Take a st. line BD = s and at D draw $\angle BDC = \frac{1}{2}\angle x$. With centre B and with radius equal to the base draw an arc of a circle to cut DC at C. At C draw $\angle ACD = \angle BDC$. Let CA cut BD at A. Join BC. Then ABC is the required triangle.

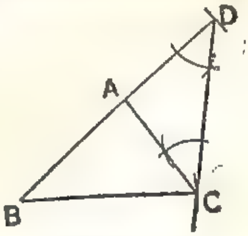


Fig. 111

Proof : $\because$ $\angle ACD = \angle ADC$, $\therefore$ AC = AD, $\therefore$ AB + AC = BD = s, and $\angle BAC = \angle ACD + \angle ADC = \frac{1}{2}\angle x + \frac{1}{2}\angle x = \angle x$.

Ex. 27. *Construct a triangle having given the base, the vertical angle and the difference of the remaining sides.*

Given the base BC, the vertical $\angle$PQR and the difference of the other two sides d.

On the base BC draw BDC a segment of a circle containing an angle equal to $90^\circ+\frac{1}{2}\angle$PQR. With centre B and radius d draw an arc of a circle to cut the arc BDC at D. Join BD and produce it to A. At C in DC draw $\angle$DCA $=\angle$ADC. Let CA and DA intersect at A. Then ABC is the required triangle.

Proof : $\because$ $\angle$ADC $=\angle$ACD, $\therefore$ AD $=$ AC,

$\therefore$ AB $-$ AC $=$ AB $-$ AD $=$ BD $=d$.

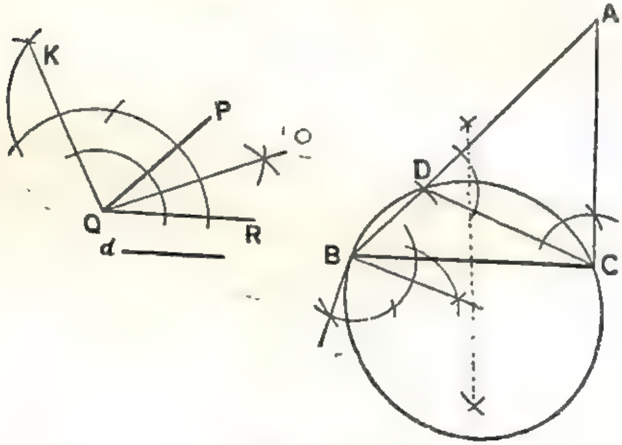


Fig. 112

Again, $\because$ $\angle$BDC$=90^\circ+\frac{1}{2}\angle$Q,

$\therefore$ $\angle$ADC $=$ supplement of $\angle$BDC $=90^\circ-\frac{1}{2}\angle$Q.

$\therefore$ $\angle$ACD $+\angle$ADC $=90^\circ-\frac{1}{2}\angle$Q $+90^\circ-\frac{1}{2}\angle$Q

$=180^\circ-\angle$Q. $\therefore$ $\angle$A $=\angle$Q.

[N. B. QO bisects the given $\angle$Q and KQ$\perp$QO.

$\therefore$ $\angle$KQO is a rt. angle and $\angle$OQR $=\frac{1}{2}\angle$Q.

$\therefore$ $\angle$KQR $=99^\circ+\frac{1}{2}\angle$Q.]

Alternative method. [See the above figure]. Take BD $=d$ and produce it. At D draw $\angle$CDA $=90^\circ-\frac{1}{2}\angle$Q (*i.e.*, $\angle$KQP). With centre B and radius equal to the base draw an arc to cut CD at C. Draw $\angle$DCA $=\angle$CDA. Then ABC is the required triangle.

Ex. 28. *Construct a triangle having given the base, the vertical angle and the median through either extremity of the base.*

Let BC be the base, $\angle$P the vertical angle and m the median from B to the opposite side. To draw the triangle.

Construction. Bisect BC at D. On DC draw a segment DEC of a circle containing $\angle$P. With centre B and radius m draw an arc of a circle to cut the segment at E. Join CE and produce it to E so that AE = CE. Join AB. Then ABC is the required triangle.

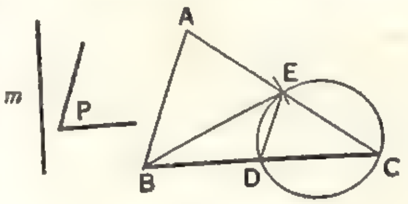


Fig. 113

Proof : $\because$ AE = CE, $\therefore$ BE is a median of $\triangle$ABC and it is equal to m. $\because$ D and E are the middle pts. of BC and CA, $\therefore$ DE $\parallel$ AB.

$\therefore$ $\angle$BAC = corresponding $\angle$DEC = $\angle$P.

Ex. 29. *Construct a triangle having given the base, the vertical angle and the perpendicular from one extremity of the base to the opposite side.*

[**Hints :** On BC draw a segment of a circle containing an angle equal to the given vertical angle and draw a semi-circle on BC as diameter. Draw an arc of a circle with centre C and radius equal to the given perpendicular to cut the semi-circle at D. Join BD.

Let BD or BD produced cut the segment of the circle at A. Join AC. Then ABC is the required triangle. Join CD. $\because$ $\angle$CDB is an angle in the semi-circle, $\therefore$ $\angle$CDB = 1 rt. angle,

$\therefore$ CD $\perp$ AB and it is equal to the given perpendicular.]

Ex. 30. *Construct a triangle having given the base, the vertical angle and the area equal to a given triangle.* [*A. U.*]

[**Hints :** On the base BC draw a segment of a circle containing the given vertical angle. $\because$ the area and the base of the triangle are given, $\therefore$ its altitude can be ascertained. (Vide Example 20 on page 40).

Now, at B draw a perp. on BC equal to that altitude and then proceed as in Example 24 (ii) for the remaining portion.

Ex. 31. *Given the vertical angle, one of the sides containing it and the perpendicular from the vertex on the base. Draw the triangle.*

Let $\angle A$ be the vertical angle, AB one of its sides and h the perp. from the vertex on the base. To draw the triangle.

Construction : At A in AB draw $\angle BAC = \angle A$. Draw a semi-circle on AB as diameter and an arc with centre A and radius h to cut the semi-circle at D. Join BD and produce it to cut AC at C. Then ABC is the required triangle.

Proof : $\angle ADB = 1$ rt. angle (being in a semi-circle).

$\therefore$ AD is perp. on the base and it is equal to h.

Ex. 32. *Construct a triangle having given the feet of the perpendiculars drawn from the vertices on the opposite sides.*

Let D, E and F be the feet of the perpendiculars drawn from the three vertices on the opposite sides. To draw the triangle. Form the $\triangle DEF$ joining D, E and F. Bisect the three angles of this triangle. Let the bisectors meet at O. At D, E and F draw perpendiculars to OD, OE and OF forming the $\triangle ABC$. Then ABC is the required triangle.

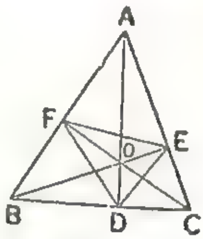


Fig. 114

Proof : Join AO, BO, CO.

$\angle ODB = \angle ODC$(rt. angles) and $\angle ODF = \angle ODE$, $\therefore \angle BDF = \angle CDE$; similarly, $\angle DEC = \angle AEF$, $\angle AFE = \angle BFD$.

Again, $\because$ $\angle OFB$ and $\angle ODB$ are each a rt. angle,

$\therefore$ the quadrilateral BDOF is cyclic, $\therefore$ $\angle BDF = \angle BOF$.

Similarly, $\angle CDE = \angle COE$; but $\angle BDF = \angle CDE$,

$\therefore$ $\angle BOF = \angle COE$. Similarly, it can be proved that $\angle BOD = \angle AOE$ and $\angle AOF = \angle DOC$,

$\therefore$ $\angle AOF + \angle BOF + \angle BOD$

$= \angle AOE + \angle COE + \angle COD = \frac{1}{2}$ of 4 rt. angles, $=$ 2rt. angles.

$\therefore$ AO and DO are in one st. line.

$\therefore$ AD is perp. to the side BC. Similarly, it may be proved that BE and CF are perps. to the other two sides.

$\therefore$ ABC is the required triangle.

Ex. 33. *Construct a triangle having given the base, the altitude and the radius of the circum-circle.*

Let BC be the base, r the circum-radius and h the height. Draw DE the perp. bisector of BC. With centre B and radius r draw an arc of a circle to cut DE at S. With centre S and radius SB draw the circum-circle.

Now from DE cut off DF equal to h. Through F draw AFA′ ∥ BC to cut the circle at A and A′. $\triangle^s$ ABC, A′BC are the required triangles. (Proof is easy).

Ex. 34. *Construct a triangle having given a vertex, the orthocentre and the circum-centre.*

Let A be a vertex, S the circum-centre and O the ortho-centre. To draw the triangle.

Construction : Join AS and draw the circum-circle with centre S and radius SA. Join AO. Produce AO to cut the circumference at E and let D be the middle pt. of OE. Through D draw the chord BDC perp. on AD. Join AB and AC. Then ABC is the required triangle.

Proof : Join BO and CO and produce them to cut AC and AB at F and G. Join BE. Now, $\triangle^s$ BDE, BOD are congruent, ∴ ∠DBE = ∠DBO. Again, ∠DBE = ∠DAC (angles at the circumference standing on the same arc). ∴ ∠DBO = ∠DAC. Now, in $\triangle^s$BDO, AOF, ∵ ∠DBO = ∠FAO, ∠BOD = ∠AOF, ∴ ∠BDO = ∠AFO, but ∠BDO = 1 right angle, ∴ BF⊥AC. Similarly, CG⊥AB.

Ex. 35. *Given the two base angles and the sum of two sides other than the base. To draw the triangle.*

[**Hints :** At any point O on a st. line XY draw ∠YOZ and ∠ZOK equal to the base angles. The remaining ∠XOK is then equal to the vertical angle of the triangle.

Take a st. line AP equal to the sum of the two given sides. At P in AP draw ∠APB = ½ of the vertical angle and at A draw ∠PAB equal to one of the base angles. At B in AB draw ∠PBC = ∠P so that BC cuts AP at C. Now, ABC is the required triangle. (Proof is easy.)]

Exercise 4

1. Construct a quadrilateral having given the lengths of the four sides and an angle. [C. U. '34]

2. Construct a triangle having-given two sides and an angle opposite to one of them. Explain the case where you get two solutions. Where does the construction fail ? [C. U. '36, '44 ; D. B. '34, '40, '43, '47, '48]

3. Construct a right-angled $\triangle$ABC, right-angled at A and having the sides AB $= 2''$ and hypotenuse BC $= 3''$. [C. U. '17]

4. Construct a triangle whose sides are 5, 12 and 13 inches. Measure the angle opposite to the longest side and the length of the perpendicular on the side from the opposite angle.

[**Hints** : Draw the figure taking 2 cm. for $1''$. The angle is a rt. angle and the perp. is $4\frac{1}{2}''$.]

5. Draw a triangle whose sides are 3, 4 and 5 inches. Bisect any two angles and draw a perpendicular from the intersection of the bisectors on any of the sides. Measure the length of the perpendicular. [C. U. '15]

6. Draw a right-angled triangle, given the hypotenuse $= 10$ cm. and the side $a = 6{\cdot}5$ cm. Measure the side b and find the value of $\sqrt{c^2 - a^2}$. [D. B. '36]

7. Construct a triangle equal in area to a given rectilineal figure. (State your construction and give a theoretical proof). [C. U. '37]

8. Describe a parallelogram equal in area to a given triangle and having an angle equal to a given angle. [C. U. '33, '35, '39 Sup., '43]

9. Draw a triangle equal in area to a given quadrilateral. [C. U. '34 ; D. B. '34, '35, '40]

10. Draw an isosceles triangle having given the vertical angle and the base.

11. Given one side and the sum of the hypotenuse and the other side of a right-angled triangle. Draw the triangle.

12. Construct a triangle having given the two base-angles and the difference of two sides other than the base.

13. Construct a triangle having given the base, the altitude and the length of the median which bisects the base.

14. Construct an isosceles triangle, having given the altitude and the base angles.

15. Divide the area of a given square into four parts from which two equal squares can be made up. [C. U. '32]

[**Hints :** Let the diagonals of the square ABCD intersect at O. Place $\triangle$BOC and $\triangle$AOD in such a way that BC and AD coincide and the pts. O are opposite to each other. Thus a square is obtained. Similarly, $\triangle$ s AOB and COD will form another square.]

16. Construct a right-angled triangle, having given the perimeter and one acute angle. [M. U.]

17. The base of a triangle is 3cm., the sum of the other two sides 5cm., and one base angle is 30°. Draw the triangle. (See Example 11]

18. Draw a triangle having given the base, one base angle and the perpendicular drawn from the vertex on the base.

[**Hints :** Given the base BC, one base-angle $\angle X$ and the length of the perpendicular P. At B in BC draw $\angle CBD = \angle X$ and draw $BE \perp BC$ so that $BE = P$. Draw $EA \parallel BC$ to cut BD at A. Join AC. ABC is the required triangle.]

19. Construct a triangle having given one of the base angles, the altitude and the sum of the other two sides.

[**Hints :** Draw $\angle XAY$ equal to the given angle. Draw $AD \perp AX$ and make AD = given altitude. Draw $DB \parallel AX$ to cut AY at B. Let AY = the sum of the two given sides. With centre B and with radius BY draw an arc of a circle to cut AX at C. Join BC. ABC is the required triangle,]

20. Find a point equidistant from three given points.

21. Bisect a given arc of a circle. [C. U. '26, '27]

22. Construct an isosceles triangle of given altitude equal in area to a given triangle.

23. Construct a square equal in area to a given rectangle. [C. U, '14, '20, '30 Addl,]

24. Describe a right-angled isosceles triangle equal to a given parallelogram.

25. Inscribe a circle in a given triangle. [C.U. '17, '19, '24 Addl.]

26. Circumscribe a circle about a given triangle. [C. U. '12, '13]

27. Construct a $\triangle ABC$, having $BC=4''$, $\angle B=30°$, $\angle A=90°$, forming the angles by a geometrical construction. Describe a circle to pass through the mid points of the sides. Measure the radius of the circle. Measure also AB and AC. [D. B. '37]

28. Draw a tangent to a circle from a given external point. [C. U. '11, '19, '24, '31, '38]

29. Draw a common tangent to two given circles and justify your construction. [C.U. Addl. '17, '19, '31]

30. On a given straight line draw a segment of a circle which shall contain an angle of 120°.

31. Draw a square equal to the sum of two squares whose sides are 2″ and 1·5″ respectively and measure its side.

Vide Ex. 25, on Page 42. side=2·5″] [D. B. '51]

32. In a given circle inscribe a traingle equiangular to a given triangle.

33. Construct the locus of points at which a given straight line subtends an angle of 30°. (Traces, of construction only are required). [W. B. S. F. '53]

[Hints : Let BC be the given st. line. On BC draw an equilateral $\triangle ABC$. With centre A and with radius AB draw an arc of a circle. Let BDC be the arc on the same side of A. The arc is the required locus.]

[N. B. No credit will be given if you draw an angle of 30° with the help of the protractor.]

34. Construct a par^m equivalent to a given triangle and having the same perimeter. [M.U.]

[Hints : Let ABC be the triangle and DC be half the base. Draw AEF ∥ BC. Draw a per^m CDEF on CD, so that its side $DE=\frac{1}{2}(AB+AC)$. Then CDEF is the required parallelogram.]

CONSTRUCTION OF CIRCLES

In order to draw a circle satisfying given conditions we must first know (i) the position of the centre and (ii) the length of the radius.

To find the position of the centre, two independent conditions or data are needed, each giving a locus on which the centre must lie ; so the point or points of intersection of the two loci are possible positions of the required centre.

The position of the centre being thus fixed, the radius is determined if we know (or can find) any point on the circumference. Thus one data is needed to find the length of the radius.

Hence in order to draw a circle three independent data are required.

Under certain conditions or data the locus of the centre of a circle will be either a straight line or the circumference of a circle. Before attempting the construction of circles the students should make themselves familiar with the following conclusions :

(1) The locus of the centres of circles which pass through two given points is the perpendicular bisector of the st. line joining the two points.

(2) The locus of the centres of circles which touch two intersecting st. lines is the bisector of the angles between the st. lines.

(3) The centres of the circles which touch two parallel st. lines will lie on a st. line parallel to and equidistant from them.

(4) The centres of circles which touch a given st. line at a given point lie on the perpendicular drawn on the st. line at the point.

(5) The locus of the centres of circles which touch a given circle at a given point is the radius (or radius produced drawn through the given point.

(6) If a circle of given radius touch a given circle, its centre will lie on the circumference of the circle which is concentric with the given circle and whose radius is equal to the sum or difference of two radii of the two circles.

(7) The locus of the centres of circles which touch two concentric circles is a concentric circle.

Examples

Ex. 1. *Draw a circle to touch a given st. line at a given point and to pass through another point.*

Let A be a fixed point on the given st. line RQ and B another given pt. To draw a circle to touch RQ at A and to pass through B.

Construction : Draw AO$\perp$RQ. Join AB and draw its perp. bisector PO. Let AO and PO meet at O. Now, draw a circle with centre O and with radius OA. This is the required circle.

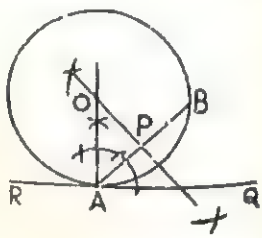


Fig. 115

Proof : $\because$ O lies on the perp. bisector of AB, $\therefore$ OA = OB ; so the circle drawn with centre O and with radius OA will pass through A and B.

Again, $\because$ RQ is perpendicular to the radius OA at the pt. A, $\therefore$ RQ is a tangent to the circle at A.

Ex. 2. *Draw a circle to touch a given circle at a given point and to pass through a given point outside the given circle.*

Let O be the centre of a given circle, A be a fixed pt. on its circumference and B be a pt. outside the circle.

To draw a circle to touch the given circle (O) at the given point A and to pass through B.

Construction : Join AB and draw PQ the perp. bisector of AB. Let PQ cut OA produced at Q.

With centre Q and radius QA draw a circle, which is the required circle.

Proof : Join BQ. $\because$ PQ is the perp. bisector of AB, $\therefore$ QA=QB. $\therefore$ the circle drawn with centre Q and with radius QA will pass through B. Again $\because$ the two circles meet at A on their line of centres, $\therefore$ they touch each other at A. $\therefore$ the circle drawn with centre Q and radius QA is the required circle.

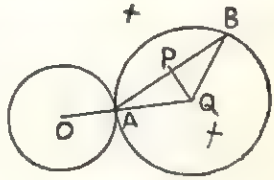


Fig. 116

Ex. 3. *Draw a circle of given radius to touch a given circle and to pass through a given point.*

Let O be the centre of the given circle, A the fixed point and r the given radius.

Construction : Suppose R to be the radius of the given circle. Draw an arc of a circle with centre O and radius R+r and draw another arc of a circle with centre A and with radius r cutting the former arc at P. Join OP cutting the circle at B. Now, with centre P and radius PB draw a circle, which will be the required circle.

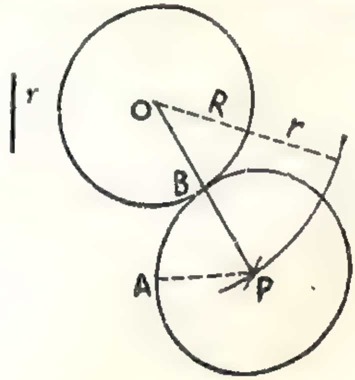


Fig. 117

Proof : OP=R+r, and OB=R, $\therefore$ PB=r=PA.

$\therefore$ the circle will pass through B and A. Again, the two circles meet each other at B on their line of centres OP, $\therefore$ they touch each other at B.

[How many such circles are possible ?]

Ex. 4. *Drow a circle of given radius to touch two intersecting straight lines.*

Let AX and AY be two intersecting st. lines and r the given radius.

Construction : Bisect $\angle$A by OA. Draw AR$\perp$AX so that AR=r. From R draw RO $\parallel$ AX cutting AO at O. Draw OP$\perp$AX

and $OQ \perp AY$. The circle drawn with centre O and with radius OP will be the required circle.

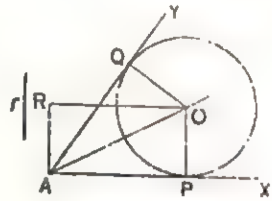


Fig. 118

Proof : $\because$ AO is the bisector of $\angle A$, $\therefore$ $OP = OQ$ and they are perp. on AX and AY. $\therefore$ the circle drawn with centre O and radius OP will pass through P and Q and touch AX and AY at those points. Now, $\because$ APOR is a par^{m}, $\therefore$ $OP = AR = r$.

[N. B. Four such circles can be drawn. Draw the other three circles.]

Ex. 5. *Describe a circle which will touch two parallel st. lines and pass through a given point.*

Let PQ and XY be two parallel st. lines and A be a fixed point. Describe a circle passing through A and touching PQ and XY.

Construction : At any pt. M on XY draw $MR \perp XY$. Let MR cut PQ at R. Draw CD the perp. bisector of MR. With centre A and with radius CM draw an arc of a circle cutting CD at O. With centre O and radius OA draw a circle, which will be the required circle.

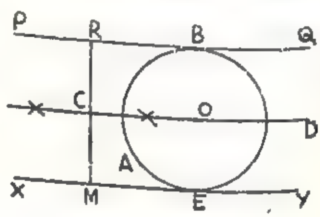


Fig. 119

Proof : $\because$ RM is the distance between PQ and XY and CD is the perp. bisector of RM, $\therefore$ the centres of the circles touching PQ and XY must lie on CD and their radii will be equal to CM. Here the radius $OA = CM$ and the centre is on CD. $\therefore$ ABE is the required circle.

[N. B. (1) $\because$ the arc of the circle drawn with centre A and with radius OA will cut CD at another point, two such circles are possible.

(2) It is impossible to draw the circle, if the point A is outside PQ and XY.]

Ex. 6. *Describe a circle to touch a given circle, have its centre in a given st. line and pass through a given point in the st. line.*

Let O be the centre of the given circle and A a fixed point on a given st. line PQ.

Construction. Draw radius OB ∥ QP and join BA cutting the circle at C. Join OC and produce the it to meet PQ at D. With centre D and radius DA draw a circle, which will be the required circle.

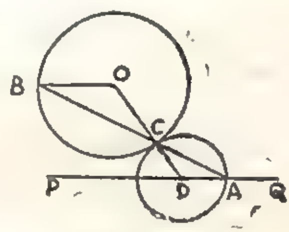


Fig. 120

Proof: ∵ OB ∥ QP, ∴ ∠OBA = alternate ∠BAD. Again, ∠OBC = ∠OCB (∵ OB = OC) = vertically opposite ∠ACD. ∴ ∠ACD = ∠CAD, ∴ DC = DA.

∴ the circle drawn with centre D and with radius DA will pass through C. Again, this circle meets the given circle (O) at a point C on their line of centres, ∴ the two circles will touch each other at C.

Ex. 7. *To draw a circle to touch a given circle and a given straight line at a given point.*

Let P be a fixed point on a given st. line XY and O be the centre of the given circle.

Construction: Draw OD⊥XY and let OD produced cut the circumference at C and F.

Join CP to cut the circumference at E. Draw PQ⊥XY.

Join OE and produce it to cut PQ at Q. With centre Q and with radius QP draw a circle. This will be the required circle.

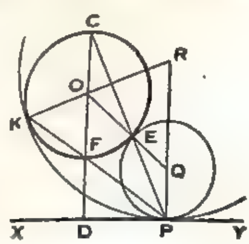


Fig. 121

Proof: ∵ CD and QP are both perpendiculars. on XY, ∴ CD ∥ PQ,

∴ ∠OCE = alternate ∠EPQ.

Again, ∠OCE = ∠OEC (∵ OC = OE) = ∠QEP

∴ ∠QEP = ∠EPQ, ∴ QE = PQ.

$\therefore$ the circle drawn with centre Q and with radius QP will pass through the pts. P and E.

$\because$ the radius $QP \perp XY$, $\therefore$ the circle will touch XY at P.

$\because$ the two circles meet each other at pt. E on their line of centres OQ, $\therefore$ they have a common tangent at E.

$\therefore$ the two circles touch each other at the pt. E.

[N. B. OD cuts the circle at F. Suppose PF cuts the circumference at K and KO produced meets PQ at R. The circle drawn with centre R and with radius RP will also be the required circle.

Ex. 8. *Draw a circle to touch a given st. line and a given circle at a given point.* [*C. U.*]

[Vide the diagram of Example 7]

Let XY be the given st. line, O the centre of the given circle and E the given point on the circumference. Draw $OD \perp XY$ and produce DO to cut the circumference at C.

Join CE and produce it to cut XY at P, draw $PQ \perp XY$. Join OE and produce it to cut PQ at Q. The circle drawn with centre Q and radius QP will be the required circle.

[There can be another such circle. Draw that circle.]

9. *Draw a circle of given radius to touch a given straight line and a given circle.*

Let r be the given radius, XY the given st. line, O the centre of the given circle whose radius is R.

To draw a circle of radius r to touch the circle (O) and the st. line XY.

Construction : Draw $PQ \parallel XY$ at a distance equal to r from XY. With centre O and with radius $R+r$ draw an arc of a circle to cut PQ at A and B. Draw $AC \perp XY$.

Now, with centre A and with radius AC draw a circle, which is the required circle.

Proof : Join OA and let it cut the given circle at D.

$\because$ $AO = R+r$ and $OD = R$, $\therefore$ $AD = r = AC$.

$\therefore$ the circle drawn with centre A and with radius AC passes through C and D.

Now, $\because$ the circle (O) and the circle (A) meet each other at D on their line of centres AO, $\therefore$ the two circles touch each other at D. Again, XY is perp. on the radius AC,

$\therefore$ the circle (A) touches XY at C.

Exercise 5

1. Describe a circle of given radius to pass through two given points. [C. U. '32]

2. Draw a circle having its centre on a given st. line and passing through two given points.

3. Draw a circle, of given radius, to touch a given st. line and a given circle.

4. AB and CD are two parallel st. lines. Describe a circle to pass through A and B and to touch CD.

5. Draw a circle of given radius to pass through a given pt. and to have its centre on a given st. line. [C. U. '26]

6. Draw two circles of radii 2 cm. and 3 cm. respectively to touch each other.

7. Draw a circle to touch a given circle and a given st. line. [C. U.]

8. Draw a circle of radius ·8″ touching two intersecting st. lines.

9. OA and OB are two intersecting st. lines and P is a point on OA. Draw a circle to touch OA at P and OB at a point. [C. U.]

10. Draw a circle to touch two given intersecting st. lines and to have its centre on another given st. line.

11. Draw a circle touching two parallel straight lines and a transversal.

12. Describe a circle to pass through a given point, to touch a given st. line and to have its centre on another given st. line.

13. Draw a circle of radius 3 cm. to pass through two given points A and B, where AB = 4·8 cm. Find by calculation and measurement the length of the perpendicular from the centre upon AB.

[Traces and statements of construction only are required.]
[Ans. 1·8 cm.] [W. B. S. F. '52]

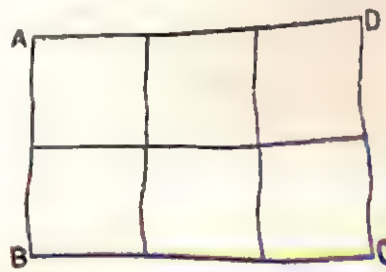
14. A point P is 4·5 cm. distant from a st. line AB. Draw two circles each of radius 3·2 cm. to pass through the point P and to touch AB.

15. Draw a circle to touch a given circle and a given st. line and to pass through a given point.

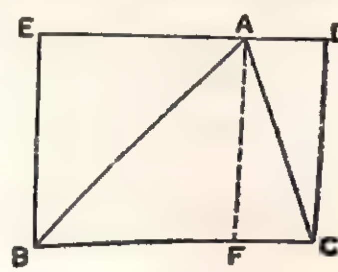
Experiments

We discuss here the methods of finding experimentally the areas of rectangles, triangles and circles.

Area of a rectangle: You know that the area of a rectangle = its length × breadth. To find it experimentally draw on a piece of paper a rectangle ABCD, 3 cm. long ued 2 cm. broad, and then fix the paper on a board. Now from a sheet of paper cut off several small square pieces, each side of them being 1 centimetre. Then evidently the area of each such square is 1 square cm. Now just cover the rectangle by placing the small square pieces side by side. You will find that the rectangle has been just covered with 6 square pieces. Hence the area of the rectangle is found to be equal to 6 sq. cm. or 3×2 sq. cm, i.e., the length × breadth of the rectangle.



Area of a triangle: Draw any triangle ABC on a piece of paper. With the help of a setsquare draw two perpendiculars on BC at B and C and through A draw EAD parallel to BC. Thus you get a rectangle BCDE. Now cut off the triangles EAB and ACD with a pair of scissors and place them on the $\triangle$ABC, so that AE falls along BC, B falls on A, and AD along CB and C on A. Then you find that the two triangles entirely coincide with the $\triangle$ABC. Hence, you understand that the area of $\triangle$ABC is just half the rectangle BCDE. You have already found that the area of a rectangle is equal to its length × breadth. You find from the diagram that the length of therectangle is equal to the base BC of the $\triangle$ABC and its breadth BE is equal to the height or altitude of the triangle.



Hence, the area of a triangle $= \frac{1}{2}$ base × altitude.

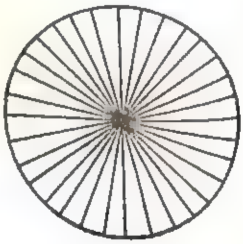
Circumference of a circle : Draw a circle on a piece of paper and paste it on a pitch-board. Now fix several pins along the circumference of the circle and then place a fine thread just round the circumference. The length of this thread is evidently the measure of the circumference.

You already know that the circumference of a circle $=2\pi r=\frac{22}{7}\times$ diameter. To find it experimentally draw two circles on two diameters, 7 cm. and $3\frac{1}{2}$ cm. in length, and measure the circumferences of the two circles as stated above. You will now find that in both cases $\frac{\text{the circumference}}{\text{the diameter}}$ is the same and is approximately equal to $\frac{22}{7}$.

So, in any circle, $\frac{\text{circumference}}{\text{diameter}}=\frac{22}{7}$ (or π).

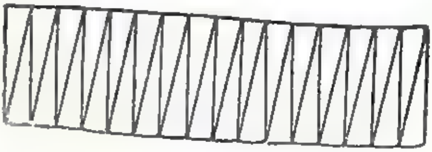
Hence the circumference of a circle $=\frac{22}{7}\times$ diameter.

Area of a circle : On a piece of paper draw a circle with radius 1 cm. long, and draw two diameters of the circle perpendicular to each other. Thus the circle is divided into 4 equal sectors, each angle of the sectors being 90°. Now bisect all the angles at the centre and then each angle of the sectors will be 45°.



Again bisect the angles at the centre twice in the same way. Thus the circles will be divided into 32 equal sectors, each of whose angles is $(11\frac{1}{4})°$

Now cut off these 32 sectors with a pair of scissors and paste them side by side in two rows, 16 in each row, as shown in the diagram. You will then find that almost a rectangle has been formed. If you go on dividing the circle into a greater number of sectors, their arcs will gradually become straight

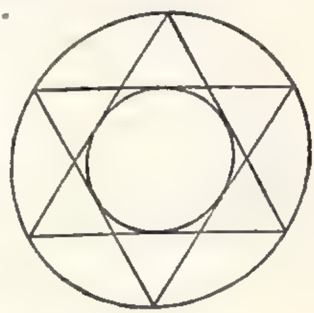


lines. The breadth of the rectangle formed is evidently equal to the radius of the circle, i.e., 1 cm. Now measure with a scale the length of the rectangle and determine its area (length × breadth). This is the area of the circle also. You will find that almost the same area is also found from the formula, area of a circle $= \pi \times (\text{radius})^2$.

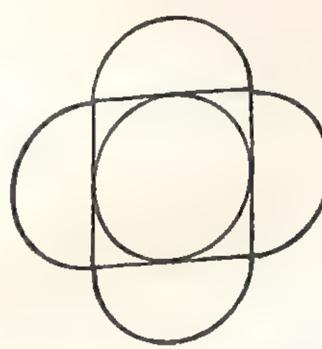
Construction of Designs with Geometrical Figures

Many designs can be constructed by drawing geometrical figures. See the following illustrations :

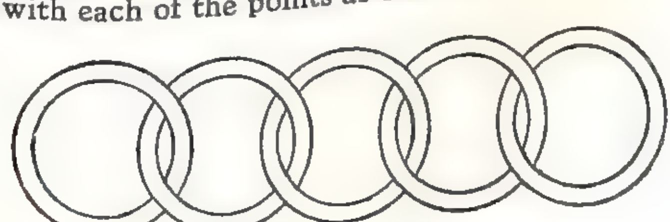
Ex. 1. Here you find the design of a star. To construct it, first draw a circle and then divide its circumference into six equal parts, each equal to the radius of the circle. Now join the points alternately. Then draw a circle, concentric with the first circle and touching the sides of the two triangles.



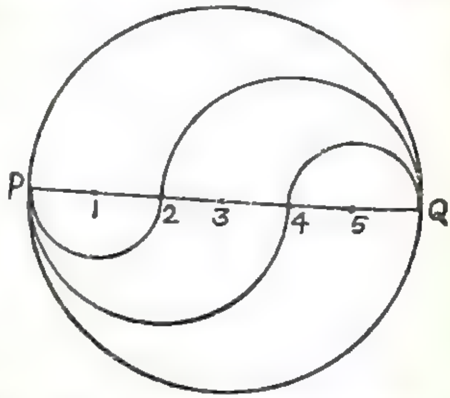
Ex. 2. To construct the design given here, first draw a square and describe 4 semi-circles on its sides as diameters. Then draw a circle touching the sides of the square.



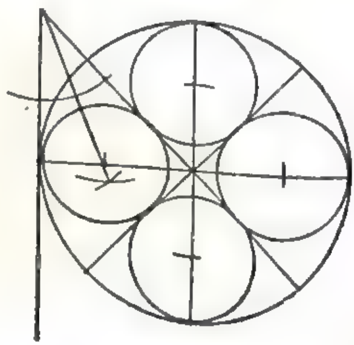
Ex. 3. To draw the chain given below, take a st. line and several points on it. Then draw a pair of concentric circles with each of the points as centre.



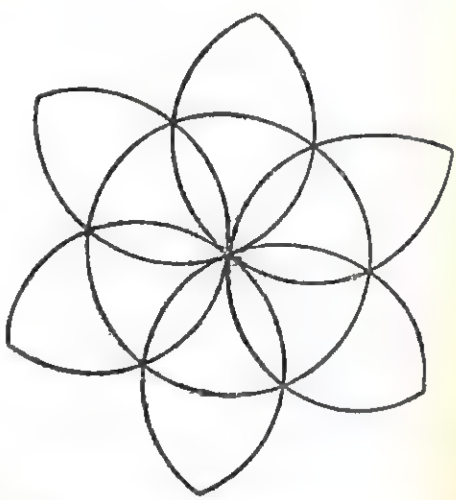
Ex. 4. Divide the st. line PQ into six equal parts. With the first and the fifth points of division as centres and with radius equal to one part draw two semi-circles, one on each side of PQ. Again with the second and fourth points of division as centres and the same radius as before draw two semi-circles, one on each side of PQ. Thus the design given here is constructed.



Ex. 5. Enlarge the design given here.



Ex. 6. Here is given the design of a lotus. First draw a circle of any radius. Then draw an arc of a circle with any point on the circumference as centre and with the same radius. It will cut the circle at two points. With these two points as centres and the same radius draw an arc cutting the circle at two points and so on. Thus the design will be constructed.



Exercise

Draw the following designs geometrically.

1

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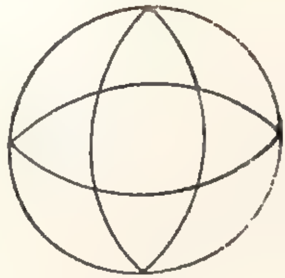

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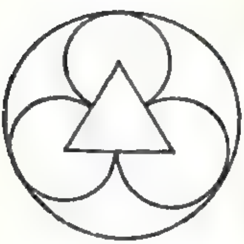

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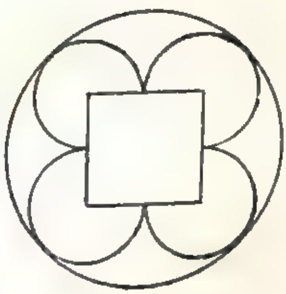

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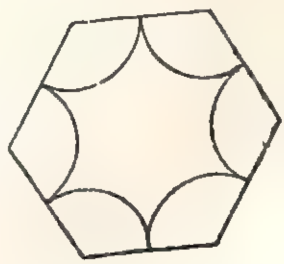

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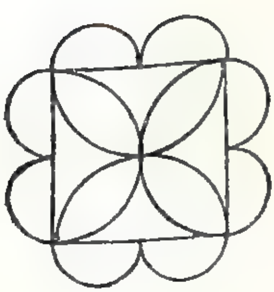

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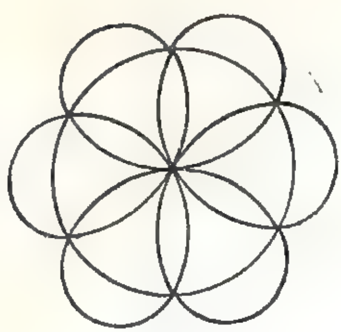
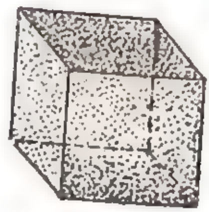


8

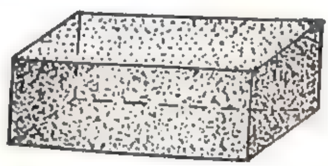
Explanation of Models

Cube : You see the model of a *cube* here. It has six faces (surfaces) and each surface has four edges (sides). Two adjacent edges of each surface meet each other at right angles. A cube has 12 edges and 8 vertices. At each vertex three edges meet at right angles to one another. The 6 surfaces of a cube are equal and each is a square. Here two surfaces meet at right angles.



Cube

Rectangular parallelopiped. Here you find the model of a rectangular parallelopiped. It has three pairs of parallel plane faces (surfaces). Each of its faces is a rectangle and each pair of opposite faces are equal. It also has 12 edges and 8 vertices.

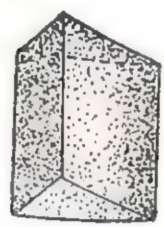


Rect. parallelopiped

Prism : A prism is a solid bounded by plane faces, two of which called the *ends*, are parallel and congruent, while the others, called *side faces*, are parallelograms.

If the *side-edges* of a prism are perpendicular to the *ends*, it is called a *right prism*.

The side-faces of a right prism are rectangles. The side-edges of any prism are parallel and equal.

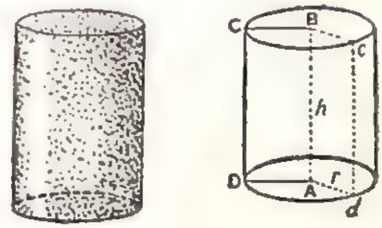


Prism

The *ends* of a prism may be triangles, quadrilaterals or polygons of any number of sides, and they are said to be triangular, quadrilateral or polygonal prism respectively. Here is given the diagram of a triangular prism, whose two ends are parallel and two congruent triangles.

A right Cricular cylinder : The solid, generated by the revolution of a rectangle about one of its sides as axis, is called a *right circular cylinder*.

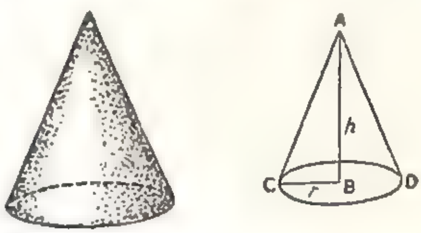
It has 3 faces of which one is a curved surface and two are plane surfaces. These two plane surfaces, called its *ends* or *bases*, are two equal and parallel circles.



Cylinder

A right circular cone : A right circular cone is a solid generated by the rotation of a right-angled triangle about one of the sides containing the right angle as axis.

In the diagram, the $\angle B$ of the $\triangle ABC$ is a right angle. If the triangle revolves about the side AB as axis, the point C will describe a circle. This circle is the *base* of the cone and A is its vertex. It appears from the given figures that a circular cone is generated by joining the points on the circumference of a circle to a point outside the circle and non-coplanar with it. The axis of a right circular cone is perpendicular to its circular base at its centre.

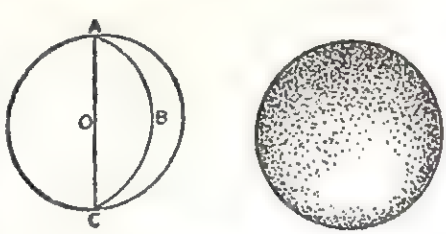


Cone

A cone has two surfaces, one curved and one plane, the base being a plane surface.

Sphere : The sphere is a solid generated by the revolution of a semi-circle about its diameter as axis.

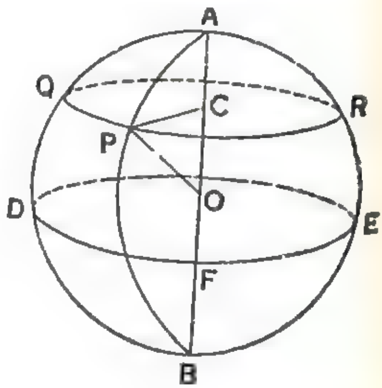
So, a sphere has only one surface, which is a curved surface. The centre and radius of the semi-circle are the centre and radius of the sphere. The st. lines drawn from the centre of a sphere to its surface are equal.



Sphere

Geometry of the Sphere

You know that a *sphere* is a solid generated by the rotation of a semi-circle about its diameter as axis.



In the figure here the sphere ADBR has been generated by revolving the semi-circle APB about its diameter AB as axis.

The surface described by the semi-circle by its revolution about AB is the *curved surface* of the sphere. All points on the circumference of the semi-circle are equidistant from the centre of the sphere (the distance being equal to r, the radius of the sphere). Hence we may say that the surface of a sphere is the locus in space of all points equidistant from a fixed point. In the figure the fixed point O is the *centre* and the constant distance OP is the *radius* (r) of the sphere.

Some Properties of the Sphere

(1) If circles are drawn on the surface of a sphere with any point on the surface as the centre and with any radius, their circumferences will be parallel to one another; for the distance between any two circumferences will remain constant.

(2) If a sphere be cut into two parts along its diameter, we get two hemispheres equal in all respects. So the sphere is a perfectly symmetrical solid. Each hemisphere has two surfaces, one plane and one curved.

(3) Every plane section of a sphere must be a circle.

Proof : In the above diagram let the plane PQR cut the sphere whose centre is O and radius OP. From O draw OC perpendicular to the cutting plane and join OP, PC.

Now, $\because$ OC is perpendicular to the cutting plane and CP is a st. line on that plane,

$\therefore$ $\angle OCP$ is a right angle.

$\therefore$ $OP^2 = OC^2 + PC^2$, or, $PC^2 = OP^2 - OC^2$,

$\therefore$ $PC = \sqrt{OP^2 - OC^2}$ = a constant.

Hence, since P is any point on the line of section and is always equidistant from C, the locus of P (*i.e.*, the curved line PQR) is a circle (*i.e.*, its circumference) whose centre is the point C.

[N. B. The perp. OC, drawn from the centre O to the cutting plane, being produced both ways cuts the sphere at two points A and B. These two points are said to be the **poles** and the diameter AB is called the **axis** of the cutting section].

(4) If any two points on the surface of a sphere be joined to the centre, the angle produced at the centre is the **angular distance** between the points.

[N. B. As the surface of the sphere is a curved one, the distance between any two points on it is expressed in angular units, degrees, minutes etc., but not in linear units.]

(5) In the given figure DEF also is a cutting circle whose centre lies on the centre of the sqhere. The centre of the other cutting section PQR, however, does not lie on the centre of the sphere.

Here you notice that the circle DEF is greater than the circle PQR, because the radius of the former is greater than that of the latter (or of any other section whose centre does not coincide with O). The radius of a plane section of a sphere diminishes as the distance of the plane from the centre increases.

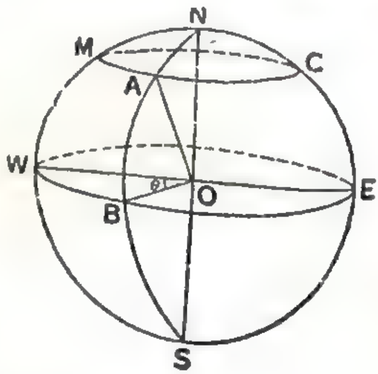
(6) The plane that is perpendicular to the diameter at any one of its extremities touches the sphere at the point and is called a **tangent plane** of the sphere.

(7) Only one tangent plane can be drawn to a sphere at a point on its surface.

Latitude and Longitude

You have already read about *latitude* and *longitude* in Geography. The earth is taken to be a sphere, as it is almost round in shape. Of course all the diameters of a sphere are equal, but the diameter of the earth joining the north pole and the south pole is a little less than its other diameters. As this small difference is negligible in comparison with the vast size of the earth, the earth is regarded as a sphere.

Latitude : As in a sphere, the diameter passing through the centre of the earth up to its surface on the North and South is called the **axis** of the earth. In the given figure NS is the axis of the earth. The points of sections of the axis and the surface of the earth are called its **Poles**. The pole on the north is called the **North Pole** and that on the south is the **South Pole**. In the figure, N is the north pole and S the south pole.



It is, however, to be remembered that actually there exists no line like NS through the earth nor is there any circle drawn on its surface. So these are *imaginary* lines or circles.

Now, if we imagine that infinite number of circles have been described on the surface of the earth with the poles as centres and with different radii, then these imaginary circles are called Latitudes. You know that the circumferences of these circles are parallel. So the circles are called **Parallels of Latitude.**

These circles may be imagined in another way. In the above figure the diameter NS is the axis of the earth. Similar circles may be drawn with points on NS as the centres.

The latitude equidistant from the two poles, i.e., the great circle drawn with centre O on the surface of the earth is called the **Equator**. In the figure, WEB is the equator.

Longitude : The circumferences of the imaginary semi-circles passing through the poles (N and S) are called the **Meridian or Longitude.**

In the figure, NABS is a longitude.

It is evident that there may be an infinite number of latitude and longitudes. You have read in Geography that the meridian that passes through Greenwich is regarded as the **Prime Meridian.**

There is a purpose for imagining these lines. In Algebra, you know, two st. lines intersecting at right angles to each other are taken as lines of reference (*i.e.*, as axes) and their point of intersection as the origin, and the position of a point on the graph paper can be located, if its distances from these axes are known. The measure of these distances are known as the co-ordinates of the point. Also the co-ordinates of a point on the graph paper may be found, if its position is known.

Similarly of the imaginary lines stated above, two such lines may be taken as the prime lines or lines of reference for ascertaining the position of a particular place on the surface of the earth. The **equator** and the **prime meridian** are the said two lines of reference.

The angular distance of any place from the equator is the latitude of that place, and the angle between the meridian of a place and the prime meridian is the longitude of the place. So the position of a place on the surface of the earth can be determined, if its latitude and longitude are known. Conversely, the latitude and longitude of a place can be determined, if its position on the earth is known.

It is to be noted that the latitude and longitude of the point of intersection of the equator and the prime meridian

are both zero degree. The latitude of any place lying on the equator is evidently zero degree. So also the longitude of any place on the prime meridian is zero degree.

Hence as the poles lie on the prime meridian the longitude of each pole is zero degree.

The latitude of a place above (*i.e.*, on the north of) the equator is called '**latitude north**' and that of any place below (*i.e.*, on the south of) the equator is called '**latitude south.**' Similarly, the longitude of any place lying on the east (on the right side) of the prime meridian is said to be '**longitude east**' and that of any place on the west (on the left side) of the prime meridian is called '**longitude west**'.

Thus the latitude of any place lying 10° north of the equator is **10° North**, and that of any places at 10° south of the equator **10° South.** Similarly the longitudes of two places lying on two meridians, one 15° east and the other 15° west of the prime meridian are 15° East and 15° West respectively.

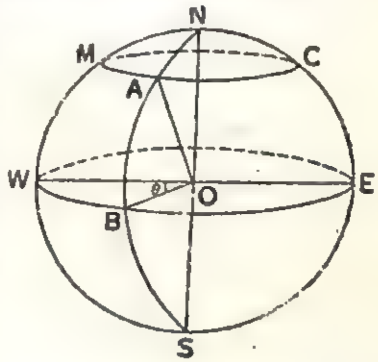
The latitude of the equator is 0° and the longitude of the prime meridian is also 0°. From the equator to the north pole as well as to the south pole, the latitudes are from 0° to 90°, for one-fourth of the circumference of a circle (here the arc from the equator to a pole) subtends a right angle at a centre of a circle. Hence, the latitude of the north pole is 90° North and that of the south pole is 90° South.

Similarly, there are longitudes from 0° to 180° to the east as well as to the west of the prime meridian.

Determination of Latitude

In the given diagram let WBE be the equator and CAM be any *parallel of latitude* on which A is any point. Suppose the meridian passing through A cuts the equator at B, and the arc AB subtends $\angle AOB$ at the centre O.

Now, the arc AN is taken to be the distance of A from N. Since at any position of A on the parallel of latitude CAM, the distance AN is constant, $\therefore$ the $\angle AON$ is always constant. Again, since ON is perpendicular to the plane WBE, the $\angle NOB$ is a right angle. So $\angle AOB = \angle NOB - \angle AON$ = a constant, and therefore the arc BA is constant. The distance of A from the equator is the arc AB, which is the latitude of the point A. Hence, the angular distance of any place from the equator is its latitude. It may be expressed otherwise. The meridian NAS passing through A cuts the equator at B. The distance of A from the pt. B along the meridian BN (i.e., the arc BA) is the latitude of A.



Determination of Longitude

The longitude of a place is the measure of the angle between the plane of the meridian passing through the place and the plane of the prime meridian. In the figure the angle WOB or θ is the angle between the plane of the prime meridian NWS and the plane of the meridian NAS passing through A. The measure of the angle θ is the longitude of A.

The position of a place on the surface of the earth can be determined if its latitude and longitude are known. Suppose the latitude of a place is 20°30′ North and its longitude is 60°12′ West. To find its position move 60°12′ to the west (to the left) of the prime meridian along the equator and therefrom move 20°30′ to the north (i.e., upwards) along the meridian and you get the position of the place.

ANSWERS

ARITHMETIC

Exercise 1

1. 312, 624 ; **2.** 6903145937 ; **3.** 144 ; **4.** 101793 ;
5. A 18, B 57, C 33 ; **6.** 74 ; **7.** 1266000 ; **8.** yes ;
9. 6480 ; **10.** Rs. 375 ; **11.** A Rs. 158, B Rs. 237, C Rs. 316 ;
12. 57980 ; **13.** 600 ; **14.** 22154 ; **15.** 685 in place of 635 ;
16. A Rs. 55 ; B Rs. 79 ; C Rs. 21 ; **17.** 96660 ; **18.** 3364647 ;
19. 3330 ; **20.** Rs. 8 ; **21.** £ 3. 10 s. **22.** 9 secs.

Exercise 2

1. 2520 ; **2.** 84 ; **3.** 6759 ; **4.** 99792 ; **5.** 100023 ;
6. 360 ; **7.** after 3 hours ; 20, 15, 10, 9, 6 times respectively ;
8. 29 yds. 2 ft. 3 in. ; **9.** 315 and 378 ; 315 and 441 ; 378 and 441 ; 315, 378 and 441 ; **10.** 3, 11, 33, 59, 177, 649, 1947 ;
11. 1836 ; **12.** 53758063, 31663 ; **13.** 99491 ; **14.** 18 and 108, or, 36 and 54 ; **15.** (i) 274 ; (ii) 2166 ; **17.** 1020 ;
18. 343, 5929 ; **19.** 4 and 224, or, 28 and 32 ; **20.** 6 hrs. ;
21. 42 ; **22.** 624, 54 ; **23.** 1015 ; **24.** 493 ; **25.** 48, 52 ;
26. 49, 56 ; **27.** 10023 ; **28.** 4a. 6p. ; **29.** 37 ; **30.** 748 ;
31. 97200 ; **32.** 9999960 ; **33.** 14403 ; **34.** 147, 777 ;
35. 15160 ; **36.** 63, 144 ; **37.** $\frac{4}{88}$, 64 ; **38.** ·7, 42.

Exercise 3

1. $\frac{19}{480}$; **2.** ·1 ; **3.** $\frac{5}{8}$; **4.** 1 ; **5.** 5 ; **6.** $\frac{11}{16}$; **7.** 75 ;
8. $\frac{1}{8}$; **9.** $1\frac{2}{3}$; **10.** $\frac{9}{10}$; **11.** 4 ; **12.** 0 ; **13.** 1 ;
14. 1 ; **15.** $\frac{1}{40}$; **16.** 1 ; **17.** $\frac{12}{49}$; **18.** $1\frac{11}{16}$; **19.** $\frac{11}{16}$;
20. 72 ; **21.** ·0075 ; **22.** $\cdot 0\dot{0}\dot{9}$; **23.** $\cdot 0348\dot{3}$; **24.** $\cdot 2\dot{4}$;
25. $\frac{7}{12}$; **26.** 2 ; **27.** 1 ; **28.** 2 ; **29.** 25 ; **30.** 2 ;
31. 2 ; **32.** 8 as. ; **33.** 8 ; **34.** 1 ; **35.** £8. 1s. ;
36. Rs. 373. 3 as. $3\frac{15}{107}$ p. ; **37.** 1 ; **38.** ·0015625 ;
39. $\cdot 00056\dot{8}\dot{1}$; **40.** $\cdot 005208\dot{3}$; **41.** $\cdot 0041\dot{6}$; **42.** ·15625 ;
43. £ 7. 10 s. ; Rs. 100 ; **44.** $11\frac{35}{41}$; **45.** 70 m. ;
46. 3 ; **47.** 1 ; **48.** $\frac{1}{24}$, $\cdot 041\dot{6}$; **49.** 1 ; **50.** 2 ;
51. 1 ; **52.** 1 ; **53.** ·03 ; **54.** 1 ; **55.** ·3 ; **56.** 1 ;
57. $\frac{1}{28}$, $\cdot 0\dot{3}\dot{6}$.

Exercise 4

1. 60 men ; 2. 6 men ; 3. 180 men ; 4. 1430 men ; 5. 25 men ; 6. $5\frac{6}{11}$ days ; 7. 14 weeks ; 8. 56 days ; 9. 216 men ; 10. 18 ; 11. 2 days ; 12. 50 ; 13. $15\frac{5}{8}$ days ; 14. 30 g. 15. Rs. 760. 50 P. ; 16. 270 men ; 17. 19s. 3d. ; 18. 125 days ; 19. Rs. 246.

Exercise 5

1. 12 yrs. ; 2. 9 yrs. ; 3. 26 yrs. ; 4. 1·0094 ; 5. 40 m. p. h. ; 6. 1032 ; 7. 119 Kg. ; 8. $34\frac{2}{7}$ Km. per hr. ; 9. 7·28 yrs. (app.) ; 10. 25.

Exercise 6

1. 1234 ; 2. 1679 ; 3. 31623 ; 4. 2501317 ; 5. 2307 ; 6. 12000 ; 7. 13·057 ; 8. 5·403 ; 9. 54·0321 ; 10. ·0325 ; 11. $1\frac{7}{9}$; 12. $\frac{6}{7}$; 13. $3\cdot\dot{3}$; 14. $\cdot\dot{6}$; 15. (*a*) $10\frac{13}{27}$, (*b*) 1·4142136 ; (*c*) ·5641 ; 16. ·632 ; 17. 1·0000 ; 18. 1·00015 ; 19. 6. 20. ·99999 ; 21. 209 ; 22. 195 ; 23. 35 ; 24. ·0006.

Exercise 7

1. 36 m. ; 2. Rs. 366. 10 as. 8 p. ; 3. 84 m. ; 4. 1100 yds. ; 5. Rs. 128. 13 as. $1\frac{1}{3}$ p. 6. (i) Rs. 1299. 6 as. ; 6. (ii) 810 ares, 360 ares ; 7. Rs. 4. 1 a. ; 8. Rs. 7776 if path is outside, Rs. 7605. 5 a. 4 p. (inside) 9. Rs. 385. 8 a. 8 p. ; 10. Rs. 4. 90 P. ; 11. Rs. 57. 12 a. ; 12. 10 m. ; 13. Rs. 17. 2 a. ; 14. 25 m. ; 15. 744 ; 16. 10 m. ; 17. 21 m., 7 m. ; 18. Rs. 3531. 19. Rs. 125 ; 20. Rs. 20. 3 a. $2\frac{2}{3}$ p. 21. 2 ft. 22. 10 m. 23. Rs. 15326. 10 a. 8 P., Rs. 7340.

Exercise 8

1. 19 sq. yd. 8 sq. ft. 16 sq. in., 5 cu. yd. 25 cu. ft. 2. 1 cu.m. 728 cu. dm. ; 3. 2 ft. 4 in. ; 4. 27 ; 5. $4\frac{1}{3}$ cm. 6. 4 ft. 6 in ; 7. 13 sq. yd. 4 sq. ft., 11 $\sqrt{3}$ ft. ; 8. 1596 ; 9. 8 cu. ft. 10. 4 m. ; 11. $2\frac{1}{12}$ cm. ; 12. 1 ft. 1 in. ; 13. 1875 gl. 14. 72 sq. m. ; 15. 6 ft. ; 16. 101 ton 5 cwt. 2 qr. 19 lb. ; 17. £1. 17s. 6d. ; 18. 43200 ;

19. £1. 17s. 4d. ; 20. 16 ft., 8 ft. ; 21. (i) 990 cu. in.
(ii) 708 sq. in. ; (ii) $2\frac{11}{24}$ sr. ; 22. 7980 c. cm. ; Rs. 1. 13·1 P. ;
23. 4608 ; 24. 19·74 in. 25. $10\frac{10}{11}$ m. p. h. ;
26. 16 m., 12 m., 12 m. ; 27. $24\frac{8}{9}$ sq. ft., Rs. 6. 3a. $6\frac{2}{3}$ p.
28. 5lb. 12 oz.

Exercise 9

1. Rs. 2000 ; 2. 44% 3. 4000 ; 4. $39\frac{1}{3}\frac{3}{8}$% ;
5. 14 srs. a rupee ; 6. $13\frac{1}{23}$% ; 7. Rs. 2500.

Exercise 10

1. Rs. 1616 ; 2. Rs. 1350 ; 3. Rs. 9000 ; 4. Rs. 400 ;
5. 4% ; 6. 8% ; 7. 24% ; 8. $4\frac{1}{2}$% ; 9. $16\frac{2}{3}$ yrs. ;
10. 7 yrs ; 11. Rs. 966. 14 a. ; 12. Rs. 550, 5% ;
13. Rs. 800, $7\frac{1}{2}$% ; 14. Rs. 12000 ; 15. Rs. 320 ;
16. 4% ; 17. $6\frac{2}{3}$% ; 18. 5% ; 19. Rs. 1140. 10 a. ;
20. 5·325% ; 21. 10 yrs. ; 22. 16 yrs. 23. £588. 10s. 10d. ;
24. Rs. 503. 25 P. ; 25. Rs. 218. 12a. ; 26. Rs. 5050 ;
27. $9\frac{11}{21}$ yrs.

Exercise 11

1. Profit $6\frac{1}{4}$% ;
3. Rs. 700 ; 3. $23\frac{1}{2}$ P. ;
4. Rs. 12. 8 a. ; 5. Rs. 40 ; 6. Rs. 4000 ; 7. Rs. 52. 8a. ;
8. 4% loss ; 9. Rs. 1200 ; 10. Rs. 279. 84 P. ;
11. 70% ; 12. Profit $2\frac{8}{11}$% ; 13. Rs. 4600 ; 14. Horse Rs. 400,
Carriage Rs. 300 ; 15 Re. 1 ; 16. Rs. 278. 9a. $1\frac{5}{7}$ p. ;
17. $12\frac{1}{2}$% ; 18. 50%.

Exercise 12

1. 30 days ; 2. 4 days, A $\frac{2}{5}$, B $\frac{1}{3}$, C $\frac{4}{15}$;
3. 8 days ;
4. First man Rs. 12, Second Rs. 10, boy Rs. 8 ; 5. 3 days ;
6. 50 days ; 7. 30 days ; 8. 25 hrs. ; 9. 30 hrs.
10. 12 hrs. ; 11. $8\frac{4}{7}$ mins. ; 12. At 7-12 P.M. ;
13. 25 days ; 14. 4 hrs. 24 mins. ; 15. 60 hrs. ;
16. man 20 days, boy 60 days ; 17. 50 days ; 18. $40\frac{10}{11}$ days ;
19. A Rs. 12, B Rs. 8, C Rs. 2.50 P.

Exercise 13

1. $14\frac{1}{2}$ hrs. ; 2. At a distance of 250 yds. ; 3. $19\frac{1}{5}$ Km. per hr. ;
4. 444 Km. ; 5. At 9-20 A.M. ; 9. 20 sec. ; 7. 5 hrs. ;

8. 10 Km. and 4 Km. per hr. ; 9. At 10-15 P. M. ; 10. At 12-30 P. M. ; 12. 160 yds. ; 13. $7\frac{1}{2}$ Km. ; 14. $25\frac{15}{22}$ min. ; 15. $31\frac{1}{2}$ Km. per hr. ; 16. 8 min. ; 17. A, by 220 yds. ; 18. 1 mile ; 19. 10 min. 11 sec. ; 20. 1100 ft. per sec. ; 21. 8 hrs. ; 22. A in 4 hr. 20 min., B in 7 hr. 35 min. ; 23. 120 Km. ; 24. $3\frac{1}{2}$ hrs. ; 25. At 5-15 o'clock ; 26. $16\frac{4}{9}$ min. ; 27. $14\frac{1}{3}$ hrs. ; 29. A $10\frac{29}{30}$ m., B $10\frac{8}{15}$ miles.

Exercise 14

1. (i) $3\frac{188}{319}$ min. past 4 and $37\frac{187}{319}$ min. past 4 (ii) $54\frac{114}{319}$ min. past 4 (iii) $20\frac{160}{319}$ min. past 4. 2. 22 ; 3. $11\frac{100}{121}$ min. slow ; 4. 60 days, 12-14 o'clock, 11-44 o'clock ; 5. 17th August, true time $21\frac{8}{17}$ min. past 2 A. M. ; 6. At $26\frac{122}{143}$ min. past 4. 7. 9 hrs. ; 8. At 2 P. M. on 23rd Aug., 1-46 P. M., 2-16 P. M.

Exercise 15

1. $1\frac{1}{3}$ litres ; 2. $\frac{1}{3}$; 3. 11 : 49 ; 4. $12\frac{1}{2}$ Kg. 6. 401 : 544 ; 6. 2 : 13 ; 7. 4s. 6d. ; 8. 2 : 5 ; 9. 4 : 5 ; 10. $\frac{1}{3}$ quart ; 11. 125 lbs. ; 12. $\frac{13}{48}$ litre ; 13. 343 : 169 ; 14. 2 : 1 ; 15. 45 gallons ; 16. 3 : 1 ; 17. 5 : 1.

Exercise 16

Arranged in descending order of magnitude :—

1. 8 : 12, 6 : 14, 5 : 25 2. $2\frac{1}{2} : 3\frac{1}{3}$, 3 : 5, $\frac{1}{3} : \frac{2}{3}$

3. already arranged

4. 3 yd. 2 ft. : 4 yd. 1 ft., 12 md. : 18 md., Rs. 6 : Rs. 10

5. 20 6. 54 7. Rs. 4. 11 a. 8. 3 yd. 1 ft. 9. Rs. 132 10. 21 yrs. 21. 9 : 10 12. A Rs. 198, B Rs. 126 13. 75 : 58 14. 8 : 15 15. 25 : 18.

Exercise 17

1. 12 2. 18 3. 12 P. 4. $\frac{1}{9}$
5. 18 6. 14 7. Re. 1. 14 as. 8. 32 metres
9. 27 10. $3\frac{9}{128}$ 11. ·2025 12. 2 hr. 5 min.
13. 15 14. 6 15. Rs. 15 16. 45
17. $3\frac{1}{7}$ 18. 2·8 g. 19. ·18 20. 18
21. 45 min. 22. No, 40 23. No, $6\frac{2}{3}$ 24. 63

25. 128 26. 16 : 24 : 30 : 35 27. 9 : 13
28. 20 : 63, 40 : 60 : 105 : 126 29. 1 : 2, 15 yrs.
30. A Rs. 306, B Rs. 408, C Rs. 510 31. 45, 60
32. Ram Rs. 600, Hari Rs. 800, 33. 115, 184
34. A 20 yrs., B 25 yrs. 35. A 48 yrs., B 60 yrs. 36. 5 : 6
37. water 72 litres, wine 108 litres 38. $2\frac{1}{2}$ Kg.
39. 7 Hg. 40. 16 : 15.

Exercise 18

1. (i) 2Kl. (ii) Rs. 24 2. 200, 250, 300
3. Rs. 80, Rs. 200, Rs. 60
4. 15 Dg. coal, 10 Dg. sulphur, 75 Dg. Saltpetre 5. 360
6. 60, 40, 30, 24 7. A Rs. 80, B Rs. 120, C Rs. 150
8. 7 cwt. 2 qr., 1 cwt., 1 cwt. 2 qr.
9. A Rs. 210, B Rs. 150, C Rs. 90.
10. House Rs. 6880, Furniture Rs. 2150
11. Rs. 480, Rs. 400, Rs. 300 12. 162, 108, 72
13. Rs. 8000, Rs. 3000, Rs. 2400
14. 80, 50, 30 respectively.
15. Rs. 112, Rs. 128, Rs. 160 respectively
16. 40, 80, 160 respectively , total sum Rs. 60.
17. 24 cm. 18. 60, 160, 400 respectively
19. 50°, 60°, 70° 20. 45, 50, 55 yrs respectively
21. Rs. 700 22. men Rs. 48, women Rs. 32, boys Rs. 20
23. A Rs. 216, B Rs. 90, C Rs. 96
24. First boy 15 P., second boy 6 P.
25. Rs. 348, Rs. 290, Rs. 232 respectively
26. Man Rs. 60, woman Rs. 40, boy Rs. 15
27. A Rs. 272, B Rs. 32, C Rs. 176
28. Man Rs. 25. 20 P., woman Rs. 15. 75 P., boy Rs. 8. 40 P.
29. 33 Kg. 6 Hg.

Exercise 19

1. Ram Rs. 80, friend Rs. 100 2. A Rs. 32, B Rs. 40
3. A Rs. 400, B Rs. 500, C Rs. 600 4. £ 2. 10 s.
5. Rs. 12. 50 P., Rs. 15, Rs. 17. 50 P. respectively
6. Rs. 400 7. Rs. 480 8. Rs. 4900
9. A Rs. 160, B Rs. 240, C Rs. 600

10. A Rs. 120, B Rs. 108, C Rs. 112
11. A Rs. 288, B Rs. 270, C Rs. 216, D Rs. 126
12. Rs. 391, Rs. 529, Rs. 1311
13. A Rs. 1386, B Rs. 693, C Rs. 2079
14. £ 23. 5 s. 9 d., £ 30. 14 s. 3 d.   15. A Rs. 250, B Rs. 270
16. A Rs. 32. 50 P., B Rs. 29. 25 P.   17. Rs. 736
18. A Rs. 230, B Rs. 300   19. after 4 months
20. A Rs. 350, B Rs. 370.

Exercise 20

1. Rs. 66   2. Rs. 9450   3. Rs. 438. 75 P.
4. Rs. 500. 25 P.   5. Rs. 1210   6. Rs. 8540
7. Rs. 1950   8. Rs. 4585. 50 P.
9. Rs. 3472   10. Rs. 558   11. Rs. 500
12. 5 P.   13. Rs. 190   14. Rs. 130.

Exercise 21

1. £ 6705. 14 s. 7 d.   2. Re. 1. 15 as.   3. Rs. 30
4. £ 1 = 4·52 dollars (App.)   5. Re. 1 = $1\frac{1}{3}$ francs
6. 25·29 dollars   7. Rs. 4500   8. Re. 1 = 1·682 fr.
9. 17 s. 6 d.   10. Rs. 15   11. Rs. 133. 5a. 4p.
12. Rs. 15516. 3 a. 9 p.   13. £ 1 = Rs. 15·625
14. £ 1231. 17 s. 6d.   15. £ 225.

Exercise 22

1. 390   2. 1230   3. 1 m. 4 dm. 1 cm. 4 mm. (App.)
4. ·001   5. 304·8 cm.   6. 24855 miles
7. 24855·36$\dot{6}\dot{1}$ miles   8. 1 Km. 47 m.
10. 11·101 lbs.   11. 239·197...sq. yd.
12. 20·40 sq. m.   13. 2020000 francs
14. 15499969   15. 61·024 cu. in. (App.)
16. l=1·875 m., b=·625 m.   17. 104550 litres
18. 240000 g.   19. 2500 Kg.
20. 7·14 Kg.   21. ·09728381300$\dot{5}71428\dot{}$
22. 330 m.   23. 1s. 11d. 1q.
24. Rs. 227. 12a. 9·6 p.   25. 10750 francs
26. 1 hr. 25 min. 20 sec.   27. 4s. 9 d.
28. 9 m., 3 m., Rs. 144   29. $6\frac{2}{3}\%$ loss   30. Rs. 51181.

Exercise 23

1. 7,000 ; 74,00   2. 10,000 ; 9700   3. 4000, 3700
4. 3·74, 3·740, 3·7404   5. 8   6. 3·27
7. 5·07   8. 72·1   9. ·00788
10. (1) ·32, ·0016, ·0049... ; ·49... ; (2) 2·45, ·003, ·0012...,·12...;
(3) ·03, ·0031, ·1152..., 11·52...,
·0004, ·0012..., ·12 ;   11. (i) ·328, (ii) ·0215, ·00005, ·0023, ·23... ;
(iii) 2·03, ·004, ·0019..., ·19...   12. ·22..., ·33... ;
13. 6 : 5   14. 37·7995, 37·80   15. 14·0942, 14·09
16. 18·7477, 18·75   17. 4·539, 4·54   18. ·049, ·0492
19. 5·485, 5·48   20. ·390625.

Exercise 24

1. Rs. 1576. 25 P.   2. £ 51   3. £ 78. 16 s. 3d.
4. Rs. 460. 2a.   5. Rs. 48. 83 P.   6. Rs. 46. 45 P.
7. £ 33. 14s. 10d. 1 q. (App.) ;   8. £ 1340. 1 s. 11d. (App.)
9. Rs. 595508   10. Rs. 815. 85 P.   11. Rs. 1940.26 P.
12. Rs. 497. 1 P. (App.)   13. Rs. 15. 25 P.
14. Rs. 3200   15. A, 10 as.   16. Rs. 30. 60 P. (App.)
17. Rs. 467. 94 P. (App.)   18. Rs. 27. 54 P. (App.)
19. Rs. 3. 5 P. (App.)   20. Rs. 12500   21. Rs. 10000
22. Rs. 3750   24. Rs. 1576. 25 P.   25. Rs. 157. 10 as.
26. Rs. 625   27. Rs. 625   28. 27783   29. 108160
30. Rs. 400, 5%   31. Rs. 1250, 4%   32. Rs. 2205
33. Rs. 464. 1 a. 7·2p.   34. Rs. 5618   35. Rs. 104. 50 P.

STATISTICS

Exercise 1

8. (i) 7, (ii) 75 ;   9. 23 – 27(*f*6), 28 – 32(*f*11)...etc.
10. 23 – 26(*f*4), 27 – 30 (*f*9),...etc. ;
11. 73, 73, 74, 74, 75, 75, 76, 76, 77, 77.
12. limits 74·5 – 79·5, 79·5 – 84·5,...etc. ;
mid points 77, 82,...etc. ;
13. Limits of interval...mid-points

82·5 – 87·5	85
77·5 – 82·5	80
72·5 – 77·5	75

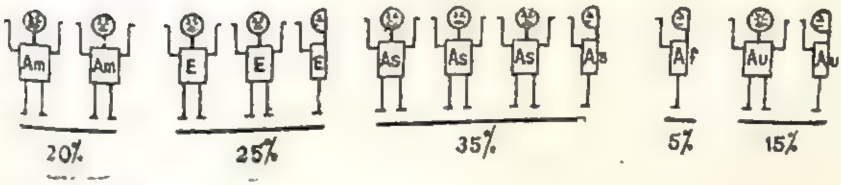
14. 50·75 – 52·75
52·75 – 54·75
54·75 – 56·75

15. (1) 54 Kg. and under (f7)
55 Kg. and under (f9), etc.
(2) 62 Kg. and above (f12),
61 Kg. and above (f26), etc.

16. 8 – 10(f30), 11 – 13(f21), 14 – 16(f48), 17 – 19(f45).

Exercise 2

2.



11. Rs. 450.

Exercise 3

1. (i) $627{\cdot}\dot{3}$ (ii) $18\frac{3}{16}$ (iii) 114·25
2. 26·1, 25·5, 24·3 3. $\frac{1}{2}(n+1)$ 4. Rs. 13·7
5. 334·25 6. Rs. 21·96 7. 119·554, 119·33, 118·89
8. (*a*) 43·2 (*b*) 44·68 (*c*) 45·5 (*d*) 48·62
9. 0 10. 3 yrs. 11. £ 23 (App.) 12. first
13. 58·03 (App.), $57{\cdot}\dot{0}\dot{9}$ 14. 45·25
15. 9·72 (App.) 16. $67{\cdot}\dot{3}$

Exercise 4

1. 1·2 2. 9·3 3. 2·57 4. 11·12 5. 1·41 6. 3·38
7. 11·7° 8. 1·117″ 9. 6 md. 10. 101·02, 13.

Exercise 5

2. 73·1 4. 7 5. 8·67, 8·67 6. 75·17 (App.).
7·58 (App.) 7. 143·125, ·6 9. 1·5, ·68 10. 1, 1
11. ·7 12. 4·6, 1·75.

ALGEBRA

Exercise 1

1. $x^3+y^3+z^3-3xyz$ 2. $1-x^8$ 3. $a^3+2ab-ab^2+b-1$
4. $a^8+a^4b^4+b^8$ 5. $a^3+b^3-1+3ab$
6. $-6a^6-9a^5+19a^4+4a^3-11a^2+8a-5$
7. x^8+x^4+1 8. $4x^3+3x+1$
9. $x^2+y^2+a^2$ 10. a^4-a^2+a
11. a^3+2a^2+4a+2 12. $a^2+b^2+c^2-ab+ac+bc$
13. Product $=4x^4+y^4-5x^2y^2$, Quotient $=2x^2+xy-y^2$
14. $(a^2-b^2)(a^2+ab+b^2)$ 15. $x^2+y^2+1-xy+x+y$
16. $x^4+x^3y-xy^3-y^4-\dfrac{y^5}{x}+\dfrac{y^7}{x^3}+\dfrac{y^8}{x^4}$,
17. $a^2+b^2+c^2+ab-ac+bc$ 18. 7.

Exercise 2

1. 1 2. 9 3. 7 4. 18 5. 1 6. 5
7. 5 8. (i) 69 (ii) 25 9. 14, 52
10. 0 11. 110 12. 27 13. 35 14. 125
16. 0 17. (i) 34, (ii) 2 18. 1 20. c^3+3c
21. $4y^2$ 22. $\left(\dfrac{a+b}{2}\right)^2-\left(\dfrac{a-b}{2}\right)^2$
23. $(x+z)^2-(y-z)^2$
25. 1 26. 6 24. $(2a+b+c)^2-(b-a-c)^2$
28. $(ac+bd)^2+(bc-ad)^2$ 27. $(a+b)^2-(b+c)^2$
29. 174.

Exercise 3

1. $(2x+3y)(2x-3y)(4x^2+9y^2)$
3. $(m^2+mn+n^2)(m^2-mn+n^2)$ 2. $(x+y)(x-y)(x^2+y^2)$
5. $(x^2+xy+y^2)(x^2-xy+y^2)$ 4. $(x^2+2x+2)(x^2-2x+2)$
7. (i) $(a+b-c)(a-b+c)$ 6. $(2x+z)(2x-2y-z)$
(ii) $(a+b-c-d)(a-b+c-d)$
8. $(2x^2+6x+9)(2x^2-6x+9)$
10. $(x^2+4x+8)(x^2-4x+8)$ 9. $(x^2+2x+3)(x^2-2x+3)$
11. $(a^2+2b^2)(a^2-2b^2)(a^2+2ab+2b^2)(a^2-2ab+2b^2)$
12. $(x^2+2x-2)(x^2-2x-2)$
14. $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$ 13. $(x^2+x+1)(x^2-x+1)$
15. $(x-3)(x^2+3x+9)$
16. $(x+4y)(x^2-4xy+16y^2)$

17. (i) $(x+3)(x^2-3x+9)(x-3)(x^2+3x+9)$
(ii) $(2a+3b)(4a^2-6ab+9b^2)$

18. $(x-3)(x+2)$ 19. (i) $(x+4)(x-7)$ (ii) $(3+4x)(4-5x)$

20. $x^2y^2(5x-3y)(25x^2+15xy+9y^2)$ 21. $(x-7)(x+5)$

22. $(2-3a)(3+4a)$ 23. $(x+y-4a)(x-y-2a)$

24. $(3x+11)(4x+7)$ 25. $(3x+7y)(7x-3y)$

26. $(2a^2+5)(2a+3)(2a-3)$

27. $(a^2+2a-4)(a^4-2a^3+8a^2+8a+16)$

28. $(a+b-2c)(a^2+b^2+4c^2-ab+2ac+2bc)$

29. $(a-1)(a^2-a-4)$ 30. $(x+2)(x-1)(x+15)$

31. $(x+2)(x+3)(x+4)$ 32. $(x+1)(x-5)(x-2)^2$

33. $(a^2+b^2)(x^2+y^2)$ 34. $(c+d+a-b)(c+d-a+b)$

35. $(5a-3b)(15b-13a)$ 36. $(x-1)(x+1)(x-3)$

37. $(a-5)(a+2)(a^2-3a+12)$

38. $(x^2-3x-5)(x^2-3x-17)$ 39. $(x+y)(y+z)(z+x)$

40. $(x-y-1)(x^2+y^2+1+xy+x-y)$

41. $(a-b)(b-c)(c-a)(a+b+c)$

42. $-(a-b)(b-c)(c-a)(ab+ac+bc)$ 43. $(a+b)(b+c)(c+a)$

44. $3(2x-3y)(3y-z)(z-2x)$ 45. $(x^2+5x+3)(x^2+5x+7)$

46. $(x+a+3)(x-a-1)$ 47. $(x-3)(2x+3)(2x^2-3x+7)$

48. $3(a+b)(b-c)(a+2b-c)$

49. $(a+d+b+c)(a+d-b-c)(b-c+a-d)(b-c-a+d)$

50. $2(a+c)(1+a)(1-c)$

51. (i) $(x+5)(5x+1)(5x^2+14x+20)$
(ii) $(a^2+3a-2)(a^2+3a-3)$

52. $(x^2+1)(x^2+x+1)$ 53. $(x-b)(x-c)(b-c)$.

Exercise 4

1. $2x+3$ 2. $x-3$ 3. $3x+1$ 4. $2x^2+7x+3$
5. $x+5$ 6. $x-2$ 7. $x+1$ 8. x^3+3x+5
9. x^2+4x+3 10. $a-2b$ 11. x^2-3x-4
12. $x+2$ 13. x^2-3x+4 14. x^3-2x-1
15. $x+3$ 16. $x+5$ 17. $x-1$
18. $x-1$ 19. a^2+3a+1 20. $3x-5$
21. x^2+x-2 22. $x-2$ 23. x^2+2x+3
24. $3x-7$ 25. $x-1$ 26. $a-1$
27. x^2-x+1 28. $2x(x-2)$

30. $(x+1)(x-1)(x+2)(x-2)$ 31. $(x+1)(x-1)(2x+1)$
32. $(x-a)(x+c)(x-c)$ 33. $(x-1)(x-2)(x-3)$
34. $(a+2)(a+3)(a+4)(a^2+a+1)$
35. $(x+1)(x-1)(x-5)(x-7)$
36. $x^2(x-2)(x-1)(x+2)(x+3)$
37. $(a+1)(a-1)^2(a-2)(a^2+1)$
38. $(a+b+c)(a-b+c)(a+b-c)^2$
39. $(2x+3)(4x^2-6x+9)(4x^2+6x+9)(7x^2-5x-6)$
40. (i) $36(x^2-1)(x^2-4)(x^2-9)$ (ii) $x^2(x+2)(x-2)(x+4)$
41. $(x-2)(x^2+2)(x^2+x+1)$ 42. x^2+4x+3
43. x^2-1 and x^2+2x-3.

Exercise 5

1. 1 2. x 3. $\dfrac{7x+5}{(x+1)(x-1)(x+2)}$ 4. $\dfrac{-64ax^3}{a^4-16x^4}$
5. 1 6. 1 7. 1 8. 0 9. 0
10. $\dfrac{3}{(x-1)(x-3)}$ 11. 0 12. 0 13. 3
14. $\dfrac{1}{2}$ 15. $\dfrac{1}{xyz}$ 16. 1
17. $\dfrac{1}{abc}$; 18. 0 ; 19. 0 ; 20. 0 ; 21. $-\dfrac{1}{1-x}$;
22. $\dfrac{16x^{15}}{x^{16}-1}$; 23. $\dfrac{(a-b)^2}{ab}$.

Exercise 6

1. 3 ; 2. 3 ; 3. $\frac{1}{3}$; 4. $\dfrac{6a}{7}$; 5. 1 ; 6. 5 ; 7. $\frac{5}{2}$;
8. $1\frac{1}{2}$; 9. $2\frac{21}{29}$; 10. $-1\frac{1}{3}$; 11. 7 ; 12. 3 ;
13. $\dfrac{ab}{2(a+b)}$ 14. (i) $\dfrac{ab}{a+b}$, (ii) $\dfrac{1}{ab}$; 15. $\frac{1}{2}(a+b)$; 16. $3\frac{1}{2}$;
17. b ; 18. $-\frac{1}{2}(a+b)$; 19. $a(a+b+c)$;
20. $-(a^2+b^2+c^2)$; 21. $\dfrac{a^2+b^2+c^2}{abc}$ 22. $10\frac{3}{8}$;
23. $\dfrac{ac+b^2}{b^2+c^2}$; 24. $2\frac{1}{3}$; 25. $2\frac{2}{5}$; 25. (a). $-6\frac{1}{2}$;
26. $-5\frac{1}{3}$; 27. 9 ; 28. $-b$; 29. 2 ; 29 (a). $\dfrac{ab}{a-2b}$;
30. $\dfrac{ab-cd}{a+b+c-d}$; 31. $\dfrac{a+b}{2}$; 32. $\dfrac{ab}{a+b}$.

Exercise 7

Respective values of x and y, or, of x, y, z are :—

1. 13, 6 ; 2. 5, 2 ; 3. 6, 2 ; 4. -3, 3, 1 ;
5 (a). 4, 10 ; 5 (b). 1, 2 ; 6. 3, 1 ; 7. 0, 2, 4 ;
8. 5, 1 ; 9. $-\frac{1}{2}, \frac{1}{7}$; 10. $\frac{1}{4}, -4$; 11. 3, 2 ;
12. 3, 3, 3 ; 13. 1, 3, 5 ; 14. 20, 15, 12 ; 15. 4, 6, 7 ;
16. $\frac{1}{2a}, \frac{1}{8a}, \frac{1}{4a}$; 16 (a). 2, -1, 3 ; 17. 1, 1, 1 ;
18. a, b, c ; 19. $bc(b-c), ca(c-a), ab(a-b)$; 20. 4, 3, 5 ;
21. 1, $\frac{1}{2}, \frac{1}{3}$; 22. 1, $\frac{2}{3}, \frac{1}{2}$;
23. $bc+ab-ca, ab+ca-bc, ca+bc-ab$;
24. 1, 2, 3 ; 25. $\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}$; 26. 1, $\frac{1}{2}, \frac{1}{3}$.
27. $\frac{ac-b-bc}{a^2-b^2}, \frac{a+ac-bc}{a^2-b^2}$; 28. $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$.

Exercise 9

7. 0 ; 21 (a). 216, (b) 104 ; 22. 2 ; 30. 36.

Exercise 10

1 (a). 12 ft., (b) Father's 38 yrs., Son's 14 yrs. ; 2. $3\frac{2}{3}m.$;
3. 35, 15 ; 4. 19, 42 ; 5. 20 ; 6. 35 m., 45 m. ;
7. 69 ; 8. 1215, 15 ; 9. 91 ; 10. $\frac{15}{26}$;
11. 80 ; 12. 93 ; 13. 72 ; 14. 54 ;
15. (1) $10\frac{10}{11}$ min. past 2. (2) $43\frac{7}{11}$ min. past 2.
(3) $2\frac{8}{11}$ min. past 2. (4) 24 mins. past 2. 16. $32\frac{4}{13}$ mins. past 5 ;
17. $22\frac{2}{319}$ mins. past 2 ; 18. $3\frac{3}{4}$ cubits ; 19. 600 ;
20. 960 ; 21. 84 d. ; 22. 2 m.p.h. ; 23. 2 : 3 ;
24. 21 tolas. 25. 10 hrs. ; 26. 28. 27. 2 mi. and 4 mi. per hr.
28. Boat 4 mi., current 1 mi. per hr. 29. 84 30. In 1800
31. $\frac{3}{4}d$; 512 32. $21\frac{9}{11}$ min. past 1 and $54\frac{6}{11}$ min. past 1.
33. tea 2s., cofee $1\frac{1}{3}$s. 34. 123 35. 180 miles
36. 7 37. $\frac{4}{15}$.

Exercise 11

15. 5, $-3{\cdot}5$ 16. 1, 1 17. 3, 4 18. 2, 4 19. 0, 5
20. 7, 1 21. $x=3, y=1$ 22. $x=5, y=1$
23. $x=-1$ 24. $x=1$ 25. $x=1, y=1$

26. $x=1\frac{1}{3}$, $y=3\frac{1}{2}$ 29. 31·5, ·5 30. $y=1$
31. $(2\frac{1}{3}, 2\frac{2}{3})$, (7, 5), (7, −2) 32. 90° 33. 13 units or 1·3 in. 34. ·225 sq. in. 35. 12 sq. units or ·12 sq. in. 37. ·1 sq. in. 38. 52·3 units of length.
39. (*i*) $3x+2y=12$ (*ii*) $y+2x=0$ (*iii*) $x+y+2=0$ (*iv*) $x+2y=8$.
43. right angle, co-ordinates (1, 1). 45. $10\frac{1}{2}$ as., 14.
46. at 5 P. M., at a distance of 35 miles. 47. (1) 50 mins. after P starts, 25 miles from Howrah, (2) 6 miles.
48. 4 hr. 12 min. after A starts, at a distance of 13·2 miles from the starting place. 49. at 9 mins. past 9, at a distance of 26 miles from Howrah. 50. 80 miles. 51. 2·4 hrs.
52. (*a*) at 24 mins. past 5, 24 miles from P, (*b*) 4 miles.
53. (i) Rs. 80, (ii) Rs. 240 per month. 54. 24500.

Exercise 12

1. ± 5 2. ± 1 3. $\pm\sqrt{2}$ 4. ± 2
5. $\frac{4}{5}, -\frac{5}{14}$ 6. $3, \frac{1}{3}$ 7. $14, 2\frac{1}{4}$ 8. $\frac{5}{3}, -\frac{3}{5}$
9. $7, -4\frac{9}{17}$ 10. $9\frac{15}{17}, -11$. 11. $5\frac{2}{3}, 9\frac{1}{2}$ 12. 3, 23
13. $2, -\frac{9}{2}$ 14. $12, \frac{3}{4}$ 15. $3\pm\sqrt{7}$ 16. −3, 2
17. $\sqrt{13}\pm 3$ 18. 6, 9 19. $4, -2\frac{1}{4}$ 20. 3, −4
21. $5\pm\sqrt{17}$ 22. $3, -\frac{68}{9}$ 23. $9, \frac{15}{13}$ 24. $6, \frac{1}{6}$
25. −4, 2 26. $5, \frac{6}{5}$ 27. 0, −7 28. $0, 2\frac{1}{4}$
29. $-2a, -3a$ 30. $\frac{3\pm\sqrt{10}}{2}$ 31. $\sqrt{3}\pm 4$ 32. $c, c-b$
33. $0, a+b$ 34. $\frac{b\pm\sqrt{b^2+4ac}}{2a}$ 35. $-a, -b$
36. $-\frac{2}{3}(1\pm\sqrt{7})$ 37. 43, −42 38. $2\frac{3}{7}, -1\frac{4}{5}$
39. $1, \frac{1}{a}$ 40. 1·85, −1·12.

Exercise 13

1. 4 2. $\frac{2}{3}, \frac{4}{5}$ 3. 5, 7, or −7, −5 4. 10, 11, or −11, −10 5. 7 or −6 6. 11, 13 or −13, −11
7. 121, 100 (in the second answer −10 is taken as sq. foot)
8. 6, 8, or −8, −6 9. 8″, 15″ 10. 5″, 12″
11. A 2 hr., B 3 hr. 12. 9 d. 13. 20
14. 76 15. 6 Km. 16. 8 cm. 17. 20m., 13m.
18. 2 Km., 4 Km. 19. 19, 20 20. 7 in.

MENSURATION

Exercise 1

1. 168 2. Rs. 1771 3. 3·84 acres, 2·16 acres
4. 110400 5. Rs. 35. 1a. 4p. 6. Rs. 4. 1a.
7. 360 8. 117 ft.

Exercise 2

1. 25 cm., 150 sq. cm. 2. 150 ft. 3. 36 cms., 15 cms.
4. 12637 ft., 12012 ft. 5. 15 m. 6. $7\frac{1}{3}$ ft.
7. $4\sqrt{3}$ cm. 8. 11·6 ft. (App.) 9. 30 ft.
10. 5000 sq. yds. 11. 176 yds. 12. $2\frac{1}{3}$ sq. ft.
13. 300 sq. cm. 14. 57·19 ft. (App.)
15. 34·64 ft., 519·6 sq. ft. (App.)
16. $200\sqrt{3}$ sq. ft. 17. 9841·3 sq. ft. (App.).

Exercise 3

1. 7 ft. 4 in. 2. 1 ft. 9 in. 3. 180 4. 26 ft. 3 in.
5. 16 s. 6 d. 6. 7 ft., $3\frac{1}{2}$ ft. 7. 30 miles
8. $10\frac{1}{2}$ in. 9. 10 miles 10. $10\frac{1}{3}$ in., 66 in.
11. 15 ft. 9 in. 12. 7 metres 13. 110 yds.
14. 19 yd. 1 ft. $11\frac{1}{7}$ in. 15. 154 sq. cm. 16. 1 ft. 2 in.
17. 1 yd. 8 in. 18. 2 sq. ft. $58\frac{1}{3}$ sq. in. 19. 17sq. ft. 16sq. in.
20. 4840 yds. 21. 27·74 yds.
22. 12 cm. 23. £ 246. 8 s. 24. 14 m.
25. 7 in. 26. 99·2 in. (App.) 27. $3\frac{1}{2}$ ft.
28. Rs. 233. 12a. 3p. 29. Rs. 577. 8a., 308 ft., 14 secs.
30. $407\frac{11}{68}$ sq. ft.

Exercise 4

1. 1032 sq. cm., 2160 cu. cm. 2. 5 cm., $5\sqrt{3}$ cm.
3. 96 sq. cm., 64 cu. cm. 4. 2 Kg. 5 Hg.
5. 1152 cu. m., 26 m.
6. length 27 m., breadth 15 m., width 12 m.
7. 5·196 cm. 8. $\frac{1}{4}$ m. 9. $8\sqrt{3}$ cm., 512 cu. cm.
10. 13 m. 11. 5 m. 12. 8 m.
13. 245 sq. cm. 14. 4320 sq. cm.
15. 376 sq. cm. 16. 18 cm.
17. 25 cm. 18. 125 19. 32000 20. 512 cu. cm.

Exercise 5

1. 220 sq. m. 2. 165 sq. ft. 3. 1232 sq. dm.
4. 341 sq. cm. 5. 3 m. 6. Rs. 27. 8 as.
7. $1\frac{3}{4}$ m. 8. 396 cu. m. 9. 7 m.
10. Rs. 440 11. 462 sq. cm. 12. $339\frac{3}{7}$ lb.
13. $\frac{1}{4}$ cm. 14. $1018\frac{2}{7}$ cu. in.

Exercise 6

1. 98 sq. dm. 56 sq. cm. 2. 154 sq. dm.
3. $1437\frac{1}{3}$ cu. cm. 4. $3\frac{1}{2}$ cm. 5. $\frac{\sqrt{2}}{4}$ m.
6. 7 m. 7. $14\frac{1}{7}$ cu. ft. 8. 3 m. 9. 216
10. 840 11. 1299·87 cu. in. 12. 4 cm.
14. 201 lb. $13\frac{3}{8}$ oz. 15. 12 in.
16. 1257·14 sq. cm. (App.).

SCHOOL FINAL EXAMINATION—1965

MATHEMATICS (Compulsory)

1. (*a*) Simplify: $\dfrac{\text{Rs. 2.20nP.}}{\text{Re. 1.4 nP.}} - \dfrac{\frac{2}{3} \div \frac{2}{5} \text{ of } \frac{2}{3}}{\frac{2}{3} \div \frac{2}{5} \times \frac{2}{3}} \times \frac{2}{3} + \left(\dfrac{\frac{14}{15}}{1\frac{1}{15}} + 7\frac{6}{13} - 6\frac{56}{91} \right)$.

[Ans. 1]

Or, (*a*) If one yard be equivalent to 0·914 metre, find correct to two decimal places, the number of cubic centimetres, equivalent to one cubic foot. [Ans. 28279·70 cm.]

(*b*) A person has an annual income of Rs. 1,915.50 nP. For the first 20 weeks of a year, he spends at the rate of Rs. 45.15nP. per week; assuming that there are 365 days in the year, what should be his expenses per day for the rest of the year, so that he may not find himself in debt at the end of the year.

[Ans. Rs. 4.50nP.]

2. A gentleman's salary is Rs. 625 per month. For the first 3,000 rupees of his annual income, he has not to pay any income tax; for the next 2,000 rupees, the rate of income tax is 7 nP. per rupee and above that, the rate is 9 nP. per rupee. Find the amount of income tax he has to pay in a year.

[Ans. Rs. 365]

Or, The total number of pupils in three classes of a school is 333. The number of pupils in classes I and II are in the ratio 3 : 5 and those in classes II and III are in the ratio 7 : 11. Find the number of pupils in each class.

[Ans. 63, 105, 165]

3. Find the difference between the simple interest and the compound interest (compounded annually) on a sum of Rs. 2,000 lent out for 3 years at 5 per cent per annum. [Ans. Rs. 15.25nP.]

Or, The following table shows the frequency distribution of marks of 34 students in a certain subject. Calculate the mean of the marks.

[Ans. 70·35]

Number of students	4	2	3	5	7	5	4	3	1
Marks	50	55	62	68	73	75	81	85	91

4. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) :

(*a*) Resolve into factors :

(*i*) $x(x-4)-y(y-4)$; (*ii*) $12x^2-7x-10$.

[Ans. (*i*) $(x-y)(x+y-4)$; (*ii*) $(3x+2)(4x-5)$]

(*b*) Find the L. C. M. of :

$2x^2-9x+9$, $6x^2-x-12$ and $3x^2-2x-8$.

[Ans. $(x-2)(x-3)(2x-3)(3x+4)$]

(*c*) Find the H. C. F. of :

$6x^3-8x^2-40x+30$ and $2x^2-x-15$. [Ans. $(x-3)$]

(*d*) If $a+b+c=0$, prove that

$$\frac{1}{(a+b)(a+c)}+\frac{1}{(b+c)(b+a)}+\frac{1}{(c+a)(c+b)}=0.$$

5. (*a*) Solve : (*i*) $\frac{1}{x-1}+\frac{1}{x-4}=\frac{2}{x-3}$. (*ii*) $\left.\begin{array}{l}2x+\ y=8\\3x-2y=5\end{array}\right\}$.

[Ans. (*i*) $x=7$; (*ii*) $x=3, y=2$]

Or, (*a*) A number cosists of two digits ; the digit in the tens' place is twice that in the units' place, and if 18 be subtracted from the number, the digits interchange their places. Find the number.

[Ans. 42]

6. Using the same unit and the same axes of co-ordinates, draw the graphs of the following two equations, plotting at least three points on each graph :—

$$x+2y=6, \qquad x+y+1=0$$

and from the graphs, find the co-ordinates of their point of intersection. [Ans. $(-8, 7)$]

7. Answer *either* (*a*) and (*b*), *or*, (*c*) and (*d*) :

(*a*) Prove that a quadrilateral is a parallelogram if one pair of opposite sides are equal and parallel.

(*b*) ABCD is a parallelogram ; AB and AD are produced to P and Q respectively, so that BP=AB and DQ=AD. Prove that P, C and Q lie on the same straight line.

(*c*) Prove that parallelograms on the same base and between the same parallels are equal in area.

(*d*) ABCD is a rectangle and PQRS, the figure formed by joining in order the middle points of the sides. Prove that PQRS is a rhombus.

8. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :—

(*a*) Prove that the angles in the same segment of a circle are equal.

(*b*) ABCD is a quadrilateral inscribed in a circle whose centre is O ; if the diagonals AC, BD intersect at E, prove that

$$\angle AOB + \angle COD = 2\angle AEB.$$

(*c*) If two circles touch, prove that the point of contact lies in the straight line through the centres.

(*d*) The length, breadth and height of a rectangular parallelopiped are in the ratio 6 : 5 : 4 and its whole surface is 33,300 sq. cms. Find the length, breadth and height of the parallelopiped. [Ans. 90 cm., 75 cm., 60 cm.]

9. Answer any *two* of the following questions (*a*) to (*f*) :—

(*a*) What two numbers between 50 and 100 have 16 for their G. C. M. ? [Ans. 64, 80 ; and 80, 96]

(b) If one rupee be equivalent to 1s. $3\frac{3}{8}d.$, how much a person has to pay in order to buy a Bank Draft £ 1030. 7s. 6d. on a London Bank ?

[Ans. Rs. 15826·56]

(c) If $2s = a+b+c+d$, prove that

$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

(d) Resolve into factors :—

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

(e) P, Q, R are the middle points of the sides of a triangle and X is the foot of the perpendicular let fall from one vertex on the opposite side ; prove that the four points P, Q, R, X lie on a circle.

(f) Draw a triangle ABC whose sides BC, CA, AB are respectively 7 cms., 5 cms., 6 cms. long. From A, draw the perpendicular AD on BC and measure its length. Hence calculate the area of the triangle ABC.

[Ans. AD = 4·19 cm., Area = 14·69 sq. cm.]

SCHOOL FINAL EXAMINATION—1966

MATHEMATICS (Compulsory)

1. (a) Simplify : $\dfrac{\frac{2}{3}\div\frac{3}{4}\text{ of }\frac{5}{6}}{\frac{2}{3}\div\frac{3}{4}\times\frac{5}{6}}-\dfrac{7\cdot7\times0\cdot12}{2\cdot1}+\dfrac{\text{Rs. }5.84\text{nP.}}{\text{Rs. }2.19\text{nP.}}$

[Ans. $3\frac{2}{3}$]

Or, (a) If 39 inches = 99 cm., express 13 miles in kilometres, metres and centimetres. [Ans. 20 Km. 908 m. 80 cm.]

(b) A Railway train passes a telegraph post in 5 seconds and a bridge 450 metres long, in 15 seconds, find the length and the speed of the train. [Ans. length = 22 m., 45 m. per sec.]

2. A fruit-seller bought a number of mangoes at the rate of 15 per rupee, and again an equal number at 12 per rupee. He mixes the whole lot and sells them at the rate of 13 per rupee. What will be the percentage of profit or loss ?

[Ans. $2\frac{22}{39}$% profit]

Or, A contractor undertakes to dig a canal, 12 miles long in 350 days and employs 45 men ; he finds that in 200 days, he has completed $4\frac{1}{2}$ miles. How many additional men must he employ to get the undertaking finished in time ? [Ans. 55]

3. *A* lends out a sum of Rs. 5,000 at the rate of 5% compound interest ; *B* lends the same amount at $5\frac{1}{4}$% simple interest. At the end of 3 years, they collect the interest due. Who will be the gainer and by what amount ?

[Ans. A, by Rs. 625]

Or, The following table gives the frequency distribution of the heights of 52 students in a school. Calculate the mean height.

No. of students	4	7	10	16	8	5	3
Heights (in inches)	30	33	35	40	43	45	48

[Ans. 38·73″ (app.)]

4. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :—

(*a*) Resolve into factors :

(*i*) $8a^4+2a^2-45$, (*ii*) $x^2-y^2-6xa+2ya+8a^2$

[Ans. (*i*) $(2a^2+5)(2a+3)(2a-3)$; (*ii*) $(x+y-4a)(x-y-2a)$]

(*b*) Find the L. C. M. of :

$6x^2-13xa+6a^2$, $6x^2+11xa-10a^2$ and $6x^2+2xa-4a^2$.

[Ans. $2(2x-3a)(3x-2a)(2x+5a)(x+a)$]

(*c*) Find the H. C. F. of :

$3x^3+17x^2-62x+14$ and $21x^3+156x^2-138x+24$.

[Ans. x^2+8x-2]

(*d*) If $3\,(a^2+b^2+c^2)=(a+b+c)^2$, prove that $a=b=c$.

5. (*a*) Solve : (*i*) $\dfrac{3x+2}{x-1}+\dfrac{2x-4}{x+2}=5$. (*ii*) $\left.\begin{array}{r}2(x-\ \ y=\ 3\\ 5x+8y=14\end{array}\right\}$.

[Ans (i) $x=6$; (ii) $x=2$, $y=\frac{1}{2}$]

Or, (*a*) A farmer bought equal numbers of two kinds of sheep, one kind at Rs. 6 each, the other at Rs. 8 each ; if he had expended his money equally in the two kinds, he would have three sheep more than he had. How many of each kind did he buy ?

[Ans. 72 sheep of each kind]

6. Using the same unit and the same axes of co-ordinates, draw the graphs of the following two equations, plotting at least three points on each graph :—

$$y-x=2, \quad 3x-2y=5 ;$$

and from the graphs, find the co-ordinates of their point of intersection.

[Ans. (9, 11)]

7. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*a*) Prove that triangles on the same base and between the same parallels are equal in area,

(*b*) ABCD is a trapezium with its sides AD and BC parallel. If X is the middle point of DC, prove that the triangle AXB is half of the trapezium.

(*c*) If a triangle is such, that the square on one side is equal to the sum of the squares on the other two sides, prove that the angle contained by these other two sides is a right angle.

(*d*) Prove that the sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

8. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*) :

(*o*) Prove that the chords of a circle which are equidistant from its centre, are equal.

(*b*) If two equal chords of a circle intersect within the circle, prove that the segments of the one are equal respectively to the segments of the other.

(*c*) Prove that the two tangents to a circle, from an external point are equal and they subtend equal angles at the centre.

(*d*) · The curved surface of a right circular cylinder is 1,000 sq. cm. and the diameter of its base is 20 cm. ; find the volume of the cylinder.

[Ans. 5000 c. c.]

9. Answer any *two* of the following questions (*a*) to (*f*) :—

(*a*) Reduce $\frac{80025}{937893}$ to its lowest terms. [Ans. $\frac{25}{293}$]

(*b*) A person pays Rs. 51,000 to a bank for a bill of exchange payable in London. The rate of exchange is 1*s.* $10\frac{1}{2}d$ for the rupee and the bank charge is 2 per cent on the amount payable in England. How much will his agent in London receive ?

[Ans. £4687. 10*s.*]

(*c*) If $x=\frac{4ab}{a+b}$, prove that $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=2.$

(*d*) Express $(x+7)(x+9)(x+11)(x+13)$ as the difference of two squares. [Ans. $(x^2+20x+95)^2-(4)^2$]

(*e*) ABC is a triangle ; its sides AB and AC are produced to D and E respectively. Prove that the bisectors of the three angles DBC, BCE and BAC are concurrent.

(*f*) Construct a triangle having given its base, and its area equal to that of a given triangle. (Statement of construction as well as proof must be given.)